Name: ____________________________

Math 6122 – Algebra II Spring 2006

Final Exam
Due Wednesday, May 3 at 2pm

There are 100 total points on this test. You may consult your class notes, the course textbook, class handouts, and your own homework solutions when working on this exam. Do not discuss the problems on this exam with anyone else, and do not consult any other materials (including other textbooks and web sites). Please type or write your answers neatly on separate sheets of paper, justify all answers, and show all of your work. You may quote results proved in the book or in assigned homework problems. Staple your answers together with this page as a cover sheet.

By signing your name below, you agree to the conditions of this exam.

Signature: ____

Final Exam Problems

In the problems on representation theory, G denotes a finite group and all representations of G are finite-dimensional complex representations.

1. (10 points) Let R be a commutative ring, let $M = R[x]$, and let $N = R[x^2]$. Prove that $M/N \cong M$ as R-modules.

2. (10 points) Let F be a field, and let n be a positive integer. Find the Jordan Canonical Form of the $n \times n$ matrix all of whose entries are 1. [**Hint:** There are two cases, depending on whether or not $\mathrm{char}(F) \mid n$.]

3. (10 points) Find the Galois group of the splitting field for $f(x) = x^3 - 7$ over $K = \mathbf{Q}(\sqrt{-3})$.

4. (15 points) A polynomial is called *reciprocal* if whenever α is a root, $1/\alpha$ is also a root.

   a. If $f(x) \in \mathbf{Q}[x]$ is irreducible of degree at least 2 and has a complex root lying on the unit circle, show that $f(x)$ is reciprocal and has even degree.

   b. If $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbf{Q}[x]$ is irreducible, monic, reciprocal, and has even degree, show that $a_0 = 1$ and $a_i = a_{n-i}$ for $i = 1, \ldots, n-1$.

5. (10 points) If ψ is a degree 1 character of G and χ is any irreducible character of G, show that $\psi\chi$ is also an irreducible character of G.

6. (10 points)

   a. Suppose $g, h \in G$. If $\chi(g) = \chi(h)$ for all irreducible characters χ of G, prove that g and h belong to the same conjugacy class.

   b. For any character χ of G, define $K_\chi = \{g \in G \;:\; \chi(g) = \chi(1)\}$. Prove that

   $$\bigcap_\chi K_\chi = \{1\}\ ,$$

   where the intersection is taken over all irreducible characters χ of G.

7. (15 points) Let ρ be a representation of G with character χ, and define $Z_\chi = \{g \in G \;:\; |\chi(g)| = \chi(1)\}$.

   a. Show that $g \in Z_\chi$ if and only if $\rho(g) = \lambda I$ for some $\lambda \in \mathbf{C}$. Deduce that if $g \in Z_\chi$, then $[g, h] \in K_\chi$ for all $h \in G$.

   b. If ρ is irreducible and $\rho(g)$ commutes with $\rho(h)$ for all $h \in G$, prove that $\rho(g) = \lambda I$ for some $\lambda \in \mathbf{C}$. [**Hint:** Use Schur's Lemma.]

   c. Prove that
   $$\bigcap_\chi Z_\chi = Z(G) \; ,$$
   where the intersection is taken over all irreducible characters χ of G.

8. (20 points) Let G be a finite group of order n.

   a. Prove that $\chi(g) \in \mathbf{Q}(\zeta_n)$ for all $g \in G$ and all characters χ of G.

   b. If $\sigma_a \in \mathrm{Gal}(\mathbf{Q}(\zeta_n)/\mathbf{Q})$ corresponds to $a \in (\mathbf{Z}/n\mathbf{Z})^*$ under the usual isomorphism, prove that $\sigma_a(\chi(g)) = \chi(g^a)$ for all $g \in G$ and all characters χ of G.

   c. Prove that the following are equivalent:

      (i) Every character of G takes values in $\mathbf{Z}$.

      (ii) Every irreducible character of G takes values in $\mathbf{Q}$.

      (iii) For all integers a with $(a, n) = 1$ and all $g \in G$, g and g^a are conjugate in G.

   d. (Bonus) Show that all characters of the symmetric group S_n are integer-valued.