Appendix D

Solutions

Solution of Exercise. (Exercise 3.1) Find the Euclidean length length$[\alpha]$ and Riemannian length length$_{\mathcal{B}}[\alpha]$ of the path $\alpha : [0, a] \to B_1(\mathbf{0})$ by $\alpha(t) = t(\cos\theta, \sin\theta)$ where $\theta \in \mathbb{R}$ and $a > 0$.

In this case, $|\alpha| = t$ and $|\alpha'| = 1$.

$$\text{length}[\alpha] = \int_0^a 1\, dt = a,$$

and

$$\begin{aligned}
\text{length}_{\mathcal{B}}[\alpha] &= \int_0^a \frac{4}{4+t^2}\, dt \\
&= \int_0^a \frac{1}{1+(t/2)^2}\, dt \\
&= 2\int_0^{a/2} \frac{1}{1+u^2}\, du \\
&= 2\tan^{-1}\left(\frac{a}{2}\right).
\end{aligned}$$

The interesting observation here is that this manifold $\mathcal{B}$ is apparently contained in a larger manifold obtained by extending the matrix assignment

$$(g_{ij}) = \frac{4}{(4+|\mathbf{x}|^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

to the entire plane $\mathbb{R}^2$. The notion of length$_{\mathcal{B}}$ for these paths then extends to entire rays defined for $a > 0$ by the same formula, and these rays have lengths bounded by π.

Since $\text{length}[\alpha] = a$, one can consider $\text{length}_{\mathcal{B}}[\alpha]$ as a function of the Euclidean length $\text{length}[\alpha]$. Notice

$$\frac{d}{da}\text{length}_{\mathcal{B}}[\alpha] = \frac{1}{1+(a/2)^2} < 1 \qquad \text{and} \qquad \frac{d}{da}\text{length}_{\mathcal{B}}[\alpha]\Big|_{a=0} = 1.$$

Thus, these paths start at $\mathbf{0}$ with Riemannian length and Euclidean length essentially equal, but as the ray extends, the Riemannian length becomes shorter and shorter relative to the Euclidean length, so much so that the total Riemannian length is always less then π as indicated on the right in Figure D.1. For radial segments contained in $\mathcal{B}$ the Riemannian lengths satisfy

$$2\tan^{-1}\left(\frac{1}{2}\right)\ \text{length}[\alpha] < \text{length}_{\mathcal{B}}[\alpha] < \text{length}[\alpha],$$

and

$$b = 2\tan^{-1}\left(\frac{1}{2}\right) \doteq 0.927295.$$

See Figure D.1 (left).

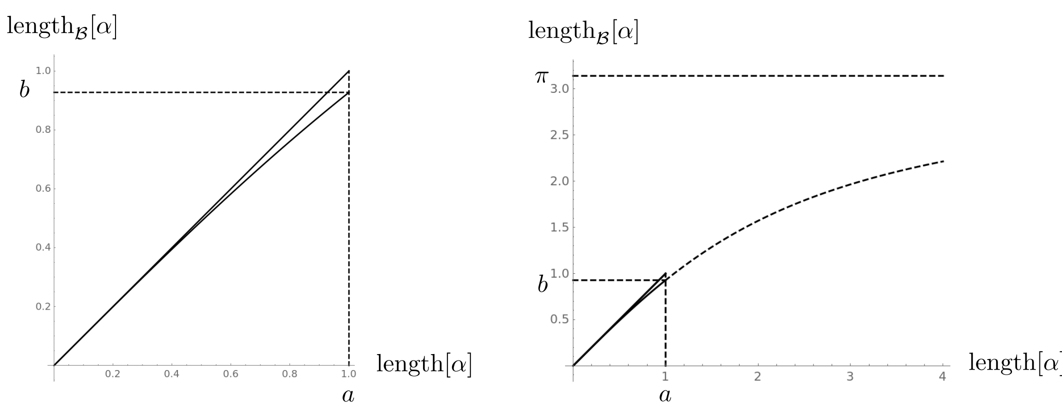


Figure D.1: Comparison of Riemannian length and Euclidean length for rays in $\mathcal{B}$. In this illustration $a = 1$ is the Euclidean radius of $B_1(\mathbf{0})$ and $b = 2\tan^{-1}(1/2)$ is the Riemannian radius of $\mathcal{B}$.

Solution of Exercise. (Exercise 7.1) Show the following: If X is a Hausdorff topological space and

1. $P \in X$,
2. V is an open set in X with $P \in V$, and
3. for some natural number $n \in \mathbb{N}$, there is a homeomorphism $\mathbf{p} : \mathbb{R}^n \to V$,

then X is locally Euclidean at P.

Let $\xi = \mathbf{p}^{-1} : V \to \mathbb{R}^n$ denote the coordinate function determined by $\mathbf{p}$, and let $\mathbf{x}_0 = \xi(P) \in \mathbb{R}^n$. First observe that there exists a homeomorphism

$$\psi_0 : B_2(\mathbf{0}) \to \mathbb{R}^n \qquad \text{with} \qquad \psi_0(\mathbf{0}) = \mathbf{x}_0.$$

In fact, $\psi_1 : B_1(\mathbf{0}) \to \mathbb{R}^n$ by

$$\psi_1(\mathbf{x}) = \begin{cases} \tanh^{-1}(|\mathbf{x}|)\, \dfrac{\mathbf{x}}{|\mathbf{x}|}, & \mathbf{x} \neq \mathbf{0} \\ \mathbf{0}, & \mathbf{x} = \mathbf{0} \end{cases}$$

is a homeomorphism, and in the solution of Exercise 7.3 below a homeomorphism $\phi_1 : B_1(\mathbf{0}) \to B_1(\mathbf{0})$ with $\phi_1(\mathbf{0})$ any prescribed point in $B_1(\mathbf{0})$ is discussed. In particular, taking the point $\psi_1^{-1}(\mathbf{x}_0)$, we can take $\psi_0 : B_2(\mathbf{0}) \to \mathbb{R}^n$ by

$$\psi_0(\mathbf{x}) = \psi_1 \circ \phi_1 \left(\frac{\mathbf{x}}{2} \right)$$

where $\phi_1(\mathbf{0}) = \psi_1^{-1}(\mathbf{x}_0)$.

Consider then $\mathbf{q} = \mathbf{p} \circ \psi_0 : B_2(\mathbf{0}) \to V$. We have then $\mathbf{q} : B_2(\mathbf{0}) \to V$ is a homeomorphism with $\mathbf{q}(\mathbf{0}) = P$. We can also consider the associated coordinate function $\eta = \mathbf{q}^{-1} = \psi_0^{-1} \circ \xi : V \to B_2(\mathbf{0})$.

Let $U = \mathbf{q}(B_1(\mathbf{0}))$. Then it is easy to see (or almost immediate) that

1. U is open in X ($U = \eta^{-1}(B_1(\mathbf{0}))$),
2. $P \in U$ ($P = \mathbf{q}(\mathbf{0})$), and
3. $$\mathbf{q}\big|_{B_1(\mathbf{0})} : B_1(\mathbf{0}) \to U$$
   is a homeomorphism.

The last assertion follows simply because the restriction of a continuous function is always continuous, and

$$\left(\mathbf{q}\big|_{B_1(\mathbf{0})}\right)^{-1} = (\mathbf{q}^{-1})\big|_U : U \to B_1(\mathbf{0}).$$

In order to show X is locally Euclidean at P it remains to show

$$\overline{U} = \mathbf{q}(\overline{B_1(\mathbf{0})}) \subset V \tag{D.1}$$

and

$$\mathbf{q}\big|_{\overline{B_1(\mathbf{0})}} : \overline{B_1(\mathbf{0})} \to \overline{U} \tag{D.2}$$

is a homeomorphism.[1]

A key point is that the continuous image of a compact set is always compact. It follows from this that

$$\mathbf{q}(\overline{B_1(\mathbf{0})}) \quad \text{is compact in } X.$$

Also a compact subset of a Hausdorff space is closed. Therefore,

$$\mathbf{q}(\overline{B_1(\mathbf{0})}) \quad \text{is also closed in } X.$$

By definition

$$U = \mathbf{q}(B_1(\mathbf{0})) \subset \mathbf{q}(\overline{B_1(\mathbf{0})}).$$

Therefore,

$$\overline{U} \subset \mathbf{q}(\overline{B_1(\mathbf{0})}) \subset V. \tag{D.3}$$

On the other hand, if $Q \in V \backslash \overline{U} = V \backslash \overline{\mathbf{q}(B_1(\mathbf{0}))}$, then there is some open set W in V (and in X) for which $Q \in W \subset V \backslash \overline{U}$.

Note that $\mathbf{q}^{-1}(W)$ is an open set in $B_1(\mathbf{0})$. Furthermore, we claim

$$\mathbf{q}^{-1}(W) \cap \overline{B_1(\mathbf{0})} = \phi. \tag{D.4}$$

To see this, assume to the contrary that $\mathbf{q}^{-1}(W) \cap \overline{B_1(\mathbf{0})} \neq \phi$. Then $\mathbf{q}^{-1}(W) \cap B_1(\mathbf{0}) \neq \phi$. But if $\mathbf{x} \in \mathbf{q}^{-1}(W) \cap B_1(\mathbf{0})$, then $\mathbf{q}(\mathbf{x}) \in W \cap U$, and this is a contradiction because $W \subset V \backslash \overline{U}$.

[1] Once (D.1) is established, the assertion concerning (D.2) can be obtained from the nice topological fact that a continuous bijection from a compact space into a Hausdorff space is a homeomorphism. It does not seem, however, that the equality in (D.1) is immediate or follows immediately from this topological fact.

We know from (D.4) that $Q \notin \mathbf{q}(\overline{B_1(\mathbf{0})})$. We have therefore shown

$$V \backslash \overline{U} \subset V \backslash \mathbf{q}(\overline{B_1(\mathbf{0})}),$$

and this implies $\mathbf{q}(\overline{B_1(\mathbf{0})}) \subset \overline{U}$. In view of (D.3) we have established (D.1). The last assertion concerning (D.2) now follows easily as before from the fact that restrictions of continuous functions are continuous. Specifically,

$$\mathbf{q}\big|_{\overline{B_1(\mathbf{0})}} : \overline{B_1(\mathbf{0})} \to \overline{U} \qquad \text{and} \qquad \left(\mathbf{q}\big|_{\overline{B_1(\mathbf{0})}}\right)^{-1} = \eta\big|_{\overline{U}} : \overline{U} \to \overline{B_1(\mathbf{0})}$$

are both continuous bijections. $\square$

Solution of Exercise. (Exercise 7.2) Under the assumptions of Exercise 7.1 show there exists a homeomorphism $\mathbf{q} : B_1(\mathbf{0}) \to V$ where $B_1(\mathbf{0}) \subset \mathbb{R}^n$.

As suggested in the previous solution we can take $\mathbf{q} = \mathbf{p} \circ \psi$ where $\psi : B_1(\mathbf{0}) \to \mathbb{R}^n$ by

$$\psi(\mathbf{x}) = \begin{cases} \tanh^{-1}(|\mathbf{x}|)\, \dfrac{\mathbf{x}}{|\mathbf{x}|}, & \mathbf{x} \neq \mathbf{0} \\ \mathbf{0}, & \mathbf{x} = \mathbf{0}, \end{cases}$$

as long as ψ is a homeomorphism. Continuity of ψ follows because

$$\lim_{\mathbf{x} \to \mathbf{0}} \left| \tanh^{-1}(|\mathbf{x}|)\, \frac{\mathbf{x}}{|\mathbf{x}|} \right| = \lim_{\mathbf{x} \to \mathbf{0}} \tanh^{-1}(|\mathbf{x}|) = 0.$$

Furthermore, the function $\phi : \mathbb{R}^n \to B_1(\mathbf{0})$ by

$$\phi(\mathbf{x}) = \begin{cases} \tanh(|\mathbf{x}|)\, \dfrac{\mathbf{x}}{|\mathbf{x}|}, & \mathbf{x} \neq \mathbf{0} \\ \mathbf{0}, & \mathbf{x} = \mathbf{0}, \end{cases}$$

is also similarly continuous because

$$\lim_{\mathbf{x} \to \mathbf{0}} \left| \tanh(|\mathbf{x}|)\, \frac{\mathbf{x}}{|\mathbf{x}|} \right| = \lim_{\mathbf{x} \to \mathbf{0}} \tanh(|\mathbf{x}|) = 0.$$

Finally, it can be checked directly that $\psi \circ \phi(\mathbf{x}) = \mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$ and $\phi \circ \psi(\mathbf{x}) = \mathbf{x}$ for $\mathbf{x} \in B_1(\mathbf{0})$. $\square$

Exercise D.1. What happens if you take $\psi : B_1(\mathbf{0}) \to \mathbb{R}^n$ by $\psi(\mathbf{x}) = \tanh^{-1}(|\mathbf{x}|)\,\mathbf{x}$? (Is this a homeomorphism?)

Exercise D.2. What happens if you take $\phi : \mathbb{R}^n \to B_1(\mathbf{0})$ by $\phi(\mathbf{x}) = \tanh(|\mathbf{x}|)\,\mathbf{x}$? (Is this a homeomorphism?)

Solution of Exercise. (Exercise 7.3) Given $\mathbf{p} \in B_1(\mathbf{0}) \subset \mathbb{R}^n$, show there exists a homeomorphism $\psi : B_1(\mathbf{0}) \to B_1(\mathbf{0})$ with $\psi(\mathbf{p}) = \mathbf{0}$. In fact, show $\phi : B_1(\mathbf{0}) \to B_1(\mathbf{0})$ by

$$\phi(\mathbf{x}) = \begin{cases} \mathbf{p} + \left(-\mathbf{p}\cdot\mathbf{x} + \sqrt{(\mathbf{p}\cdot\mathbf{x})^2 + |\mathbf{x}|^2(1-|\mathbf{p}|^2)}\right)\dfrac{\mathbf{x}}{|\mathbf{x}|}, & \mathbf{x} \neq \mathbf{0} \\ \mathbf{p}, & \mathbf{x} = \mathbf{0} \end{cases}$$

is the continuous inverse of such a function.

These homeomorphisms are based on the observation that the Euclidean geodesic rays

$$\Gamma_{\mathbf{v}} = \{\mathbf{p} + t\mathbf{v} : t > 0\}$$

from a point $\mathbf{p} \in \mathbb{R}^n$ where $\mathbf{v} \in \mathbb{S}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}$ constitute a **foliation** of $\mathbb{R}^n \backslash \{\mathbf{p}\}$. That is,

1. Each $\Gamma_{\mathbf{v}}$ is a one-dimensional submanifold of $\mathbb{R}^n$,
2. If $\mathbf{v}, \mathbf{w} \in \mathbb{S}^{n-1}$ with $\mathbf{v} \neq \mathbf{w}$, then $\Gamma_{\mathbf{v}} \cap \Gamma_{\mathbf{w}} = \phi$, and
3. Each point $\mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{p}\}$ lies in one of the rays $\Gamma_{\mathbf{v}}$.

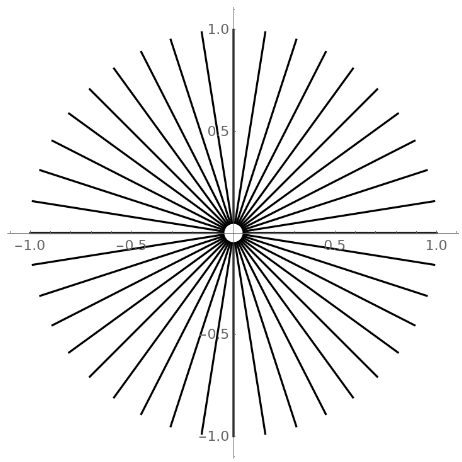
Specifically, $\mathbf{x} \in \Gamma_{\mathbf{v}}$ where

$$\mathbf{v} = \frac{\mathbf{x} - \mathbf{p}}{|\mathbf{x} - \mathbf{p}|}.$$

Furthermore, each $\mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{p}\}$ determines not only a unique $\mathbf{v} \in \mathbb{S}^{n-1}$ but also a unique $t \in [0, \infty)$ for which

$$\mathbf{x} = \mathbf{p} + t\mathbf{v}.$$

Taking the special case, $\mathbf{p} = 0$, the foliation restricts to a foliation of $B_1(\mathbf{0}) \backslash \{\mathbf{0}\}$ as indicated on the left in Figure D.2 for the case $n = 2$. More generally, for any point $\mathbf{p} \in B_1(\mathbf{0})$ one obtains a foliation of $B_{\mathbf{0}}(\mathbf{0}) \backslash \{\mathbf{p}\}$ as indicated on the right in Figure D.2. The assignment of points on parallel segments via convex combination is continuous. Specifically, given a point



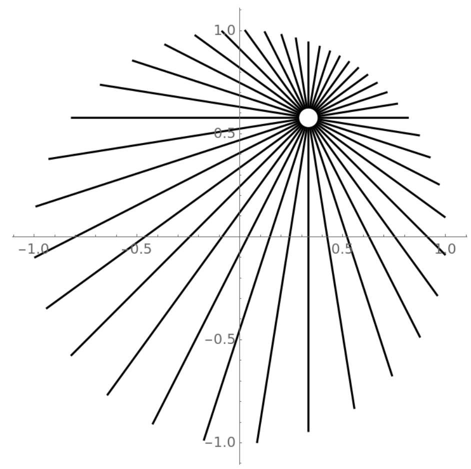


Figure D.2: Foliations of punctured unit disks. Centered at $\mathbf{0} \in B_1(\mathbf{0})$ (left) and centered at a point $\mathbf{p} \in B_1(\mathbf{0})$ (right). This kind of foliation by one dimensional affine submanifolds may be used in any dimension.

$\mathbf{x} \in B_1(\mathbf{0})\backslash\{\mathbf{0}\}$ we first find a point $\mathbf{b} = \mathbf{p} + t\mathbf{x}/|\mathbf{x}|$ with $|\mathbf{b}| = 1$. Thus, we find that t must satisfy $t > 0$ and

$$t^2 + 2\frac{\mathbf{p} \cdot \mathbf{x}}{|\mathbf{x}|} - (1 - |\mathbf{p}|^2) = 0.$$

Note that if $\mathbf{p} = 0$, we can take $t = 1$, and set

$$\phi(\mathbf{x}) = \tau \, \frac{\mathbf{x}}{|\mathbf{x}|} = \mathbf{x}$$

where

$$\frac{\tau}{|\mathbf{b} - \mathbf{p}|} = \tau = \frac{|\mathbf{x}|}{1}.$$

More generally,

$$t = |\mathbf{b} - \mathbf{p}| = \frac{-\mathbf{p} \cdot \mathbf{x} + \sqrt{(\mathbf{p} \cdot \mathbf{x})^2 + (1 - |\mathbf{p}|^2)|\mathbf{x}|^2}}{|\mathbf{x}|}$$

and we set

$$\phi(\mathbf{x}) = \mathbf{p} + \tau \, \frac{\mathbf{x}}{|\mathbf{x}|}$$

where
$$\frac{\tau}{|\mathbf{b}-\mathbf{p}|} = |\mathbf{x}|$$
or
$$\phi(\mathbf{x}) = \mathbf{p} + \left(-\mathbf{p}\cdot\mathbf{x} + \sqrt{(\mathbf{p}\cdot\mathbf{x})^2 + (1-|\mathbf{p}|^2)|\mathbf{x}|^2}\right)\,\frac{\mathbf{x}}{|\mathbf{x}|}.$$
Notice that
$$\lim_{\mathbf{x}\to\mathbf{0}} |\phi(\mathbf{x}) - \mathbf{p}| = 0,$$
so $\phi \in C^0(B_1(\mathbf{0}) \to B_1(\mathbf{0})$ and
$$\phi\Big|_{B_1(\mathbf{0})\backslash\{\mathbf{0}\}} \in C^\infty(B_1(\mathbf{0})\backslash\{\mathbf{0}\} \to B_1(\mathbf{0})\backslash\{\mathbf{p}\})$$
where ϕ is given in the statement of the exercise.

Similarly, we can take $\mathbf{x} \in B_1(\mathbf{0})\backslash\{\mathbf{p}\}$ and find a point $\mathbf{b} = \mathbf{p} + t\mathbf{v}$ for a unique $t > 0$ satisfying $|\mathbf{b}| = 1$ where
$$\mathbf{v} = \frac{\mathbf{x}-\mathbf{p}}{|\mathbf{x}-\mathbf{p}|}.$$
Here we are finding the function ψ which takes the foliating segments on the right in Figure D.2 to the foliating segments illustrated on the left. In this case, we find
$$t^2 + 2\mathbf{p}\cdot\frac{\mathbf{x}-\mathbf{p}}{|\mathbf{x}-\mathbf{p}|}\,t - (1-|\mathbf{p}|^2) = 0$$
or
$$t = |\mathbf{b}-\mathbf{p}| = \frac{-\mathbf{p}\cdot(\mathbf{x}-\mathbf{p}) + \sqrt{(\mathbf{p}\cdot(\mathbf{x}-\mathbf{p}))^2 + (1-|\mathbf{p}|^2)|\mathbf{x}-\mathbf{p}|^2}}{|\mathbf{x}-\mathbf{p}|}.$$
We then set
$$\psi(\mathbf{x}) = \tau\,\frac{\mathbf{x}-\mathbf{p}}{|\mathbf{x}-\mathbf{p}|}$$
where
$$\frac{|\psi(\mathbf{x})|}{1} = \tau = \frac{|\mathbf{x}-\mathbf{p}|}{|\mathbf{b}-\mathbf{p}|}.$$
That is,
$$\tau = \frac{|\mathbf{x}-\mathbf{p}|^2}{-\mathbf{p}\cdot(\mathbf{x}-\mathbf{p}) + \sqrt{(\mathbf{p}\cdot(\mathbf{x}-\mathbf{p}))^2 + (1-|\mathbf{p}|^2)|\mathbf{x}-\mathbf{p}|^2}}$$

and

$$\psi(\mathbf{x}) = \frac{|\mathbf{x}-\mathbf{p}|(\mathbf{x}-\mathbf{p})}{-\mathbf{p}\cdot(\mathbf{x}-\mathbf{p}) + \sqrt{(\mathbf{p}\cdot(\mathbf{x}-\mathbf{p}))^2 + (1-|\mathbf{p}|^2)|\mathbf{x}-\mathbf{p}|^2}}.$$

Again, it is easily checked that

$$\lim_{\mathbf{x}\to\mathbf{p}} |\psi(\mathbf{x})| = 0.$$

Thus, we obtain $\psi \in C^0(B_1(\mathbf{0}) \to B_1(\mathbf{0}))$ by

$$\psi(\mathbf{x}) = \begin{cases} \dfrac{|\mathbf{x}-\mathbf{p}|(\mathbf{x}-\mathbf{p})}{-\mathbf{p}\cdot(\mathbf{x}-\mathbf{p}) + \sqrt{(\mathbf{p}\cdot(\mathbf{x}-\mathbf{p}))^2 + (1-|\mathbf{p}|^2)|\mathbf{x}-\mathbf{p}|^2}}, & \mathbf{x} \neq \mathbf{p} \\ \mathbf{0}, & \mathbf{x} = \mathbf{p}. \end{cases}$$

It is also clear that

$$\psi\big|_{B_1(\mathbf{0})\backslash\{\mathbf{p}\}} \in C^\infty(B_1(\mathbf{0})\backslash\{\mathbf{p}\} \to B_1(\mathbf{0})\backslash\{\mathbf{0}\}).$$

Direct substition gives $\psi \circ \phi(\mathbf{x}) = \mathbf{x}$ and $\phi \circ \psi(\mathbf{x}) = \mathbf{x}$ for $\mathbf{x} \in B_1(\mathbf{0})$, so ϕ and ψ are homeomorphisms.

In order to investigate the potential higher regularity of ϕ and ψ we consider directional derivatives of the form

$$D_{\mathbf{v}}\psi^j(\mathbf{p}) = \lim_{t\searrow 0} \frac{1}{t}\psi^j(\mathbf{p}+t\mathbf{v}) \qquad \text{and} \qquad D_{\mathbf{v}}\phi^j(\mathbf{0}) = \lim_{t\searrow 0} \frac{1}{t}[\phi^j(t\mathbf{v}) - \mathbf{p}]$$

for $j = 1, 2, \dots, n$ and $\mathbf{v} \in \mathbb{S}^{n-1}$ where $\psi = (\psi^1, \psi^2, \dots, \psi^n)$ and $\phi = (\phi^1, \phi^2, \dots, \phi^n)$.

Observe first that

$$\frac{1}{t}\psi^j(\mathbf{p}+t\mathbf{v}) = \frac{v_j}{-\mathbf{v}\cdot\mathbf{p} + \sqrt{(\mathbf{v}\cdot\mathbf{p})^2 + 1 - |\mathbf{p}|^2}} \tag{D.5}$$

is independent of t but has interesting dependence on the direction $\mathbf{v}$. In particular, if we take a special case $\mathbf{p} = a\mathbf{e}_1$ for some $a > 0$ and $\mathbf{v} = \pm\mathbf{e}_i$ with $i \neq j$, then

$$D_{\pm\mathbf{e}_i}\psi^j(a\mathbf{e}_1) = 0.$$

This might appear promising if we wish to imagine ψ is differentiable (or at least partiall differentiable) at $\mathbf{p} = a\mathbf{e}_1$. However, taking $\mathbf{v} = \mathbf{e}_j$ gives

$$D_{\mathbf{e}_j}\psi^j(a\mathbf{e}_1) = \frac{1}{a^2(\delta_{1j} - 1) - a\delta_{1j} + 1}$$

while taking $\mathbf{v} = -\mathbf{e}_j$ we obtain

$$D_{-\mathbf{e}_j}\psi^j(a\mathbf{e}_1) = \frac{-1}{a^2(\delta_{1j}-1)+a+1}.$$

Since $0 < a < 1$, it can never be the case that

$$D_{-\mathbf{e}_j}\psi^j(a\mathbf{e}_1) = -D_{\mathbf{e}_j}\psi^j(a\mathbf{e}_1), \tag{D.6}$$

that is to say

$$\frac{\partial\psi^j}{\partial x_j}(a\mathbf{e}_1) = \lim_{v\to 0}\frac{\psi^j(a\mathbf{e}_1+v\mathbf{e}_j)-\psi^j(a\mathbf{e}_1)}{v}$$

can never exist. (Should this partial derivative exist, it would have to take the common value (D.6).)

In order to understand the situation somewhat better, let us generalize the calculation of the directional derivative and observe that for $s > 0$ and $\mathbf{p} + s\mathbf{v} \in B_1(\mathbf{0})$

$$\begin{aligned}
D_{\mathbf{v}}\psi^j(\mathbf{p}+s\mathbf{v}) &= \lim_{t\searrow 0}\frac{1}{t}[\psi^j(\mathbf{p}+(s+t)\mathbf{v}) - \psi^j(\mathbf{p}+s\mathbf{v})\\
&= \lim_{t\searrow 0}\frac{1}{t}\left(\frac{(s+t)(s+t)v_j}{-(s+t)\mathbf{p}\cdot\mathbf{v}+\sqrt{(s+t)^2(\mathbf{p}\cdot\mathbf{v})^2+(1-|\mathbf{p}|^2)(s+t)^2}}\right.\\
&\qquad\left.-\frac{s^2v_j}{-s\mathbf{p}\cdot\mathbf{v}+\sqrt{s^2(\mathbf{p}\cdot\mathbf{v})^2+(1-|\mathbf{p}|^2)s^2}}\right)\\
&= \lim_{t\searrow 0}\frac{1}{t}\left(\frac{(s+t)v_j - sv_j}{-\mathbf{p}\cdot\mathbf{v}+\sqrt{(\mathbf{p}\cdot\mathbf{v})^2+(1-|\mathbf{p}|^2)}}\right)\\
&= \frac{v_j}{-\mathbf{v}\cdot\mathbf{p}+\sqrt{(\mathbf{v}\cdot\mathbf{p})^2+1-|\mathbf{p}|^2}}
\end{aligned}$$

which is the same constant value we found in (D.5). It follows that the rate of change of ψ^j in the direction $\mathbf{v}$ is constant along every foliating segment $\Gamma_{\mathbf{v}} = \{\mathbf{p}+s\mathbf{v} : s > 0,\ |\mathbf{p}+s\mathbf{v}| < 1\}$. The graph of ψ^j

$$\{(\mathbf{x}, \psi^j(\mathbf{x})) \in \mathbb{R}^{n+1} : \mathbf{x} \in B_1(\mathbf{0})\}$$

punctured at $(\mathbf{p}, 0)$ is thus seen to be foliated by straight line segments emanating from the puncture and forming a **conical hypersurface** over a smooth $n-2$ dimensional submanifold in $\mathbb{R}^{n+1}$. In the case $n = 2$, the graphs of ψ^1 and ψ^2 are illustrated in Figure D.3

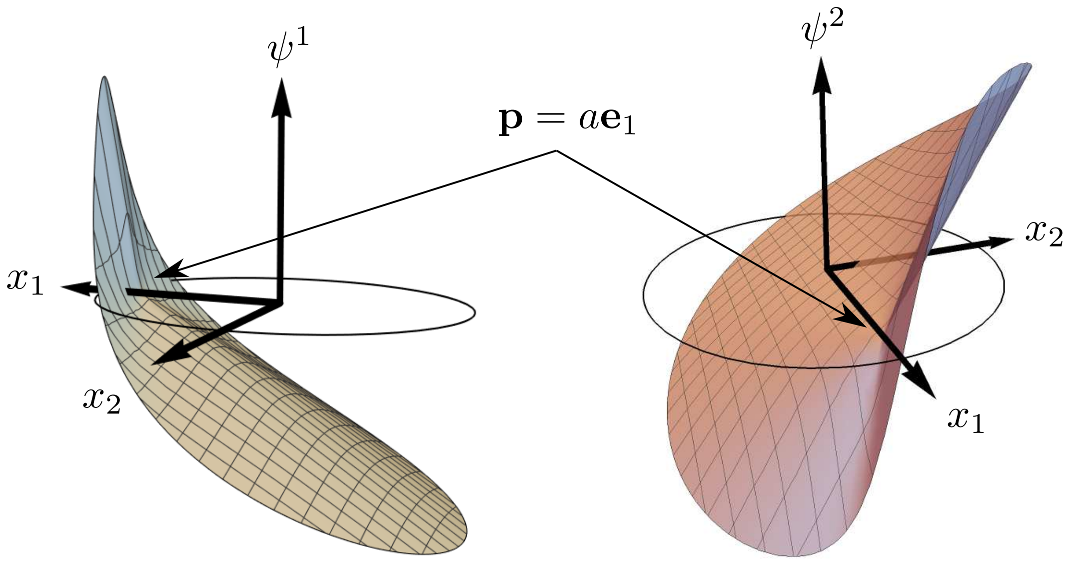


Figure D.3: Conical graphs of coordinate functions $\psi = (\psi^1, \psi^2)$ of a homeomorphism $\psi : B_1(\mathbf{0}) \to B_1(\mathbf{0})$ with $\psi(\mathbf{p}) = \mathbf{0}$ having singularities at $\mathbf{p} = a\mathbf{e}_1 \in \mathbb{R}^2$. Each graph is a **cone** centered at $\mathbf{p}$ over a smooth curve in the cylinder $\{\mathbf{x} = (x_1, x_2, x_3) : x_1^2 + x_2^2 = 1\}$. The general situation for $\mathbf{p} \in B_1(\mathbf{0})\backslash\{\mathbf{0}\}$ with $|\mathbf{p}| = a$ may be understood by rotating the conical graph(s) in this special case.

Exercise D.3. What happens to the conical graph(s) and the $n-2$ dimensional submanifolds generating them as $\mathbf{p} \to \mathbf{0}$?

The singular point of $\phi = (\phi^1, \phi^2)$ is at $\mathbf{x} = \mathbf{0}$, and one should now expect similar regularity holds. In particular, the coordinate functions ψ^j and ϕ^j satisfy

$$\psi^j, \phi^j \in \text{Lip}(B_1(\mathbf{0})) \qquad \text{for } j = 1, 2, \ldots, n.$$

Exercise D.4. Find the Lipschitz constants of the coordinate functions of ψ and ϕ.

Proceeding to a consideration of ϕ with the formula given in the statement of the exercise,

$$
\begin{aligned}
D_{\mathbf{v}}\phi^j(s\mathbf{v}) &= \lim_{t\searrow 0}\frac{1}{t}[\phi^j((s+t)\mathbf{v}) - \phi^j(s\mathbf{v})] \\
&= \lim_{t\searrow 0}\frac{1}{t}\left(\frac{\left(-(s+t)\mathbf{p}\cdot\mathbf{v} + \sqrt{(s+t)^2(\mathbf{p}\cdot\mathbf{v})^2 + (s+t)^2(1-|\mathbf{p}|^2)}\right)(s+t)v_j}{(s+t)}\right. \\
&\qquad\left. \frac{\left(-s\mathbf{p}\cdot\mathbf{v} + \sqrt{s^2(\mathbf{p}\cdot\mathbf{v})^2 + s^2(1-|\mathbf{p}|^2)}\right)sv_j}{s}\right) \\
&= v_j\left(-\mathbf{p}\cdot\mathbf{v} + \sqrt{(\mathbf{p}\cdot\mathbf{v})^2 + (1-|\mathbf{p}|^2)}\right).
\end{aligned}
$$

As expected the quantity is independent of s for $0 < s < 1$ but depends on $\mathbf{v}$. The partial derivative

$$\frac{\partial\psi^j}{\partial x_i}(\mathbf{0})$$

for $i \neq j$ is well-defined and takes the value 0. More generally, the directional derivatives in directions $\mathbf{v}$ for which $\mathbf{p}\cdot\mathbf{v} = 0$, that is the intersection of the $(n-1)$ dimensional subspace orthogonal to $\mathbf{p}$ intersected with $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, satisfy

$$D_{-\mathbf{v}}\phi^j(\mathbf{0}) = -D_{\mathbf{v}}\phi^j(\mathbf{0}).$$

All other directional derivatives satisfy

$$D_{-\mathbf{v}}\phi^j(\mathbf{0}) \neq -D_{\mathbf{v}}\phi^j(\mathbf{0}),$$

so that ψ^j is globally Lipschitz but clearly not differentiable (or even partially differentiable) at $\mathbf{0} \in \mathbb{R}^n$. □

Exercise D.5. Characterize all directions $\mathbf{v} \in \mathbb{S}^1 \subset \mathbb{R}^n$ for which

$$D_{-\mathbf{v}}\psi^j(\mathbf{p}) = -D_{\mathbf{v}}\psi^j(\mathbf{p}).$$

That is, for $j = 1, 2, \ldots, n$, find

$$\{\mathbf{v} \in \mathbb{S}^{n-1} \subset \mathbb{R}^n : D_{-\mathbf{v}}\psi^j(\mathbf{p}) = -D_{\mathbf{v}}\psi^j(\mathbf{p})\}.$$

Exercise D.6. Make illustrations of the graphs of ϕ^1 and ϕ^2 in the case $n = 2$ and explain how those graphs depend on $\mathbf{p} \in B_1(\mathbf{0})$.