

Name: _____

Math 6121 – Algebra I

Fall 2005

FINAL EXAM
DUE WEDNESDAY, DEC. 14 AT 5PM

There are 120 total points on this test. You may consult your class notes, the course textbook, class handouts, and your own homework solutions when working on this exam. Do not discuss the problems on this exam with anyone else, and do not consult any other materials (including other textbooks and web sites). Please type or write your answers neatly on separate sheets of paper, justify all answers, and show all of your work. You may quote results proved in the book or in assigned homework problems. Staple your answers together with this page as a cover sheet.

By signing your name below, you agree to the conditions of this exam.

Signature: _____

PROBLEMS

Do *six* of the following eight problems (20 points each).

If you turn in solutions to more than 6 problems, please indicate which ones you want graded.

1. Let p be a prime number, and let $q = p^n$ with $n \geq 1$ a positive integer. Prove that every p -Sylow subgroup of $\mathbf{GL}_2(\mathbf{F}_q)$ is isomorphic to $(\mathbf{Z}/p\mathbf{Z})^n$.
2. Let n be a positive integer. If $\gcd(n, \phi(n)) > 1$, prove that there exists a non-cyclic group of order n . (Here $\phi(n)$ denotes Euler's ϕ -function.)
3. Prove that if G is a group having no subgroup of index 2, then any subgroup of index 3 is normal in G .
4. Let G be a group, and let H be a normal subgroup of G .
 - a. If $G = H \rtimes_{\varphi} K$ for some subgroup K of G and some homomorphism $\varphi : K \rightarrow \text{Aut}(H)$, show that $K \cong G/H$, and that under this isomorphism, the map $H \rtimes_{\varphi} K \rightarrow K$ sending (h, k) to k can be identified with the natural projection π of G onto G/H .
 - b. Show that $G \cong H \rtimes_{\varphi} K$ for some subgroup K of G and some homomorphism $\varphi : K \rightarrow \text{Aut}(H)$ if and only if there is a homomorphism $\sigma : G/H \rightarrow G$ such that $\pi \circ \sigma$ is the identity map on G/H .
5. Let I be the ideal $(n, x^3 + 2x + 2)$ in $\mathbf{Z}[x]$. For which n with $1 \leq n \leq 7$ is I a maximal ideal?
6. Suppose L/K is an algebraic field extension, and that R is a subring of L containing K . Prove that R is a field.
7. Find a splitting field for $x^8 + x^6 + x^4 + x^2 + 1$ over \mathbf{Q} , and determine its degree.
8. Let ℓ be a prime number, and let $\Phi_{\ell}(x) = x^{\ell-1} + x^{\ell-2} \cdots + x + 1 \in \mathbf{Z}[x]$. Let p be a prime number different from ℓ , and let f be the order of p in the group $(\mathbf{Z}/\ell\mathbf{Z})^*$. Show that f is the degree of the smallest extension of \mathbf{F}_p in which $\Phi_{\ell}(x)$ has a root.