Name: ____________________________

Math 6122 – Algebra II Spring 2006

Midterm exam # 1
Due Wednesday, Feb. 22 at 5pm

There are 100 total points on this test. You may consult your class notes, the course textbook, class handouts, and your own homework solutions when working on this exam. Do not discuss the problems on this exam with anyone else, and do not consult any other materials (including other textbooks and web sites). Please type or write your answers neatly on separate sheets of paper, justify all answers, and show all of your work. You may quote results proved in the book or in assigned homework problems. Staple your answers together with this page as a cover sheet.

By signing your name below, you agree to the conditions of this exam.

Signature: ____

Midterm #1 Problems

1. (15 points) Let α be a complex number.
   a. Show that α is an algebraic integer iff α is an eigenvalue of a matrix $A \in M_{n\times n}(\mathbf{Z})$ for some $n \geq 1$.
   b. If all entries of $A \in M_{n\times n}(\mathbf{C})$ are algebraic integers, prove that every eigenvalue of A is an algebraic integer.

2. (20 points) Let $q = p^k$ be a prime power. Use the rational canonical form to determine the number of conjugacy classes in the groups $\mathrm{GL}_2(\mathbf{F}_q)$ and $\mathrm{GL}_3(\mathbf{F}_q)$.

3. (20 points) Let $g(x) = x^p - x + a \in \mathbf{F}_p[x]$, where p is a prime and $a \in \mathbf{F}_p^*$. Let E be a splitting field over $\mathbf{F}_p$ for $g(x)$.
   a. Let $\alpha \in E$ be a root of $g(x)$. Show that there is an automorphism of E taking α to $\alpha + 1$.
   b. Prove that $g(x)$ is irreducible over $\mathbf{F}_p$.
   c. Prove that $[E : \mathbf{F}_p] = p$ and that $\mathrm{Aut}(E/\mathbf{F}_p)$ is cyclic of order p.

4. (15 points) If $f(x) \in \mathbf{Q}[x]$ is an irreducible polynomial with roots $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$ and $i \neq j$, prove that $\alpha_i - \alpha_j \notin \mathbf{Q}$.

5. (15 points) For which pairs $(p(x), m(x))$ of monic polynomials over $\mathbf{C}$ does there exist $A \in M_{n\times n}(\mathbf{C})$ whose characteristic polynomial is $p(x)$ and whose minimal polynomial is $m(x)$?

6. (15 points) Let R be a Noetherian commutative ring with 1, and let $I \subset R$ be a proper ideal with the property that $1 + a$ is not a zero-divisor in R for any $a \in I$. Prove that if $a \in I^n$ for all $n \geq 1$ then $a = 0$. [**Hint:** You may use that fact from Commutative Algebra that if M is the R-module $M = \cap_{n\geq 1} I^n$, then $IM = M$.]