# Appendix C

# Solutions

**Solution of Exercise. (Exercise 3.1)** Find the Euclidean length length$[\alpha]$ and Riemannian length $\text{length}_{\mathcal{B}}[\alpha]$ of the path $\alpha : [0, a] \to B_1(\mathbf{0})$ by $\alpha(t) = t(\cos\theta, \sin\theta)$ where $\theta \in \mathbb{R}$ and $a > 0$.

In this case, $|\alpha| = t$ and $|\alpha'| = 1$.

$$\text{length}[\alpha] = \int_0^a 1\,dt = a,$$

and

$$\begin{aligned}\text{length}_{\mathcal{B}}[\alpha] &= \int_0^a \frac{4}{4+t^2}\,dt \\ &= \int_0^a \frac{1}{1+(t/2)^2}\,dt \\ &= 2\int_0^{a/2} \frac{1}{1+u^2}\,du \\ &= 2\tan^{-1}\left(\frac{a}{2}\right).\end{aligned}$$

The interesting observation here is that this manifold $\mathcal{B}$ is apparently contained in a larger manifold obtained by extending the matrix assignment

$$(g_{ij}) = \frac{4}{(4+|\mathbf{x}|^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

to the entire plane $\mathbb{R}^2$. The notion of $\text{length}_{\mathcal{B}}$ for these paths then extends to entire rays defined for $a > 0$ by the same formula, and these rays have lengths bounded by $\pi$.

Since length$[\alpha] = a$, one can consider length$_{\mathcal{B}}[\alpha]$ as a function of the Euclidean length length$[\alpha]$. Notice

$$\frac{d}{da}\text{length}_{\mathcal{B}}[\alpha] = \frac{1}{1+(a/2)^2} < 1 \qquad \text{and} \qquad \frac{d}{da}\text{length}_{\mathcal{B}}[\alpha]\Big|_{a=0} = 1.$$

Thus, these paths start at $\mathbf{0}$ with Riemannian length and Euclidean length essentially equal, but as the ray extends, the Riemannian length becomes shorter and shorter relative to the Euclidean length, so much so that the total Riemannian length is always less then $\pi$ as indicated on the right in Figure C.1. For radial segments contained in $\mathcal{B}$ the Riemannian lengths satisfy

$$2\tan^{-1}\left(\frac{1}{2}\right)\ \text{length}[\alpha] < \text{length}_{\mathcal{B}}[\alpha] < \text{length}[\alpha],$$

and

$$b = 2\tan^{-1}\left(\frac{1}{2}\right) \doteq 0.927295.$$

See Figure C.1 (left).

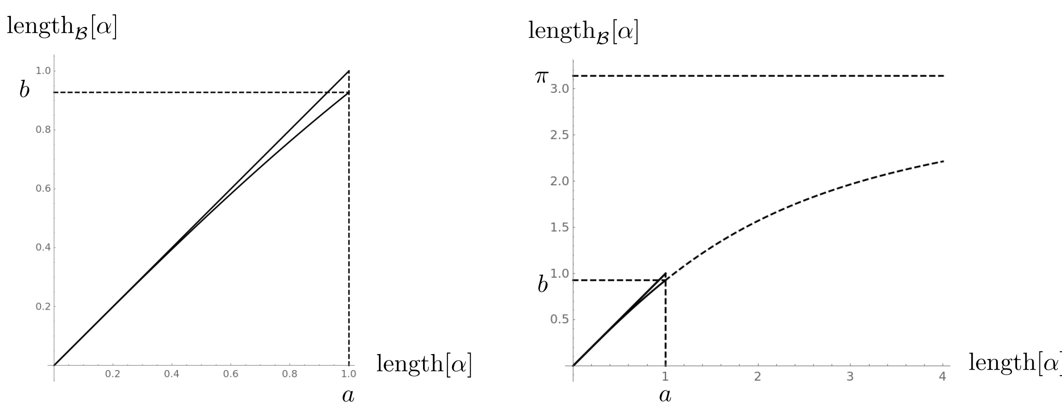


Figure C.1: Comparison of Riemannian length and Euclidean length for rays in $\mathcal{B}$. In this illustration $a = 1$ is the Euclidean radius of $B_1(\mathbf{0})$ and $b = 2\tan^{-1}(1/2)$ is the Riemannian radius of $\mathcal{B}$.