# Chapter 7

# Poincaré conjecture

> Know the unknown. Hear the unheard. See the unseen.
>
> —Yogi Tea (teabag tag/tab)

This chapter picks up, in a certain sense, from the discussion of topological spaces in Chapter 13. Some minimal additional structure and an essentially purely intrinsic viewpoint are required. We also use the discussion of paths from section 3.3 in Chapter 3.

**Note:** It is apparently difficult to find a precise, reasonable, and correct statement of the Poincaré conjecture. For example, in 2024 the web page

`https://www.claymath.org/millennium/poincare-conjecture/`

starts by saying:

> In 1904 the French mathematician Henri Poincaré asked if the three dimensional sphere is characterized as the unique simply connected three manifold.

As far as I know, $\mathbb{R}^3$ is a simply connected three manifold that is decidedly not the three-sphere. The web page

`https://en.wikipedia.org/wiki/Poincare_conjecture`

says the "standard form of the conjecture" is the following:

> Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

It may be correct that this is the "standard form of the conjecture" and this may also be a technically correct statement of the conjecture. Unfortunately, the reason it is technically correct is because the word "closed" in the statement does not mean closed in the topological sense of being a closed set. Also, the actual intended meaning is not mentioned except by giving a "link" to the word "closed." Not to be too critical, this is just one more instance in which I view the terminology to be somewhat poorly chosen.

My main objective in this section is to give a precise, reasonable, and correct statement along with certain other details. Above all one may note[1] from the webpage

`https://en.wikipedia.org/wiki/Manifold`

that

> There are many different kinds of manifolds. In geometry and topology, all manifolds are topological manifolds, possibly with additional structure.

Thus, a real question of importance if one wants to know what the conjecture is actually asserting is

> "What kind of manifold is mentioned in the conjecture?"

We attempt to present one answer to this question in the following section.

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[1] Something which actually is correct.

## 7.1 Locally Euclidean manifolds

Recall that the open ball of radius $r > 0$ centered at the origin $\mathbf{0}$ in $\mathbb{R}^n$ is denoted by

$$B_r(\mathbf{0}) = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : |\mathbf{x}| < r\}.$$

**Definition 5.** (locally Euclidean) Given a point $P$ in a topological space $X$, if there is some natural number $n \in \mathbb{N}$ and a pair $(U, V)$ of open sets in $X$ for which

**(i)** $V$ is the codomain of a homeomorphisms $\mathbf{p} : B_2(\mathbf{0}) \to V$,

**(ii)** $P \in U \subset \overline{U} \subset V$, and

**(iii)** the restrictions

$$\mathbf{p}\Big|_{B_1(\mathbf{0})} : B_1(\mathbf{0}) \to U \qquad \text{and} \qquad \mathbf{p}\Big|_{\overline{B_1(\mathbf{0})}} : \overline{B_1(\mathbf{0})} \to \overline{U}$$

are both homeomorphisms,

then the topological space $X$ is said to be **locally Euclidean at** $P$.

**Definition 6.** (Poincaré manifold) Given a fixed natural number $n \in \mathbb{N}$, a topological space $X$ along with a specified collection

$$\mathcal{A} = \{\mathbf{p}_\alpha\}_{\alpha \in \Gamma}$$

of homomeomorphisms $\mathbf{p}_\alpha : B_2(\mathbf{0}) \to V_\alpha$ is said to be a **Poincaré manifold of dimension** $n$ if for each $P \in X$, there exists a pair $(U_\alpha, V_\alpha)$ of open sets in $X$ such that

**(i)** $V_\alpha$ is the codomain of one of the homeomorphisms $\mathbf{p}_\alpha \in \mathcal{A}$,

**(ii)** $P \in U_\alpha \subset \overline{U_\alpha} \subset V_\alpha$, and

**(iii)** the restrictions

$$\mathbf{p}\Big|_{B_1(\mathbf{0})} : B_1(\mathbf{0}) \to U_\alpha \qquad \text{and} \qquad \mathbf{p}\Big|_{\overline{B_1(\mathbf{0})}} : \overline{B_1(\mathbf{0})} \to \overline{U_\alpha}$$

are both homeomorphisms.

The collection $\mathcal{A}$ in Definition 6 is called the **covering atlas** of $X$, and the set $X$ alone is often referred as the Poincaré manifold (or PM manifold) for reasons that should become clear below. Each homeomorphism

$$\mathbf{p} : B_2(\mathbf{0}) \to V$$

in the covering atlas $\mathcal{A}$ is called a **chart function** and the Euclidean ball $B_2(\mathbf{0})$ in this context is referred to as a **chart** or **coordinate chart**. The codomain $V$ is called a **patch** or **coordinate patch** in $X$. The inverse $\xi = \mathbf{p}^{-1} : V \to B_2(\mathbf{0}) \subset \mathbb{R}^n$ is called a **coordinate function**, and each coordinate $\xi^j : V \to \mathbb{R}$ in $\xi = (\xi^1, \xi^2, \ldots, \xi^n)$ naturally bears the same name.

**Theorem 1.** A Poincaré manifold is a Hausdorff space.

Proof: Let $P, Q \in X$ where $P \neq Q$ and $X$ is a Poincaré manifold. Let $\mathbf{p} : B_2(\mathbf{0}) \to V$ and $\mathbf{q} : B_2(\mathbf{0}) \to W$ be chart functions in the covering atlas $\mathcal{A}$ with $P \in U_\alpha \subset \overline{U_\alpha} \subset V$ and $Q \in U_\beta \subset \overline{U_\beta} \subset W$ as in the definition of the chart functions. If $U_\alpha \cap U_\beta = \phi$, then the open sets $U_\alpha$ and $U_\beta$ separate the points $P$ and $Q$ in $X$.

Consider the situation in which $U_\alpha \cap U_\beta \neq \phi$. If $Q \in V \backslash \overline{U_\alpha}$, then the open sets $U_\alpha$ and $V \backslash \overline{U_\alpha}$ separate $P$ and $Q$.

Alternatively, $Q \in \overline{U_\alpha} \subset V$. In this case, $\mathbf{x} = \mathbf{p}^{-1}(P)$ and $\mathbf{y} = \mathbf{p}^{-1}(Q)$ are both well-defined distinct points in $B_2(\mathbf{0}) = \mathbf{p}^{-1}(V)$. There exist disjoint open balls $B_a(\mathbf{x})$ and $B_b(\mathbf{y})$ in $B_2(\mathbf{0})$, and

$$\mathbf{p}(B_a(\mathbf{x})) \qquad \text{and} \qquad \mathbf{p}(B_b(\mathbf{y}))$$

are disjoint open sets in $X$ separating $P$ and $Q$. $\square$

**Exercise 7.1.** Show the following: If $X$ is a Hausdorff topological space and

1. $P \in X$,

2. $V$ is an open set in $X$ with $P \in V$, and

3. for some natural number $n \in \mathbb{N}$, there is a homeomorphism $\mathbf{p} : \mathbb{R}^n \to V$,

then $X$ is locally Euclidean at $P$.

**Exercise 7.2.** Under the assumptions of Exercise 7.1 show there exists a homeomorphism $\mathbf{q} : B_1(\mathbf{0}) \to V$ where $B_1(\mathbf{0}) \subset \mathbb{R}^n$.

**Exercise 7.3.** Given $\mathbf{p} \in B_1(\mathbf{0}) \subset \mathbb{R}^n$, show there exists a homeomorphism $\psi : B_1(\mathbf{0}) \to B_1(\mathbf{0})$ with $\psi(\mathbf{p}) = \mathbf{0}$. In fact, show $\phi : B_1(\mathbf{0}) \to B_1(\mathbf{0})$ by

$$\phi(\mathbf{x}) = \begin{cases} \mathbf{p} + \left(-\mathbf{p}\cdot\mathbf{x} + \sqrt{(\mathbf{p}\cdot\mathbf{x})^2 + |\mathbf{x}|^2(1-|\mathbf{p}|^2)}\right)\dfrac{\mathbf{x}}{|\mathbf{x}|}, & \mathbf{x} \neq \mathbf{0} \\ \mathbf{p}, & \mathbf{x} = \mathbf{0} \end{cases}$$

is the continuous inverse of such a function.

**Exercise 7.4.** What can you say about the higher regularity of the function $\psi = \phi^{-1}$ from Exercise 7.3?

**Exercise 7.5.** Given $\mathbf{p} \in B_1(\mathbf{0}) \subset \mathbb{R}^2$, show there exists a function $\psi \in C^\infty(\overline{B_1(\mathbf{0})} \to \mathbb{R}^2)$ with $\psi(\mathbf{p}) = \mathbf{0}$, the restrictions

$$\psi\big|_{B_1(\mathbf{0})} : B_1(\mathbf{0}) \to B_1(\mathbf{0}) \qquad \text{and} \qquad \psi\big|_{\overline{B_1(\mathbf{0})}} : \overline{B_1(\mathbf{0})} \to \overline{B_1(\mathbf{0})}$$

both homeomorphisms, and

$$D\psi(\mathbf{x}) = c(\mathbf{x}) \begin{pmatrix} \cos\theta(\mathbf{x}) & -\sin\theta(\mathbf{x}) \\ \sin\theta(\mathbf{x}) & \cos\theta(\mathbf{x}) \end{pmatrix}$$

where $c \in C^\infty(B_1(\mathbf{0}) \to (0,\infty))$ is a positive smooth function and $\theta \in C^\infty(B_1(\mathbf{0}))$ is a smooth real valued function. In this case, we can say $A : B_1(\mathbf{0}) \to SL_2(\mathbb{R})$ by

$$A(\mathbf{x}) = \begin{pmatrix} \cos\theta(\mathbf{x}) & -\sin\theta(\mathbf{x}) \\ \sin\theta(\mathbf{x}) & \cos\theta(\mathbf{x}) \end{pmatrix}$$

satisfies $A \in C^\infty(\overline{B_1(\mathbf{0})} \to SL_2(\mathbb{R}))$.

**Exercise 7.6.** A Poincaré manifold is a topological space which is locally Euclidean at each point and for which the chart functions all have domains of the same dimension. (True or false?)

**Exercise 7.7.** (change of coordinates) Let $X$ be a topological manifold of dimension $n$. Assume $\mathbf{p} : B_1(\mathbf{0}) \to U$ and $\mathbf{q} : B_1(\mathbf{0}) \to V$ are chart functions that is, $\mathbf{p}$ and $\mathbf{q}$ are homeomorphisms onto their respective codomains $U$ and $V$ which are open sets in $X$. Let $\xi = \mathbf{p}^{-1} : U \to B_1(\mathbf{0})$ and $\eta = \mathbf{q}^{-1} : V \to B_1(\mathbf{0})$. If $Z = U \cap V \neq \phi$, show the **changes of coordinates**

$$\psi = \eta \circ \mathbf{p}\big|_{\xi(Z)} : \xi(Z) \to \eta(Z) \qquad \text{and} \qquad \phi = \xi \circ \mathbf{q}\big|_{\eta(Z)} : \eta(Z) \to \xi(Z)$$

are homeomorphisms.

**Exercise 7.8.** The changes of coordinates in Exercise 7.5 are examples of **conformal mappings**, and those particular conformal mappings are sometimes called **Möbius transformations of the unit disk**. The function $c : B_1(\mathbf{0}) \to (0,\infty)$ is called the **conformal factor**. Extend the result of Exercise 7.5 to higher dimensions: Given $\mathbf{p} \in B_1(\mathbf{0}) \subset \mathbb{R}^n$, show there exists a function $\psi \in C^\infty(\overline{B_1(\mathbf{0})} \to \mathbb{R}^n)$ with $\psi(\mathbf{p}) = \mathbf{0}$, the restrictions

$$\psi\Big|_{B_1(\mathbf{0})} : B_1(\mathbf{0}) \to B_1(\mathbf{0}) \qquad \text{and} \qquad \psi\Big|_{\overline{B_1(\mathbf{0})}} : \overline{B_1(\mathbf{0})} \to \overline{B_1(\mathbf{0})}$$

both homeomorphisms, and

$$D\psi(\mathbf{x}) = c(\mathbf{x})A(\mathbf{x})$$

where $c \in C^\infty(B_1(\mathbf{0}) \to (0,\infty))$ is a positive smooth function and $A : B_1(\mathbf{0}) \to SL_n(\mathbb{R})$ satisfies $A \in C^\infty(\overline{B_1(\mathbf{0})} \to SL_n(\mathbb{R}))$.

**Exercise 7.9.** (atlas; overlap condition) Let $\mathcal{A} = \{\mathbf{p}_\alpha : B_2(\mathbf{0}) \to V_\alpha\}_{\alpha\in\Gamma}$ be any collection of chart functions as specified in the definition of a PM manifold satisfying

$$X \subset \bigcup_{\alpha\in\Gamma} V_\alpha.$$

Such a collection $\mathcal{A}$ is called an **atlas** for $X$, each open set $V_\alpha$ is called a **covering patch**, and each homeomorphism $\mathbf{p}_\alpha \in \mathcal{A}$ is called a **chart function**.

The domain $B_2(\mathbf{0})$ associated with a specific homeomorphism $\mathbf{p} \in \mathcal{A}$ is called a **chart**. Verify the following patch overlap condition: If $\mathbf{p} : B_2(\mathbf{0}) \to V$ and $\mathbf{q} : B_2(\mathbf{0}) \to W$ are chart functions in $\mathcal{A}$ for which $Z = V \cap W \neq \phi$, then the functions

$$\mathbf{p}^{-1} \circ \mathbf{q}\Big|_{\mathbf{q}^{-1}(Z)} : \mathbf{q}^{-1}(Z) \to \mathbf{p}(Z) \quad \text{and} \quad \mathbf{q}^{-1} \circ \mathbf{p}\Big|_{\mathbf{p}^{-1}(Z)} : \mathbf{p}^{-1}(Z) \to \mathbf{q}(Z)$$

are homeomorphisms.

## 7.2 Connected and simply connected spaces

Recall (or consider) the following definitions:

**Definition 7.** (path connected) A topological space $X$ is **path connected** if given any two points $x, y \in X$ there exists a path $\alpha \in C^0([0,1] \to X)$ with $\alpha(0) = x$ and $\alpha(1) = y$.

Given a topological space $X$ and numbers $a, b \in \mathbb{R}$ with $a < b$, a path $\alpha \in C^0([a,b] \to X)$ is called a **loop** if $\alpha(a) = \alpha(b)$.

**Definition 8.** (simply connected) A path connected topological space $X$ is **simply connected** if given any continuous path $\alpha : [0,1] \to X$ with $\alpha(0) = \alpha(1)$, there exists a function $h \in C^0([0,1] \times [0,1] \to X)$ satisfying

**(i)** $h(0, s) = \alpha(0) = h(1, s)$ for $0 \le s \le 1$,

**(ii)** $h(t, 0) \equiv \alpha(t)$ for $0 \le t \le 1$, and

**(iii)** $h(t, 1) \equiv \alpha(0)$ for $0 \le t \le 1$.

The function $h$ in the definition of simply connected is called a **homotopy** or continuous **deformation**. The first argument of the deformation $h$ is called the parameter or parameterization variable. The second argument is the deformation variable. It is sometimes convenient to have one or both of these variables defined on an interval other than $[0, 1]$.

**Exercise 7.10.** For this exercise, let $X$ be a topological space and let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$.

**(a)** Given a continuous path $\alpha \in C^0([a,b] \to X)$, use the change of variable

$$\tau = (1 - t)a + tb$$

to find a path $\alpha_0 \in C^0([0,1] \to X)$ with

$$\{\alpha_0(t) : t \in [0,1]\} = \{\alpha(\tau) : \tau \in [a,b]\}.$$

**(b)** Given a loop $\alpha : [a,b] \to X$ how there exists a homotopy $h \in C^0([a,b] \times [c,d] \to X)$ satisfying

**(i)** $h(a, s) = \alpha(a) = h(b, s)$ for $c \le s \le d$,

**(ii)** $h(t, c) \equiv \alpha(t)$ for $a \leq t \leq b$, and

**(iii)** $h(t, d) \equiv \alpha(a)$ for $a \leq t \leq b$.

if and only if there exists a homotopy $h_0 \in C^0([0,1] \times [0,1] \to X)$ satisfying

**(i)** $h_0(0, s) = \alpha_0(0) = h_0(1, s)$ for $0 \leq s \leq 1$,

**(ii)** $h_0(t, 0) \equiv \alpha_0(t)$ for $0 \leq t \leq 1$, and

**(iii)** $h_0(t, 1) \equiv \alpha_0(0)$ for $0 \leq t \leq 1$

where $\alpha_0$ is the path you found in part **(a)** above.

## 7.3 Compact manifolds

Recall (or consider) the following definition:

**Definition 9.** A topological space $X$ is **compact** if every open cover of $X$, that is a collection $\{U_\alpha\}_{\alpha \in \Gamma}$ where $\Gamma$ is some indexing set, the sets $U_\alpha$ are open in $X$, and

$$X = \bigcup_{\alpha \in \Gamma} U_\alpha,$$

contains a finite subcover, that is for some $k \in \mathbb{N}$ there exist indices $\alpha_1, \alpha_2, \ldots, \alpha_k$ for which

$$X = \bigcup_{j=1}^{k} U_{\alpha_j}.$$

## 7.4 Statement of the conjecture

**Theorem 2.** (Poincaré conjecture) A compact simply connected Poincaré manifold of dimension 3 is homeomorphic to $\mathbb{S}^3$.

Here $\mathbb{S}^3 = \{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : |\mathbf{x}| = 1\}$ is called the three-sphere.

**Exercise 7.11.** Show the three-sphere with the inherited topology from $\mathbb{R}^4$ is a compact, simply connected, Poincaré manifold of dimension 3.

A conjecture/theorem may be considered in every dimension $n \geq 2$:

**Theorem 3.** (Poincaré conjecture in dimension $n \geq 2$) A compact simply connected Poincaré manifold of dimension $n$ with $n \geq 2$ is homeomorphic to $\mathbb{S}^n$.

In one dimension, the circle $\mathbb{S}^1$ is not simply connected, so it doesn't really make sense to ask a question quite like the Poincaré conjecture.

**Exercise 7.12.** $\mathbb{S}^1$ is a compact connected Poincaré manifold but is not simply connected.

**Exercise 7.13.** A compact connected Poincaré manifold $X$ of dimension $n = 1$ is homeomorphic to $\mathbb{S}^1$. In this case, the assumption that $X$ is simply connected is both unnecessary and not possible.

**Exercise 7.14.** A noncompact connected Poincaré manifold $X$ of dimension $n = 1$ is homeomorphic to $\mathbb{R}^1$. In this case, the assumption that $X$ is noncompact and connected implies $X$ is simply connected.

**Exercise 7.15.** A simply connected Poincaré manifold $X$ of dimension $n = 1$ is homeomorphic to $\mathbb{R}^1$. In this case, the assumption that $X$ is simply connected implies $X$ is noncompact.

The Poincaré conjecture for dimension $n = 2$ is closely related to the famous Riemann mapping theorem and/or the uniformization theorem from complex analysis, and is worth having an entire section devoted to it below.

## 7.5 Topological manifolds

**Definition 10.** (potentially naughty manifolds) Given a fixed natural number $n \in \mathbb{N}$, a **PN topological manifold** (of dimension $n$) is a topological space $X$ satisfying the following condition:

> For each $P \in X$, there exist open sets $V \subset X$ with $P \in V$ and $U \subset \mathbb{R}^n$ and a homeomorphism $\mathbf{p} : U \to V$.

The condition involving $U$, $V$ and $\mathbf{p}$ in Definition 10, or some minor variant of that condition, is often said to express that $X$ is (topologically) **locally Euclidean** at each point. We have take a somewhat different point of view above, and thus we may wish to introduce alternative terminology for this condition now. Let us say a topological space $X$ is **loco-ly Euclidean** at $P \in X$ if there exist open sets $V \subset X$ with $P \in V$ and $U \subset \mathbb{R}^n$ and a homeomorphism $\mathbf{p} : U \to V$.

In view of the common dimension $n$ for the sets $U$ in Definition 10, another way to refer to a PN topological manifold is as a **a loco-ly Euclidean topological space with uniform dimension** $n$.

**Exercise 7.16.** Show a loco-ly Euclidean topological space with uniform dimension $n$ has the following property:

> For each $P \in M$, there exists an open set $V \subset M$ with $P \in V$ and a homeomorphism $\mathbf{p} : B_1(\mathbf{0}) \to V$

where $B_1(\mathbf{0}) = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : |\mathbf{x}| < 1\}$ as usual.

**Exercise 7.17.** Show any open interval $I = (a, b)$ with $a, b \in \mathbb{R}$ and $a < b$ is homeomorphic to $\mathbb{R}$ in several ways:

**(a)** $\tanh^{-1} : (-1, 1) \to \mathbb{R}$ and $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ are both $C^\infty$ homeomorphisms, so $\phi : (a, b) \to \mathbb{R}$ by

$$\tanh^{-1}\left[\frac{1}{b-a}(2x - a - b)\right]$$

and $\psi : (a, b) \to \mathbb{R}$ by

$$\tan\left[\frac{\pi}{2(b-a)}(2x - b - a)\right]$$

are both $C^\infty$ homeomorphisms.

**(b)** Note that $q_1 : (-1, 1) \to \mathbb{R}$ by

$$q_1(x) = \frac{1}{2}\left[\frac{1}{1-x} - \frac{1}{x-1}\right] = \frac{x}{1-x^2}$$

is a $C^\infty$ rational homeomorphism. Consider $q : (a, b) \to \mathbb{R}$ by

$$q(x) = \frac{1}{b-x} - \frac{1}{x-a}.$$

**Exercise 7.18.** Show a loco-ly Euclidean topological space with uniform dimension $n$ has the following property:

> For each $P \in M$, there exists an open set $V \subset M$ with $P \in V$ and a homeomorphism $\mathbf{p} : \mathbb{R}^n \to V$.

Essentially all definitions of any kind of topological manifold contain some kind of loco-ly Euclidean condition like the ones in the definiton of PN topological manifolds given above or in Exercises 7.16 or 7.18. Many references use the one in Exercise 7.18.

The manifold(s) under consideration in the Poincaré conjectures are not so potentially naughty. Aside from being locally Euclidean, they have the additional features of being compact, connected, simply connected, and Hausdorff. The last condition follows, as we have seen in Theorem 1, from being locally Euclidean at each point.

It is interesting to see a counterexample of sorts.[2]

Consider the class of compact and connected loco-ly Euclidean topological spaces with uniform dimension $n$. These are still potentially naughty, and indeed they can be. They do share a nice property with the open connected subsets in $\mathbb{R}^n$.

**Exercise 7.19.** Show a compact connected loco-ly Euclidean topological space $X$ with uniform dimension $n$ is path connected. Hint: Lettig $P_0$ be fixed in $X$, show

$$\{P \in X : \text{there exists a path connecting } P_0 \text{ to } P\}$$

is both open and closed in $X$.

[2]This example might be of even more interest before the definition of Poincaré manifolds is given as a motivation for that definition and the associated terminology. In particular, this kind of example is precisely why the potentially naughty topological manifolds should not be called locally Euclidean. One might even say that to do so would be loco.

## Two sphere with an extra north pole

Recall that

$$\mathbb{S}^2 = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$$

is a simply connected topological subspace of $\mathbb{R}^3$. Consider $X = \mathbb{S}^2 \cup \{N^*\}$ where the open sets in $X$ include all the open sets $U$ in $\mathbb{S}^2$ as a subset/topological subspace of $\mathbb{R}^3$ along with the open sets of the following forms:

$$V = (V_1 \backslash \{(0,0,1)\}) \cup \{N^*\} \tag{7.1}$$

and

$$V = V_1 \cup \{N^*\} \tag{7.2}$$

where $V_1$ is an open subset of $\mathbb{S}^2$ satisfying $(0,0,1) \in V_1$. This space is also called the "bug-eyed sphere."

Note that $N^*$ is not equal to any point in $\mathbb{S}^2$. We have incorporated a new point. However, given a homeomorphism $\mathbf{p} : U \to V_1 = \mathbf{p}(U) \subset \mathbb{S}^2$ with $V_1$ open in $\mathbb{S}^2$, the set $U$ an open subset of $\mathbb{R}^2$, and $(0,0,1) \in V = \mathbf{p}(U)$, the chart $\mathbf{q} : U \to V$ by

$$\mathbf{q}(\mathbf{x}) = \begin{cases} \mathbf{p}(\mathbf{x}), & \mathbf{x} \in U \backslash \{\mathbf{p}^{-1}(0,0,1)\} \\ N^*, & \mathbf{x} = \mathbf{p}^{-1}(0,0,1) \end{cases}$$

where $V$ is given in (7.1) is a homeomorphism. Thus, $M$ is a loco-ly Euclidean topological space.

**Exercise 7.20.** Let $X$ be the two sphere with an extra north pole as defined above. Verify the following:

**(a)** $X$ is a loco-ly Euclidean topological space of uniform dimension 2, i.e., a PN topological manifold.

**(b)** $X$ is connected.

**(c)** $X$ is compact.

According to Exercise 7.19 $X$ is also path connected.

**Exercise 7.21.** Show the two sphere with an extra north pole $X$, as featured in Exercise 7.20 above is simply connected.

**Exercise 7.22.** Show the two sphere with an extra north pole $X$, as featured in Exercise 7.20 above is not homeomorphic to $\mathbb{S}^2$. Hint: The property of being a Hausdorff topological space is preserved under homeomorphism, i.e., the property of being a Hausdorff space is a topological invariant.

There is, of course, also a three sphere with two north poles (and an $n$ sphere with two north poles). These are all simply connected (pretty naughty) topological manifolds that are not Hausdorff and are not homeomorphic to the standard sphere of the same dimension.

## 7.6 Topological surfaces

A Poincaré manifold of dimension $n = 2$ is a (topological) surface.

**Exercise 7.23.** Show

$$\mathbb{S}^2 = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$$

as a topological subspace of $\mathbb{R}^3$ is a topological surface with an atlas $\mathcal{A}_0$ consisting of two charts.

**Exercise 7.24.** Show

$$\mathbb{S}^2 = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\},$$

the topological surface featured in Exercise 7.23 is compact.