Name: _____

Math 6122 – Algebra II Spring 2006

Midterm exam # 2
Due Wednesday, April 5 at 5pm

There are 100 total points on this test. You may consult your class notes, the course textbook, class handouts, and your own homework solutions when working on this exam. Do not discuss the problems on this exam with anyone else, and do not consult any other materials (including other textbooks and web sites). Please type or write your answers neatly on separate sheets of paper, justify all answers, and show all of your work. You may quote results proved in the book or in assigned homework problems. Staple your answers together with this page as a cover sheet.

By signing your name below, you agree to the conditions of this exam.

Signature: _____

Midterm #2 Problems

1. (20 points) In the following examples, find the Galois group of the splitting field for $f(x)$ over K.
   a. $K = \mathbf{F}_7$, $f(x) = (x^2-1)(x^2-2)(x^2-3)(x^2-4)(x^2-5)(x^2-6)$.
   b. $K = \mathbf{Q}(\sqrt{2})$, $f(x) = x^5 - 10x + 5$.

2. (20 points) Let $K = \mathbf{Q}(\sqrt[4]{7}, \sqrt{-1})$.
   a. Show that K is Galois over $\mathbf{Q}$, and describe the Galois group explicitly.
   b. Give explicit generators for all proper subfields of K.

3. (20 points) Let ζ be a primitive 37th root of unity, and let $\eta = \zeta + \zeta^{10} + \zeta^{26}$. Determine the Galois group of $\mathbf{Q}(\eta)$ over $\mathbf{Q}$.

4. (20 points) Let F be a finite field, and let $a, b \in F$. Prove that at least one of the polynomials $x^3 + ax + b$ or $x^2 + 4a^3 + 27b^2$ has a root in F.

5. (20 points) Let n be a positive integer, and let a be a positive rational number for which $x^n - a$ is irreducible in $\mathbf{Q}[x]$. Let $\sqrt[n]{a}$ denote a real nth root of a.
   a. For each $d \mid n$, prove that $E = \mathbf{Q}(\sqrt[d]{a})$ is the unique subfield of $K = \mathbf{Q}(\sqrt[n]{a})$ of degree d over $\mathbf{Q}$. (**Hint:** Consider $N_{K/E}(\sqrt[n]{a})$.)
   b. Let K be as in part (a). If n is odd, prove that K has no nontrivial subfield which is Galois over $\mathbf{Q}$, and if n is even, prove that there is a unique nontrivial subfield of K which is Galois over $\mathbf{Q}$.