

GEOMETRY PROBLEMS

IN-CLASS PROBLEMS (AUGUST 13, 2007):

1. Show that every polyhedron has two faces with the same number of edges.
2. Given $2n$ points in the plane, prove that there exists a straight line having exactly n points on each side of it.
3. If every point of the plane is colored either red or blue, prove that one of these colors contains pairs of points at every possible distance.
4. Let T be a triangle with side lengths a, b, c and area A . Prove that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}A ,$$

with equality if and only if T is equilateral.

5. A “strip” is the region between two parallel lines in the plane. Prove that you cannot cover the entire plane with a set of strips whose widths have a finite sum.
6. Given a finite set S of points in the plane, not all lying on a single line, prove that there is a line passing through exactly two points of S .

HOMEWORK PROBLEMS (DUE AUGUST 28):

1. There are four lines in the plane in general position (no two parallel, no three intersecting in a common point). Four ghost beetles are crawling along these lines with constant velocity, one per line. Being ghosts, if the beetles cross paths they just continue crawling through each other uninterrupted. Suppose that five of the possible six meetings actually happen. Prove that the sixth does as well.
2.
 - a. Prove that if every point of the plane is colored using three colors, there must be two points of the same color which are exactly 1 inch apart.
 - b. If every point of the plane is colored using one of nine colors, must there be two points of the same color which are exactly 1 inch apart?
3. Consider a paper punch that can be centered at any point of the plane and that, when operated, removes from the plane precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?
4. Prove that no subset S of the real line can be partitioned into two disjoint subsets, each of which is congruent (by a translation) to the original set. [**Harder:** Does there exist a subset S of the Euclidean plane \mathbf{R}^2 which can be partitioned into two disjoint subsets, each of which is congruent (by translations and rotations) to the original set?]
5. Let P be an interior point of a regular n -gon with side length 1, and let x_1, \dots, x_n be the distances from P to the sides of the n -gon. Prove that

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} > 2\pi .$$

6. Given a finite collection of squares of total area $1/2$, show that they can be arranged (with no overlaps) to fit in a unit square.