Appendix D

Solutions

D.1 Chapter 1

D.1.1 Introduction to geodesics: $\mathcal{B}$

Solution of Exercise. (Exercise 1.45, page 61) Derive a system of ordinary differential equations for the component functions α_j, $j = 1, 2, \dots, n$ for a minimizer of Riemann's length minimization problem. (Assume as much regularity as you need, but note how much regularity that is. Specifically, explain how regular the component functions α_j, $j = 1, 2, \dots, n$ and the matrix entries g_{ij}, $i, j = 1, 2, \dots, n$ need to be in order for your calculus techniques/manipulations to be justified.)

I'm going to give a solution for a special case of this problem related to the example Riemannian manifold $\mathcal{B}$ discussed in Chapter 3. This is related to the calculation(s) in Exercise 1.41 (page 58) and gives in particular a parametric version of some of those calculation(s). See also Appendix B.

I begin with the subset

$$\mathcal{A}^2 = \left\{\alpha \in \mathfrak{P}^2(B_1(\mathbf{0})) : \alpha \in \mathcal{A}\right\}$$

where

$$\mathcal{A} = \left\{\alpha \in \mathfrak{P}^1(B_1(\mathbf{0})) : \alpha(a) = \mathbf{x}, \alpha(b) = \mathbf{y} \in B_1(\mathbf{0})\right\}.$$

In particular, I assume $\alpha \in C^2([a.b] \to B_1(\mathbf{0}))$ minimizes

$$\int_{(a,b)} \frac{4}{4 + |\alpha|^2}\, |\alpha'|$$

which I will write simply as

$$\int \frac{4}{4+|\alpha|^2}\ |\alpha'|.$$

Accordingly, I compute for some admissible vector valued perburbation $\phi \in C^2([a,b] \to \mathbb{R}^2)$ with $\phi(a) = \phi(b) = 0$

$$\begin{aligned}\frac{d}{dt}\int \frac{4}{4+|\alpha+t\phi|^2}\ |\alpha'+t\phi'|\Big|_{t=0} &= \int \left(-\frac{8\langle\alpha,\phi\rangle}{(4+|\alpha|^2)^2}\ |\alpha'| + \frac{4\langle\alpha',\phi'\rangle}{(4+|\alpha|^2)|\alpha'|}\right) \\ &= \int \left\langle -\frac{8|\alpha'|}{(4+|\alpha|^2)^2}\ \alpha - \left(\frac{4}{(4+|\alpha|^2)|\alpha'|}\ \alpha'\right)', \phi\right\rangle.\end{aligned}$$

Thus, I obtain for the parametric Euler-Lagrange equation, i.e., system of ODEs, for geodesics in $\mathcal{B}$

$$\left(\frac{1}{(4+|\alpha|^2)|\alpha'|}\ \alpha'\right)' + \frac{2|\alpha'|}{(4+|\alpha|^2)^2}\ \alpha = \mathbf{0}. \tag{D.1}$$

Under our regularity assumption $\alpha \in C^2([a,b] \to B_1(\mathbf{0}))$ we can expand the derivative to write

$$\begin{aligned}\frac{1}{(4+|\alpha|^2)|\alpha'|}\ \alpha'' - \frac{\langle\alpha',\alpha''\rangle}{(4+|\alpha|^2)|\alpha'|^3}\ \alpha' \\ - \frac{2\langle\alpha,\alpha'\rangle}{(4+|\alpha|^2)^2|\alpha'|}\ \alpha' + \frac{2|\alpha'|}{(4+|\alpha|^2)^2}\ \alpha = \mathbf{0},\end{aligned}$$

or

$$\begin{aligned}N[\alpha] = (4+|\alpha|^2)\ \alpha'' - \frac{(4+|\alpha|^2)\langle\alpha',\alpha''\rangle}{|\alpha'|^2}\ \alpha' \\ - 2\langle\alpha,\alpha'\rangle\ \alpha' + 2|\alpha'|^2\ \alpha = \mathbf{0}.\end{aligned}$$

A good way to check that this is the correct system of ODEs is to see what happens in the special case when we have a nonparametric minimizer given by $\alpha(x) = (x, h(x))$ as considered in Exercise 1.41 and in Appendix B in particular. If we have the correct system, then we should expect some kind of reduction to the single ODE

$$\frac{4+x^2+h^2}{1+h'^2}\ h'' + 2\,(h - xh') = 0. \tag{D.2}$$

In fact, we observe that in this case

$$\alpha' = (1, h') \qquad \text{and} \qquad \alpha'' = (0, h'')$$

so that

$$N[\alpha] = (4 + x^2 + h^2)\ (0, h'') - \frac{(4 + x^2 + h^2)h'h''}{1 + h'^2}\ (1, h') \\ - 2(x + hh')\ (1, h') + 2(1 + h'^2)\ (x, h).$$

Therefore, in this case the condition $N[\alpha] = \mathbf{0}$ gives the system of (two) equations

$$- \frac{4 + x^2 + h^2}{1 + h'^2}\ h'h'' - 2(x + hh') + 2x(1 + h'^2) \\ = -\frac{4 + x^2 + h^2}{1 + h'^2}\ h'h'' - 2hh' + 2xh'^2 = 0$$

and

$$(4 + x^2 + h^2)\ h'' - \frac{(4 + x^2 + h^2)h'^2h''}{1 + h'^2} - 2(x + hh')\ h' + 2h(1 + h'^2) \\ = \frac{4 + x^2 + h^2}{1 + h'^2}\ h'' + 2(h - xh') = 0.$$

The first equation reduces to (D.2) when $h' \neq 0$, and the second equation gives a nonsingular reduction to (D.2). This is pretty good circumstantial evidence that we've made the calculation correctly.

It may be recalled that we have found a nominally two-parameter family of solutions of (D.2) given by

$$h(x) = y + \sqrt{(y_a - y)^2 - x^2}$$

corresponding to points (a, y_a) in the second quadrant of $B_1(\mathbf{0})$ for the particular values

$$y = -\frac{4 - a^2 - y_a^2}{2y_a}.$$

Recall also that the point $(0, y)$ gives the center of the circular arc determined by these solutions. The radii of the circular arcs determined by these solutions my also be determined and is found to be given by

$$|(a, y_a) - (0, y)| = \sqrt{a^2 + \frac{(4 - a^2 + y_a^2)^2}{4y_a^2}} = |(\pm 2, 0) - (0, y)|.$$

This means each circular arc is given by the arc of a circle with center some point $(0, y)$ with $y < 0$, and that circle also passes through the points $(\pm 2, 0)$ which are diametrically opposed points on the circle $\partial B_2(\mathbf{0})$ of radius 2. See Figure D.1 (left).

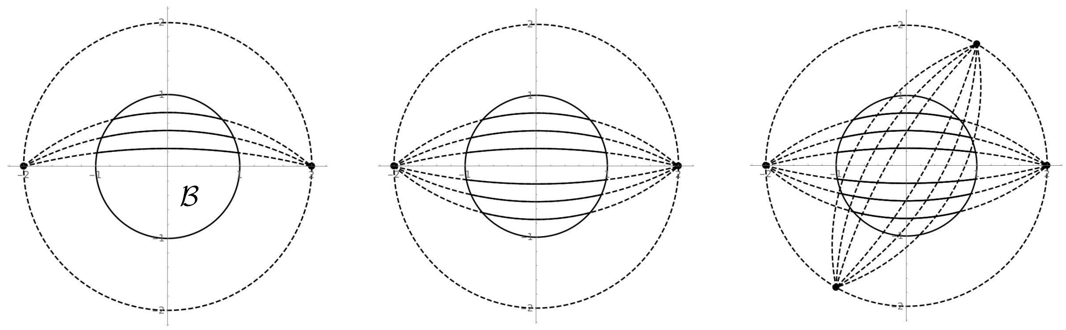

Figure D.1: Geodesics in $\mathcal{B}$. Arcs of circles through $(\pm 2, 0)$ passing through the second quadrant of $\mathcal{B}$ (left). Reflections about the x_1-axis (middle). Rotations (right)

Given the symmetry of the matrix assignment in $B_1(\mathbf{0}) \subset \mathbb{R}^2$ which determines the geodesics in $\mathcal{B}$ it should be intuitively clear that all arcs of circles passing through $(\pm 2, 0)$ and $B_1(\mathbf{0})$, as indicated in Figure D.1 (middle), or passing through any diametrically opposed points on $\partial B_2(\mathbf{0})$ and $B_1(\mathbf{0})$, as indicated in Figure D.1 (right), are also geodesics.

Exercise D.1. Argue that if an arc of a circle passing through four points $\mathbf{x}$ and $\mathbf{y}$ in $B_1(\mathbf{0})$ and $(\pm 2, 0)$ and parameterized by $\alpha : [a, b] \to B_1(\mathbf{0})$ is a minimizer for $\text{length}_{\mathcal{B}}[\alpha]$ among paths connecting $\mathbf{x}$ to $\mathbf{y}$ in $B_1(\mathbf{0})$, then $\beta : [a, b] \to B_1(\mathbf{0})$ by $\beta(t) = -\alpha(t)$ is a minimizer for $\text{length}_{\mathcal{B}}[\beta]$ among paths connecting $-\mathbf{x}$ to $-\mathbf{y}$.

Exercise D.2. (reflection argument) Argue that if an arc of a circle passing through four points $\mathbf{x}$ and $\mathbf{y}$ in $B_1(\mathbf{0})$ and $(\pm 2, 0)$ and parameterized by $\alpha : [a, b] \to B_1(\mathbf{0})$ is a minimizer for $\text{length}_{\mathcal{B}}[\alpha]$ among paths connecting $\mathbf{x}$ to $\mathbf{y}$ in $B_1(\mathbf{0})$, then $\beta : [a, b] \to B_1(\mathbf{0})$ by $\beta(t) = -\alpha(t)$ is a minimizer for $\text{length}_{\mathcal{B}}[\beta]$ among paths connecting $-\mathbf{x}$ to $-\mathbf{y}$.

Exercise D.3. (rotation argument) Argue that if an arc of a circle passing through four points $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $B_1(\mathbf{0})$ and $(\pm 2, 0)$ and parameterized by $\alpha = (\alpha^1, \alpha^2) : [a, b] \to B_1(\mathbf{0})$ is a minimizer for $\text{length}_{\mathcal{B}}[\alpha]$ among paths connecting $\mathbf{x}$ to $\mathbf{y}$ in $B_1(\mathbf{0})$, then for any fixed $\theta \in \mathbb{R}$ the path $\beta : [a, b] \to B_1(\mathbf{0})$ by

$$\beta(t) = (\cos\theta\ \alpha^1(t) - \sin\theta\ \alpha^2(t), \sin\theta\ \alpha^1(t) + \cos\theta\ \alpha^2(t))$$

is a minimizer for $\text{length}_{\mathcal{B}}[\beta]$ among paths connecting

$$(\cos\theta\, x_1 - \sin\theta\, x_2, \sin\theta\, x_1 + \cos\theta\, x_2) \quad \text{to} \quad (\cos\theta\, y_1 - \sin\theta\, y_2, \sin\theta\, y_1 + \cos\theta\, y_2).$$

I'd like to verify directly that **every rotation** of a circular arc passing through $(\pm 2, 0)$ gives a solution of

$$N[\alpha] = (4 + |\alpha|^2)\ \alpha'' - \frac{(4 + |\alpha|^2)\langle \alpha', \alpha'' \rangle}{|\alpha'|^2}\ \alpha' \\ - 2\langle \alpha, \alpha' \rangle\ \alpha' + 2|\alpha'|^2\ \alpha = \mathbf{0}.$$

Such an arc is illustrated in Figure D.2. Given any $\ell > 0$, such an arc is

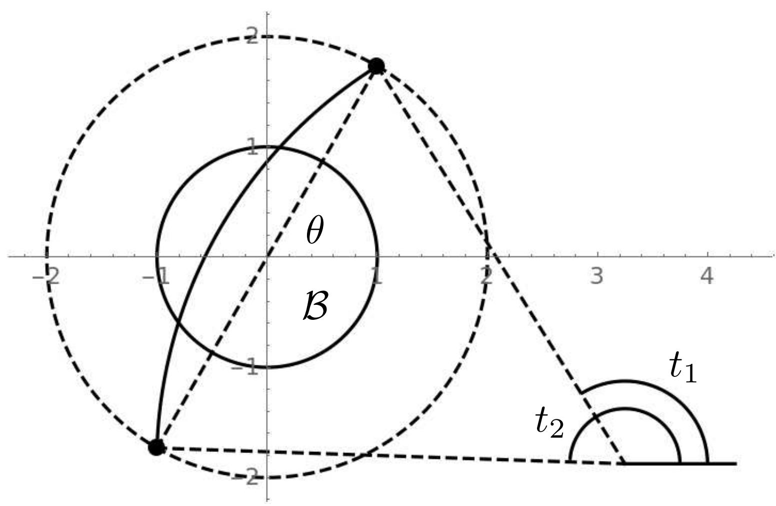

Figure D.2: Circular arc through diametrically opposed points $\pm 2(\cos\theta, \sin\theta)$.

parameterized by

$$\alpha(t) = \ell(\sin\theta, -\cos\theta) + \sqrt{\ell^2 + 4}\ (\cos t, \sin t).$$

Exercise D.4. Assume $0 < \theta < \pi/2$ and find the ranges of t for which $t_1 < t < t_2$ corresponds to

(a) $\alpha(t) \in B_2(\mathbf{0})$.

(a) $\alpha(t) \in B_1(\mathbf{0})$.

From the formula above, we calculate

$$|\alpha|^2 = 2\ell^2 + 4 + 2\ell\sqrt{\ell^2+4}\,(\sin\theta\cos t - \cos\theta\sin t),$$

$$\alpha' = \sqrt{\ell^2+4}\,(-\sin t, \cos t) \qquad \text{and} \qquad \alpha'' = -\sqrt{\ell^2+4}\,(\cos t, \sin t).$$

Thus,

$$\begin{aligned}
N[\alpha] = &-\Big(2\ell^2 + 8 + 2\ell\sqrt{\ell^2+4}\,(\sin\theta\cos t - \cos\theta\sin t)\Big)\sqrt{\ell^2+4}\,(\cos t, \sin t)\\
&\quad - 2\ell(\ell^2+4)(-\sin\theta\sin t - \cos\theta\cos t)(-\sin t, \cos t)\\
&\qquad + 2(\ell^2+4)\Big(\ell(\sin\theta, -\cos\theta) + \sqrt{\ell^2+4}\,(\cos t, \sin t)\Big)\\
= &-2\ell(\ell^2+4)(\sin\theta\cos t - \cos\theta\sin t)\,(\cos t, \sin t)\\
&\quad + 2\ell(\ell^2+4)(\sin\theta\sin t + \cos\theta\cos t)\,(-\sin t, \cos t)\\
&\qquad + 2\ell(\ell^2+4)(\sin\theta, -\cos\theta)\\
= &\ \mathbf{0}.
\end{aligned}$$

This establishes the claim that these parameterized arcs of circles satisfy the parametric Euler-Lagrange equation for the geodesics.

Consider the following observations about geodesics (paths which are solutions of the parametric equation of vanishing curvature) in the plane $\mathbb{R}^2$:

1. Connecting every two points $\mathbf{x}$ and $\mathbf{y}$, there exists a unique geodesic (straight line) path which is a minimizer of length among paths connecting those points. See Figure D.3(left).

2. There exist families of parallel geodesics (straight lines) foliating the entire space $\mathbb{R}^2$.

3. Such a family/foliation can be rotated by an angle θ so that each point has multiple geodesics passing through the point. In particular, given any one point and one geodesic line through that point, there exists a unique second geodesic line through the same point making an angle θ with the first geodesic line.

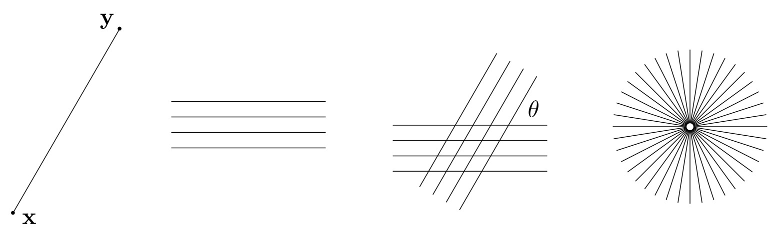

Figure D.3: Euclidean geodesics.

4. As a result, given a single point $\mathbf{x}$, the submanifold $\mathbb{R}^2\backslash\{\mathbf{x}\}$ is foliated by a "star" of geodesic lines/rays passing through $\mathbf{x}$.

5. Each of these observations generalizes to $\mathbb{R}^n$ for $n \geq 2$ with the "star foliation" parameterized on $\mathbb{S}^{n-1}$.

In short the geodesic lines in $\mathbb{R}^n$ can be organized in various interesting ways. It may also be observed that these observations may be used to characterize certain subsets of the manifold $\mathbb{R}^n$. For example, a subset of $\mathbb{R}^n$ is classified as **convex** precisely if the geodesic line segment connect each pair of its points lies intirely in the set. It is natural to seek generalizations of each of these observations (or to see that they fail and what actually does occur) in any Riemannian manifold.

In particular, verification of the first observation, about unique geodesics connecting each pair of points, would be interesting to establish for the Riemannian manifold $\mathcal{B}$. This turns out to be possible and continues to be possible for the manifolds corresponding to open balls $B_r(\mathbf{0})$ with the same matrix assignment $g_{ij} = 16\delta_{ij}/(4+|\mathbf{x}|^2)^2$ as long as $0 < r \leq 2$. For $r > 2$, the property of having unique Riemannian geodesic segments connecting pairs of points fails. For example, Any of the paths illustrated on the left in Figure D.1 are minimizers of $\text{length}_{\mathcal{B}}[\alpha]$ among paths α connecting $(-2, 0)$ to $(2, 0)$ in $B_3(\mathbf{0})$. Furthermore, as we shall see in more detail later, the sets $B_r(\mathbf{0})$ correspond to convex sets in the extension of $\mathcal{B}$ corresponding to $B_3(\mathbf{0})$ precisely for $0 < r \leq 2$.

Exercise D.5. Find the smallest natural number m for which there exists a (unique) geodesic Γ of shortest Riemannian length connecting the points $\mathbf{x} = (m\sqrt{2}, 0)$ and $\mathbf{y} = (m, m)$ in the extension of $\mathcal{B}$ corresponding to $\mathbb{R}^2$ but for which

$$\Gamma \not\subset B_{m+1}(\mathbf{0}).$$

I am going to now focus on attempting to obtain a "star foliation" associated with each point $\mathbf{x} \in \mathcal{B}$. I will start with the special case when $\mathbf{x} = (0, b)$ for some b with $0 < b < 1$. We already know what happens at $\mathbf{x} = (0, 0)$. Furthermore, I will consider a specified direction $(\cos\theta, \sin\theta)$ for paths passing through $\mathbf{x}$ with $0 < \theta < \pi/2$. For the point $\mathbf{x} = (0, b)$, we already know there is a unique circular geodesic passing horizontally through $(0, b)$ and corresponding to $\theta = 0$. Also, there is a unique geodesic passing vertically through $(0, b)$ corresponding to $\theta = \pi/2$, namely the path corresponding to the x_2-axis. There are various ways to see (at least heuristically) that one and exactly one of the circular arcs passing through two diameterically opposed points on $\partial B_2(\mathbf{0})$ passes through $(0, b)$ making an angle θ with the horizontal. I will outline one approach and then attempt to pursue another carefully.

Consider Figure D.4(left). The circle $\partial B_b(\mathbf{0})$ of radius b with center $(0, 0)$

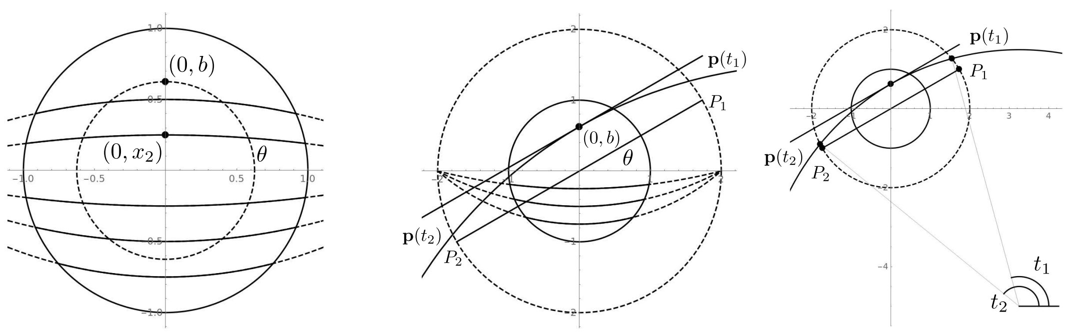

Figure D.4: Heuristic approaches to find the geodesic star at $(0, b) \in \mathcal{B}$.

is shown dashed. The arc of this circle parameterized by $\beta(t) = b(\cos t, \sin t)$ for $0 \le t \le \pi/2$ intersects each geodesic circular arc Γ passing through $(\pm 2, 0)$ and the point $\mathbf{x} = (0, x_2)$ with $0 \le x_2 \le b$. The heuristic argument is then reduced to basically two assertions:

(a) The angle θ with $0 \le \theta \le \pi/2$ at which $\partial B_b(\mathbf{0})$ and Γ meet, as indicated on the left in Figure D.4, is a smooth decreasing function of x_2 with $\theta = \pi/2$ when $x_2 = 0$ and $\theta = 0$ when $x_2 = b$.

(b) By rotating the geodesic arc Γ with respect to $(0, 0)$ so the point of intersection rotates to coincide with $(0, b)$, one obtains a geodesic circular arc Γ' passing through $(0, b)$ and making an angle θ with $\partial B_b(\mathbf{0})$ and hence with the geodesic arc Γ_0 which is horizontal at $(0, b)$. This gives the desired geodesic ray passing through $(0, b)$ and gives a one-to-one correspondence between the rays in the "geodesic star" and the rotations of the foliation of $B_1(\mathbf{0})$ given by circular arcs passing through $(\pm 2, 0)$.

The alternative approach, which I will attempt to discuss in detail, is illustrated in the middle and right in Figure D.4. Here we begin directly with an initial direction $(\cos\theta, \sin\theta)$ at $(0, b)$ and consider the family of circles tangent to the line generated by the direction. The circle with curvature $k > 0$ has center

$$(0, b) + \frac{1}{k}(\sin\theta, -\cos\theta)$$

and may be parameterized by

$$\begin{aligned}\mathbf{p} &= (0, b) + \frac{1}{k}(\sin\theta, -\cos\theta) + \frac{1}{k}(\cos t, \sin t)\\ &= \left(\frac{1}{k}(\sin\theta + \cos t), b + \frac{1}{k}(-\cos\theta + \sin t)\right).\end{aligned}$$

Given the three parameters k, b, and θ, a first task is to determine the values of t corresponding to $\mathbf{p} \in \partial B_2(\mathbf{0})$. There should be two of them satisfying $t_1 < t_2$ and $|\mathbf{p}| = 2$ or

$$2 + 2\sin\theta\cos t - 2\cos\theta\sin t + 2bk(-\cos\theta + \sin t) + b^2k^2 = 4k^2.$$

More precisely, for k small enough there should exist two solutions t_1 and t_2 of the equation

$$2\sin\theta\cos t = 2(\cos\theta - bk)\sin t + (4 - b^2)k^2 + 2bk\cos\theta - 2, \tag{D.3}$$

and the corresponding points of intersection, $\mathbf{p}(t_1)$ and $\mathbf{p}(t_2)$ should tend as $k \searrow 0$ to the corresponding intersection points of the straight line parameterized by $\ell_b(t) = (0, b) + t(\cos\theta, \sin\theta)$ with $\partial B_2(\mathbf{0})$, that is

$$\lim_{k\searrow 0}\mathbf{p}(t_1) = (0, b) + \left(\sin\theta + \sqrt{4 - b^2\cos^2\theta}\right)(\cos\theta, \sin\theta)$$

and

$$\lim_{k\searrow 0}\mathbf{p}(t_2) = (0, b) + \left(\sin\theta - \sqrt{4 - b^2\cos^2\theta}\right)(\cos\theta, \sin\theta).$$

In particular, for k small, both of the points $\mathbf{p}(t_j)$ for $j = 1, 2$ should lie in an open semicircle determined by the parallel line parameterized by $\ell(t) = t(\cos\theta, \sin\theta)$ passing through $(0, 0)$. Based on the illustration, as k increases from $k = 0$,

(a) The point $\mathbf{p}(t_2)$ will initially become closer to $P_2 = -2(\cos\theta, \sin\theta)$ with increasing k while

(b) The point $\mathbf{p}(t_1)$ will initially become closer to $P_1 = 2(\cos\theta, \sin\theta)$.

(c) The point $\mathbf{p}(t_2)$ will coincide with P_2 for a unique positive value k_a of k, and the point $\mathbf{p}(t_1)$ will not have coincided with P_1. At this point, $\mathbf{p}(t_1)$ and $\mathbf{p}(t_2)$ remain in an open semicircle of $\partial B_2(\mathbf{0})$ and are not diametrically opposed satisfying $|\mathbf{p}(t_2) - \mathbf{p}(t_1)| < 2$.

(d) As k increases from k_a, the diameter determined by $\mathbf{p}(t_2)$ will determine a diametrically opposed point $-\mathbf{p}(t_2)$ which will get closer to $\mathbf{p}(t_1)$ and eventually coincide with $\mathbf{p}(t_1)$ at a unique value of $k = k_g(\theta, b) = k_{\text{geodesic}}(\theta, b)$. Thus the points $\mathbf{p}(t_1)$ and $\mathbf{p}(t_2)$ will be diametrically opposed and lie on the unique geodesic circular arc passing through $(0, b)$ tangent to the direction $(\cos\theta, \sin\theta)$.

Computationally, the situation is somewhat more complicated than it may appear heuristically.

We return to the equation (D.3) and let

$$\begin{aligned} c_0 &= (4 - b^2)k^2 + 2bk\cos\theta - 2, \\ c_1 &= 2(\cos\theta - bk), \qquad \text{and} \\ c_2 &= 2\sin\theta \end{aligned}$$

so the equation takes the form $c_2 \cos t = c_1 \sin t + c_0$. Squaring the equation we obtain a quadratic equation in $\sin t$ (at the risk of introducing extraneous roots):

$$c_2^2(1 - \sin^2 t) = c_1^2 \sin^2 t + 2c_0c_1 \sin t + c_0^2$$

or

$$(c_1^2 + c_2^2)\sin^2 t + 2c_0c_1 \sin t + c_0^2 - c_2^2 = 0. \tag{D.4}$$

While this quadratic equation for $\sin t$ is singular as k tends to 0, it is regular as θ tends to 0. In that nonsingular limit, we note that c_2 tends to zero, and the equation becomes

$$(c_1 \sin t + c_0)^2 = 0$$

yielding the solutions

$$t_1 = \sin^{-1}\left(-\frac{c_0}{c_1}\right) < \frac{\pi}{2} < t_2 = \pi - t_1$$

satisfying

$$\frac{\pi}{2} - t_1 = t_2 - \frac{\pi}{2}$$

as expected by symmetry. Incidentally, in the limit as θ tends to 0 we also have

$$\begin{aligned} c_0 &= (4-b^2)k^2 + 2bk - 2, \qquad \text{and} \\ c_1 &= 2(1-bk). \end{aligned}$$

For k small therefore we see $c_0 < 0$ and $c_1 > 0$, so $0 < -c_0/c_1$. Also, with these values

$$c_1 + c_0 = (4-b^2)k^2 > 0.$$

This means $c_1 > |c_0| = -c_0$ so the solutions t_1 and t_2 in this limit are definitely well-defined.

D.1.2 Special Configurations

In order to understand the solutions of equation (D.3) in reference to Figure D.2 and Figure D.4(right), we focus first on some special configurations and seek in particular an interval

$$0 < k < k_{\max} = k_{\max}(b, \theta)$$

on which we expect values of t_1 and t_2 to be well-defined as corresponding to intersection points of $\partial B_2(\mathbf{0})$ with the circle Γ parameterized by

$$\gamma(t) = (0, b) + \frac{1}{k}(\sin\theta + \cos t, -\cos\theta + \sin t)$$

having radius $1/k$, passing through $(0, b)$, tangent to the line parameterized by $\ell(t) = (0, b) + t(\cos\theta, \sin\theta)$ and with center

$$(0, b) + \frac{1}{k}(\sin\theta, -\cos\theta).$$

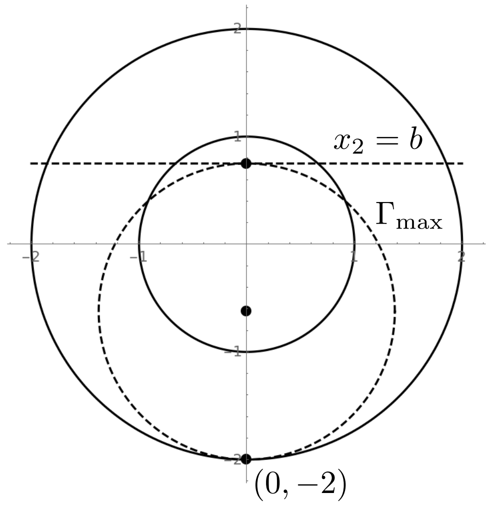

Figure D.5: Limiting configuration with $\theta = 0$. The desired circle $\Gamma_{\max}$ in this limiting case has center at $(0, (-2+b)/2)$ and radius $r_{\min} = (2+b)/2 = 1+b/2$. The unique point of intersection of Γ with $\partial B_2(\mathbf{0})$ is the point $(0, -2)$.

D.1.3 $\theta = 0$

When $\theta = 0$, the situation is illustrated in Figure D.5. The value of $k_{\max}$ in this case is

$$k_{\max} = \frac{2}{2+b} = \frac{1}{1+\frac{b}{2}}. \tag{D.5}$$

For $0 < k < k_{\max}$ or equivalently for any radius $r = 1/k$ satisfying

$$r > r_{\min} = \frac{1}{k_{\max}} = 1 + \frac{b}{2},$$

there should exist exactly two distinct intersection points

$$\gamma(t_1) = (0, b) + \frac{1}{k}(\sin\theta + \cos t_1, -\cos\theta + \sin t_1) = (0, b) + \frac{1}{k}(\cos t_1, -1 + \sin t_1)$$

and

$$\gamma(t_2) = (0, b) + \frac{1}{k}(\sin\theta + \cos t_2, -\cos\theta + \sin t_2) = (0, b) + \frac{1}{k}(\cos t_2, -1 + \sin t_2)$$

satisfying $|\gamma(t_j)| = 2$ for $j = 1, 2$. In this special configuration moreover, we should be able to find unique values t_1 and t_2 satisfying

$$-\frac{\pi}{2} < t_1 < \frac{\pi}{2} < t_2 < \frac{3\pi}{2}.$$

More specifically, the equation (D.4) which is quadratic in $\sin t$ can determine at most two distinct values for $\sin t$. For each such value, there are nominally infinitely many choices for t given by $t = t_0 + m\pi$ with $m \in \mathbb{Z}$. In the special case $\theta = 0$ as noted above, equation (D.4) becomes $(c_1 \sin t + c_0)^2 = 0$ which determines exactly one value

$$\sin t = -\frac{c_0}{c_1}$$

for $\sin t$. We next analyze the value

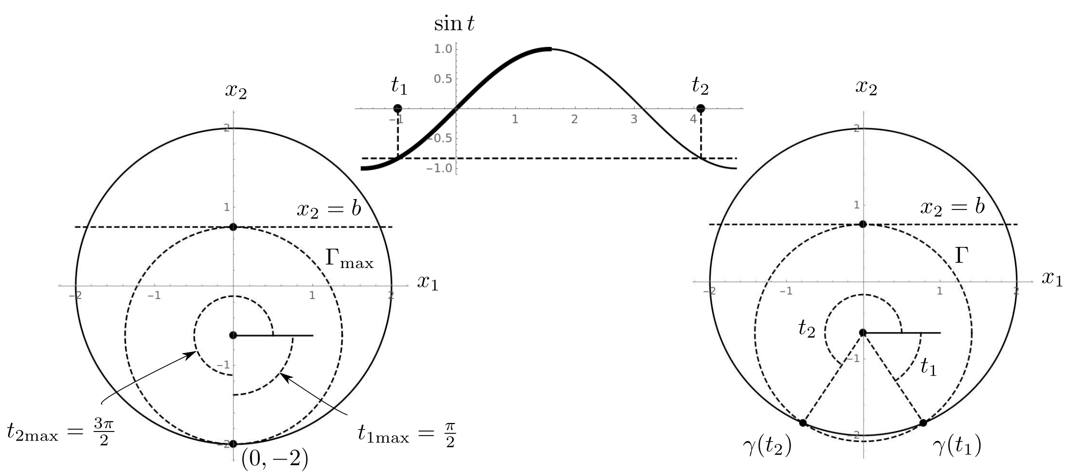

Figure D.6: Limiting configuration with $\theta = 0$. For $0 < k < k_{\max}$ we anticipate the intersection of Γ with $\partial B_2(\mathbf{0})$ consists of exactly two points.

$$-\frac{c_0}{c_1} = \frac{2(1 - bk) - (4 - b^2)k^2}{2(1 - bk)} = 1 - \frac{(4 - b^2)k^2}{2(1 - bk)} \tag{D.6}$$

for $0 \le k \le k_{\max} = 2/(2 + b)$. We observe first that $c_1 = 2(1 - bk)$ when $\theta = 0$ satisfies $c_1 > 0$. In fact, $0 < bk < 1$ when

$$k < \frac{1}{b},$$

but the condition $k < k_{\max} = 2/(2+b)$ implies $k < 1/b$ as long as

$$\frac{2}{2+b} < \frac{1}{b} \qquad \text{or} \qquad b < 2.$$

Thus, the value $-c_0/c_1$ is always well-defined and finite for $0 \le k \le k_{\max} = 2/(2+b)$. In fact, it is relatively straightforward to see that the unique value of k for which $-c_0/c_1 = 1$ in this interval is $k = 0$, and the unique value of k for which $-c_0/c_1) = -1$ is $k_{\max} = 2/(2+b)$. Here are the details: If $-c_0/c_1 = 1$, then

$$\frac{(4-b^2)k^2}{2(1-bk)} = 0,$$

but we have already shown $1 - bk > 0$ for $b < 2$. Clearly $b < 2$ also implies $4 - b^2 > 0$ as well, so the only possibility is $k = 0$. If $-c_0/c_1 = -1$, then we must have

$$\frac{(4-b^2)k^2}{2(1-bk)} = 2,$$

or

$$(4-b^2)\left(k + \frac{2b}{4-b^2}\right)^2 = \frac{16}{4-b^2}.$$

That is,

$$k = \pm\frac{4}{4-b^2} - \frac{2b}{4-b^2} = 2\frac{\pm 2 - b}{4-b^2}.$$

The choice -2 for "± 2" gives the negative root $\bar{k}_{\max} = -2/(2-b) < 0$ outside the interval $0 \le k \le k_{\max}$, and the other root is

$$k = \frac{2}{2+b} = k_{\max}.$$

Finally we claim the quantity

$$\frac{k^2}{1-bk}$$

increases from 0 to $4/(4-b^2)$ on $0 \le k \le k_{\max} = 2/(2+b)$ so that $-c_0/c_1$ decreases from 1 to -1 on the same interval. In fact,

$$\frac{d}{dk}\left(\frac{k^2}{1-bk}\right) = \frac{(1-bk)2k + bk^2}{(1-bk)^2} = k\,\frac{2-bk}{(1-bk)^2} \ge 0$$

with equality only if $k = 0$ since, as noted above $1 - bk > 0$ and also $2 - bk > 1 - bk$.

Thus, we understand exactly how the unique value of $\sin t$ determined by equation (D.4) changes as k decreases from $k_{\max} = 2/(2+b)$. We can also take

$$t_1 = t_1(b,k) = \sin^{-1}\left(-\frac{c_0}{c_1}\right) = \sin^{-1}\left(1 - \frac{(4-b^2)k^2}{2(1-bk)}\right) \tag{D.7}$$

and

$$t_2 = \pi - t_1$$

where $\sin^{-1}$ is the principal branch of arcsine giving values $-\pi/2 \le t_1 \le \pi/2$. In this way, we obtain distinct angles $t_1 \le t_2$ with strict inequality except for $k = 0$ and for which

$$\cos t_2 = -\sqrt{1 - \sin^2 t_2} = -\sqrt{1 - \sin^2 t_1} = -\cos t_1$$

with the symmetric points of intersection $\gamma(t_1)$ and $\gamma(t_2)$ indicated in Figure D.6 are parameterized explicitly as functions of b and k alone in the special case $\theta = 0$.

In order to complete the discussion of this special case, we can look among these points of intersection $\gamma(t_1)$ and $\gamma(t_2)$ with $|\gamma(t_j)| = 2$, $j = 1, 2$ for the two that are diametrically opposed. This condition should be captured in general by the condition that $\gamma(t_1)$ and $\gamma(t_2)$ are in opposite directions, that is $\gamma(t_2) = -\gamma(t_1)$. This can be phrased most precisely in terms of a system of equations

$$\begin{cases} 2\sin\theta + \cos t_1 + \cos t_2 = 0 \\ 2\cos\theta - \sin t_1 - \sin t_2 = 2bk. \end{cases} \tag{D.8}$$

Specializing this to the case $\theta = 0$, we find the first equation is always saisfied (for our choice of t_1 and t_2 in particular) and the second equation becomes

$$\sin t_1 + \sin(\pi - t_1) = 2\sin t_1 = 2(1 - bk).$$

That is,

$$\frac{2(1-bk) - (4-b^2)k^2}{2(1-bk)} = 1 - bk,$$

or

$$2 - 2bk - (4-b^2)k^2 = 2(b^2k^2 - 2bk + 1),$$

or

$$(4+b^2)k^2 = 2bk$$

according to which the desired value of $k = k_g(b) > 0$ when $\theta = 0$ must be

$$k_g(b) = \frac{2b}{4+b^2}. \tag{D.9}$$

This value appears to be the correct one as illustrated in Figure D.7.

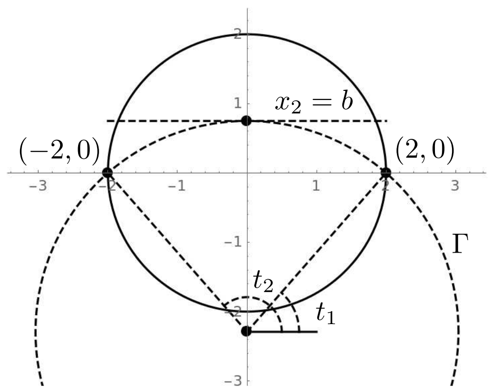

Figure D.7: Limiting configuration with $\theta = 0$. For $0 < k < k_{\max}$ the circle Γ intersects $\partial B_2(\mathbf{0})$ in exactly two points $\gamma(t_1)$ and $\gamma(t_2)$, and for exactly one value of k with $0 < k < k_{\max} = 2/(2+b)$, these two points are diametrically opposed on $\partial B_2(\mathbf{0})$. The value of k corresponding to the illustration is $k_g(b) = 2b/(4+b^2)$.

Let me see if I can gather together all the formulas in this case. With $\theta = 0$, the only parameter is b. We can take $0 < b < 1$ if we wish to remain in $B_1(\mathbf{0})$ which I have erased in Figures D.6, and D.7, or we can take $0 < b < 2$ more generally. The key value is

$$t_1 = t_1(b,k) = \sin^{-1}\left(1 - \frac{(4-b^2)k^2}{2(1-bk)}\right)$$

according to which

$$\begin{aligned}
\sin t_1 &= 1 - \frac{(4-b^2)k^2}{2(1-bk)} \\
\cos t_1 &= k\frac{\sqrt{(4-b^2)[(4-b^2)k^2 + 4(1-bk)]}}{2(1-bk)} \\
\sin t_2 &= \sin t_1 \\
\cos t_2 &= -\cos t_1
\end{aligned}$$

for $0 < k < k_* = 2/(2+b)$. With these values on the same interval

$$\gamma(t_1) = \left(\frac{1}{k}\sin\theta + \frac{\sqrt{(4-b^2)[(4-b^2)k^2+4(1-bk)]}}{2(1-bk)},\right.$$
$$\left. b - \frac{1}{k}\cos\theta + \frac{1}{k} - \frac{(4-b^2)k}{2(1-bk)}\right)$$
$$= \left(\frac{\sqrt{(4-b^2)[(4-b^2)k^2+4(1-bk)]}}{2(1-bk)}, b - \frac{(4-b^2)k}{2(1-bk)}\right)$$
$$= \frac{1}{2(1-bk)}\left(\sqrt{(4-b^2)[(4-b^2)k^2+4(1-bk)]}, 2b-(4+b^2)k\right)$$
$$\gamma(t_2) = \left(\frac{1}{k}\sin\theta - \frac{\sqrt{(4-b^2)[(4-b^2)k^2+4(1-bk)]}}{2(1-bk)},\right.$$
$$\left. b - \frac{1}{k}\cos\theta + \frac{1}{k} - \frac{(4-b^2)k}{2(1-bk)}\right)$$
$$= -\left(\frac{\sqrt{(4-b^2)[(4-b^2)k^2+4(1-bk)]}}{2(1-bk)}, \frac{(4-b^2)k}{2(1-bk)} - b\right)$$
$$= -\frac{1}{2(1-bk)}\left(\sqrt{(4-b^2)[(4-b^2)k^2+4(1-bk)]}, (4+b^2)k-2b\right).$$

These points are diametrically opposed when

$$k = k_g(b) = \frac{2b}{4+b^2}.$$

This of course implies $\gamma(t_1) = (2,0)$ and $\gamma(t_2) = (-2,0)$. One also sees the key value $\sin t_1$ becomes

$$t_1 = \sin^{-1}\left(\frac{4-b^2}{4+b^2}\right)$$

with $0 < t_1 < \pi/2$. This can be seen directly from Figure D.7 by taking $k = k_g(b) = 2b/(4+b^2)$ so the radius of the circle Γ is $r = r_g(b) = (4+b^2)/(2b)$. Then the right triangle through the center $(0, b-r)$, $(2,0)$ and the origin has angle t_1 at $(2,0)$ with opposite side length

$$r - b = \frac{4+b^2}{2b} - b = \frac{4-b^2}{2b}$$

and the sine of the angle is $(r-b)/r$.

On the one hand, all of this is not so remarkable because we already know this geodesic very well. On the other hand, the approach sets up a framework in which there is some chance to find the other geodesics passing through $(0,b)$.

Some aspects of what we have done for $\theta = 0$ can be generalized. Before considering those generalizations, I will attempt to calculate the value $k = k_g(b)$ obtained above in a nominally different way based on an alternative characterization of the condition $\gamma(t_2) = -\gamma(t_1)$ according to which the intersection points $\gamma(t_1)$ and $\gamma(t_2)$ are diametrically opposite. More properly, a condition generalizing the condition is that $\gamma(t_1)$ and $\gamma(t_2)$ are parallel. This will generally be satisfied for $k = k_{\max}$ and $k = k_g(b)$ (at the intersection). The advantage is that the system (D.8) can be replaced by a single scalar equation, namely

$$|(\gamma(t_1), 0) \times (\gamma(t_2), 0)| = 0$$

where we are using the standard cross product of vectors in $\mathbb{R}^3$. Writing this product in column notation we have

$$\left[\begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} + \frac{1}{k}\begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix} + \frac{1}{k}\begin{pmatrix} \cos t_1 \\ \sin t_1 \\ 0 \end{pmatrix}\right] \times \left[\begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} + \frac{1}{k}\begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix} + \frac{1}{k}\begin{pmatrix} \cos t_2 \\ \sin t_2 \\ 0 \end{pmatrix}\right].$$

Notice that only the third component of this cross product will be zero and takes the value

$$\begin{aligned} &\frac{1}{k}(-b\cos t_2) + \frac{1}{k^2}(\sin\theta\sin t_2 + \cos\theta\cos t_2) \\ &\qquad + \frac{1}{k}(b\cos t_t) + \frac{1}{k^2}(-\cos\theta\cos t_1 - \sin\theta\sin t_1\cos t_1\sin t_2 - \sin t_1\cos t_2) \\ &= \frac{1}{k}\left(b(\cos t_1 - \cos t_2) + \frac{1}{k}[\cos(\theta - t_2) - \cos(\theta - t_1) + \sin(t_2 - t_1)]\right). \end{aligned}$$

Thus, an equation for the vanishing of this quantity is

$$\cos(\theta - t_2) - \cos(\theta - t_1) + \sin(t_2 - t_1) = bk(\cos t_2 - \cos t_1). \qquad \text{(D.10)}$$

For the initial single intersection point at $k = k_{\max}$ we should have $\cos t_{2\max} = \cos t_{1\max}$, $\sin t_{2\max} = \sin t_{1\max}$, and we should always be able to take $t_{1\max}$ with $-\pi/2 < t_{1\max} < \pi/2$ so that (as we shall see below)

$$t_{1\max} = \sin^{-1}\left(-\frac{c_1 c_0}{c_2^2 + c_1^2}\right) \tag{D.11}$$

and we should always be able to take $t_{2\max} = t_{1\max} + 2\pi$. This may appear at odds with the choice $t_{2\max} = \pi - t_{1\max}$ in the case $\theta = 0$ suggested above. In that limiting case however we took $t_{1\max} = -\pi/2$, so $\pi - t_{1\max} = 3\pi/2 = 2\pi + t_{1\max}$.

Taking $t_{2\max} = t_{1\max} + 2\pi$ or more generally $t_2 = t_1 + 2\pi m$ for some $m \in \mathbb{Z}$ the equation (D.10) becomes an identity as expected. Thus, we are most interested in solutions t_1 and t_2 with $t_2 \neq t_1 + 2\pi m$.

In the case $\theta = 0$, the equation (D.10) becomes

$$\cos t_2 - \cos t_1 + \sin(t_2 - t_1) = bk(\cos t_2 - \cos t_1)$$

or

$$(1 - bk)(\cos t_2 - \cos t_1) + \sin(t_2 - t_1) = 0.$$

Using further the special symmetric choice $t_2 = \pi - t_1$ in the case $\theta = 0$ according to which $\cos t_2 = -\cos t_1$ and $\sin t_2 = \sin t_1$, this equation becomes

$$2(1 - bk)\cos t_1 = \sin(2t_1)$$

or

$$1 - bk = \sin t_1 = -\frac{c_0}{c_1} = \frac{2(1 - bk) - (4 - b^2)k^2}{2(1 - bk)} = 1 - \frac{(4 - b^2)k^2}{2(1 - bk)}.$$

where we have invoked also (D.6) and (D.7). Simplifying, this becomes

$$2bk(1 - bk) = k(2b - 2b^2 k) = k(4k - b^2 k)$$

according to which we obtain (D.9) a second time

$$k = k_g(b) = \frac{2b}{4 + b^2}.$$

D.1.4 Aspects of the general case

In general we start with the quadratic equation

$$(c_1^2 + c_2^2)\sin^2 t + 2c_0c_1 \sin t + c_0^2 - c_2^2 = 0 \tag{D.12}$$

from (D.4) where

$$\begin{aligned} c_0 &= (4 - b^2)k^2 + 2bk\cos\theta - 2, \\ c_1 &= 2(\cos\theta - bk), \qquad \text{and} \\ c_2 &= 2\sin\theta. \end{aligned}$$

The solution(s) of the quadratic equation (D.4) can be written as

$$\begin{aligned} \sin t &= \frac{-c_1c_0 \pm \sqrt{c_1^2c_0^2 + (c_2^2 + c_1^2)(c_2^2 - c_0^2)}}{c_2^2 + c_1^2} \\ &= \frac{-c_1c_0 \pm \sqrt{c_2^2(c_2^2 + c_1^2 - c_0^2)}}{c_2^2 + c_1^2} \\ &= \frac{-c_1c_0 \pm c_2\sqrt{c_2^2 + c_1^2 - c_0^2}}{c_2^2 + c_1^2} \end{aligned}$$

where in the last line we have used $c_2 = 2\sin\theta \geq 0$ which certainly holds for $0 \leq \theta < \pi/2$ with equality only if $\theta = 0$, but may also be extendeded to $0 \leq \theta \leq \pi$.

In order to obtain real solutions we need $c_2^2 + c_1^2 - c_0^2 \geq 0$.

Lemma D.1. The quantity $c_2^2 + c_1^2 - c_0^2$ is nonnegative precisely for

$$0 < k \leq k_{\max} = \frac{2(2 - b\cos\theta)}{4 - b^2}$$

with equality precisely for $k = k_{\max}$. In particular, there are no real roots for $k > k_{\max}$.

This should be expected in general for some positive $k_{\max}$, and we can see the expression above simplifies to the previously obtained expression $k_{\max} = 2/(2 + b)$ when $\theta = 0$.

Proof of Lemma D.1:

$$\begin{aligned}
c_2^2 + c_1^2 - c_0^2 &= 4\sin^2\theta + 4\cos^2\theta - 8bk\cos\theta + 4b^2k^2 \\
&\quad - \big[(4-b^2)^2k^4 + 4bk\cos\theta(4-b^2)k^2 - 4(4-b^2)k^2 \\
&\qquad + 4b^2k^2\cos^2\theta - 8bk\cos\theta + 4\big] \\
&= -(4-b^2)^2k^4 - 4b\cos\theta(4-b^2)k^3 + [4b^2 + 4(4-b^2) - 4b^2\cos^2\theta]k^2 \\
&= -k^2[(4-b^2)^2k^2 + 4b(4-b^2)\cos\theta\ k - 4(4-b^2\cos^2\theta)] \\
&= -(4-b^2)^2k^2\left(k^2 + \frac{4b\cos\theta}{4-b^2}\ k - 4\ \frac{4-b^2\cos^2\theta}{(4-b^2)^2}\right) \\
&= -(4-b^2)^2k^2\left[\left(k + \frac{2b\cos\theta}{4-b^2}\right)^2 - \frac{4b^2\cos^2\theta}{(4-b^2)^2} - 4\ \frac{4-b^2\cos^2\theta}{(4-b^2)^2}\right] \\
&= -(4-b^2)^2k^2\left[\left(k + \frac{2b\cos\theta}{4-b^2}\right)^2 - \frac{16}{(4-b^2)^2}\right].
\end{aligned}$$

For $k > 0$ this quantity vanishes only for

$$k_{\max} = \frac{2(2-b\cos\theta)}{4-b^2}$$

which is the value given in the statement of the lemma. For $0 < k < k_{\max}$ we see $c_2^2 + c_1^2 - c_0^2 > 0$, and for $k > k_{\max}$ we have $c_2^2 + c_1^2 - c_0^2 < 0$. □

The value of $k_{\max}$ was obtained directly in the proof above, but this value may also be obtained from the geometry of Figure D.5 adapted to the general case. The appropriate figure is indicated in Figure D.8. The center of Γ is $(0, b) + r(\sin\theta, -\cos\theta)$ where

$$r = \frac{1}{k} = \frac{1}{k_{\max}}.$$

Thus focusing on the segment connecting the origin to the point of tangency we should have

$$2 - \left|(0,b) + \frac{1}{k}(\sin\theta, -\cos\theta)\right| = \frac{1}{k}$$

or

$$\left|(0,b) + \frac{1}{k}(\sin\theta, -\cos\theta)\right| = 2 - \frac{1}{k}.$$

This implies

$$b^2 - \frac{2b\cos\theta}{k} = 4 - \frac{4}{k}$$

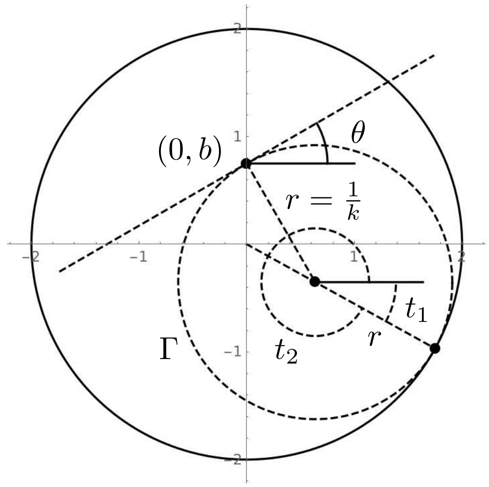

Figure D.8: Largest circle Γ tangent to the line through $(0, b)$ with inclination θ and inside $\overline{B_1(\mathbf{0})}$.

or

$$\frac{1}{k_{\max}} = \frac{4 - b^2}{2(2 - b\cos\theta)}$$

which gives a general replacement for (D.5). As expected this also turns out to be the value so that the quadratic equation for $\sin t$ has two real roots precisely when $0 < k < k_{\max}$ as described by Lemma D.1. We can use this formula for

$$k_{\max} = \frac{2(2 - b\cos\theta)}{4 - b^2}$$

to check the value obtained in other special configurations just as we have in the case $\theta = 0$. Eventually, we seek to find formulas for t_1 and t_2 as well as use the conditions that the intersection points $\gamma(t_1)$ and $\gamma(t_2)$ are diametrically opposite on $\partial B_2(\mathbf{0})$ to find the formula for the geodesic.

Nominally, $k_{\max} = k_{\max}(b, \theta)$. Howevever, having obtained this value, we can fix b with $0 < b < 1$ as originally suggested (or more generally with $0 < b < 2$) and consider $k_{\max} = k_{\max}(b)$. This gives a reasonable framework for the analysis of the general case as follows: The function $k_{\max} : [0, \pi/2] \to \mathbb{R}$ (or more generally $k_{\max} : [0, \pi] \to \mathbb{R}$) by

$$k_{\max}(\theta) = \frac{2(2 - b\cos\theta)}{4 - b^2}$$

is increasing and positive with $k_{\max}(0) = 2/(2+b)$ and $k_{\max}(\pi/2) = 4/(4-b^2)$ as indicated in Figure D.9. (The graph of $k_{\max}$ is fundamentally determined by the cosine function.) Thus, for each fixed b we start with $k = k_{\max}(\theta)$ (for

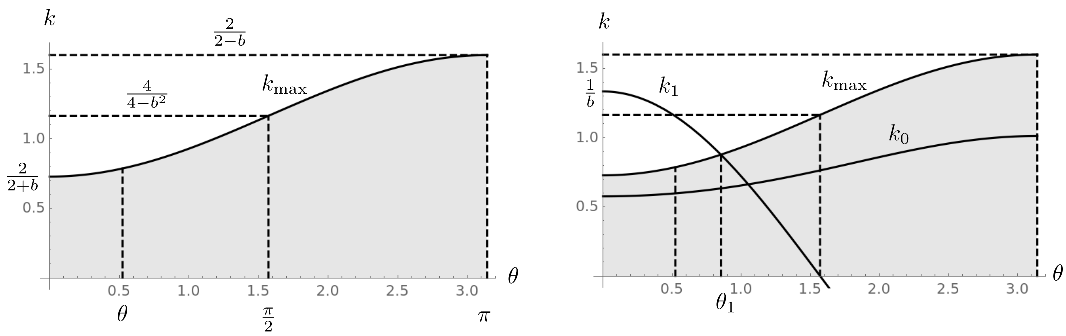

Figure D.9: Region in θ, k-parameter plane corresponding to intersections of the circle Γ with $\partial B_2(\mathbf{0})$. (This particular plot is for $b = 3/4$.)

θ also fixed) and consider values of k with $0 < k < k_{\max}$ decreasing from $k_{\max}$. This corresponds to considering a vertical segment $\{(\theta, k) : 0 < k \le k_{\max}\}$ in the shaded region in Figure D.9. For each such k we wish to solve (D.12) for two values t_1 and t_2 so that $\gamma(t_1)$ and $\gamma(t_2)$ are the intersection points of Γ with $\partial B_2(\mathbf{0})$. We then seek a unique value of k for which these points are dimatrically opposite on $\partial B_2(\mathbf{0})$ thus determining the entire configuration and a particular desired geodesic. The values

$$\sin t = \frac{-c_1c_0 \pm \sqrt{c_1^2c_0^2 + (c_2^2 + c_1^2)(c_2^2 - c_0^2)}}{c_2^2 + c_1^2} = \frac{-c_1c_0 \pm c_2\sqrt{c_2^2 + c_1^2 - c_0^2}}{c_2^2 + c_1^2}$$

must be examined carefully. We have examined the quantity under the square root in the proof of Lemma D.1 to determine the value $k_{\max}$. As a result, we know $k_{\max}$ is the unique value for which there is precisely one possibility for $\sin t$. For all other values of k with $0 < k < k_{\max}$, there will be two roots of the quadratic polynomial for $\sin t$, and we will need to determine what to do with them.

Additional information may be obtained by considering each of the quantities c_1c_0, $c_2^2 - c_0^2$, and the entire expression

$$\sin t = \frac{-c_1c_0 \pm c_2\sqrt{c_2^2 + c_1^2 - c_0^2}}{c_2^2 + c_1^2}.$$

The denominator

$$c_2^2 + c_1^2 = 4(b^2k^2 - 2b\cos\theta\ k + 1) = 4[(1-bk)^2 + 2b(1-\cos\theta)k]$$

is always positive. We can also approach the calculation in the proof of Lemma D.1 somewhat differently to obtain

$$\begin{aligned} c_2^2 + c_1^2 - c_0^2 &= -k^2[(4-b^2)^2k^2 + 4b(4-b^2)\cos\theta\ k - 4(4-b^2\cos^2\theta)] \\ &= -k^2[(4-b^2)^2k^2 + 4b(4-b^2)\cos\theta\ k - 4(2-b\cos\theta)(2+b\cos\theta)] \\ &= -k^2[(4-b^2)k - 2(2-b\cos\theta)][(4-b^2)k + 2(2+b\cos\theta)] \\ &= -(4-b^2)^2k^2(k-k_{\max})(k-\bar{k}_{\max}) \end{aligned}$$

where

$$\bar{k}_{\max} = -\frac{2(2+b\cos\theta)}{4-b^2} < 0 \qquad \text{for} \qquad 0 < b < 2.$$

I will attempt to use the "bar" notation, as in $\bar{k}_{\max}$ somewhat consistently below to denote negative roots like $\bar{k}_{\max}$ of various polynomials in k. These "bar" roots are of secondary interest, but the notation is introduced primarily to shorten the factored form of the polynomial. Notice a factor $k - \bar{k}_{\max}$ is always positive in our region of interest where $k > 0$.

For every $k < k_{\max}$ as mentioned above, the roots in the polynomial for $\sin t$ are distinct. For $k = k_{\max}$ there is one root

$$\sin t_{\max} = -\frac{c_1c_0}{c_2^2 + c_1^2}.$$

We wish to consider next the quantity c_1c_0. Initially we have

$$c_1c_0 = -2(bk - \cos\theta)[(4-b^2)k^2 + 2b\cos\theta\ k - 2].$$

This value is zero when

$$k_1 = k_1(\theta) = \frac{\cos\theta}{b}$$

and for

$$k_0 = k_0(\theta) = \frac{-b\cos\theta + \sqrt{b^2\cos^2\theta + 2(4-b^2)}}{4-b^2}.$$

Notice that k_0 is always real and positive for $0 < b < 2$.

In particular,

$$\begin{aligned}(4-b^2)k^2 + 2b\cos\theta\ k - 2 \\ &= (4-b^2)(k-k_0)\left(k + \frac{b\cos\theta + \sqrt{b^2\cos^2\theta + 2(4-b^2)}}{4-b^2}\right)\end{aligned}$$

so that

$$\begin{aligned}c_1 c_0 &= -2(bk - \cos\theta)[(4-b^2)k^2 + 2b\cos\theta\ k - 2] \\ &= -2b(k-k_1)(k-k_0) \\ &\qquad \left[(4-b^2)k + b\cos\theta + \sqrt{b^2\cos^2\theta + 2(4-b^2)}\right]\end{aligned}$$

(at least nominally) changes signs when $k = k_j$ for $j = 0, 1$.

It can be shown that

$$k_0(0) = \frac{-b + \sqrt{8-b^2}}{4-b^2} < k_{\max}(0) = \frac{2}{2+b} < k_1(0) = \frac{1}{b}$$

for every b with $0 < b < 2$. Also, k_1 is a positive multiple of $\cos\theta$ and is decreasing for $0 < \theta < \pi/2$ with $k_1(\pi/2) = 0$ as indicated on the right in Figure D.9. At least in some cases (for example when $b = 3/4$ as shown on the right in Figure D.9) we also have

$$k_0(\theta) < k_{\max}(\theta) \qquad \text{for} \qquad 0 \le \theta \le \pi.$$

Perhaps this can be shown in general. In any case, we can conclude that for θ small enough the common value $\sin t_{\max}$ will have the same sign as $-c_1 c_0$ at $k = k_{\max}$ and thus will be negative like the value of $\sin t_{\max}$ when $\theta = 0$ considered in the special configuration above.

Denoting the negative root of $c_0 = (4-b^2)k^2 + 2b\cos\theta\ k - 2$ by

$$\bar{k}_0 = -\frac{b\cos\theta + \sqrt{b^2\cos^2\theta + 2(4-b^2)}}{4-b^2}$$

we can also write

$$c_0 = (4-b^2)(k-k_0)(k-\bar{k}_0)$$

and

$$c_1 = -2b(k-k_1)$$

so that

$$-c_1c_0 = 2b(4-b^2)(k-k_1)(k-k_0)(k-\bar{k}_0).$$

Finally we consider $c_2^2 - c_0^2 = (c_2-c_0)(c_2+c_0)$:

$$\begin{aligned} c_2 + c_0 &= \sin\theta + (4-b^2)k^2 + 2b\cos\theta\ k - 2 \\ &= (4-b^2)(k-k_2)(k-\bar{k}_2) \end{aligned}$$

where

$$k_2 = \frac{-b\cos\theta + \sqrt{b^2\cos^2\theta + 2(4-b^2)(1-\sin\theta)}}{4-b^2} \geq 0$$

and $\bar{k}_2 \leq 0$.

$$\begin{aligned} c_2 - c_0 &= \sin\theta - (4-b^2)k^2 - 2b\cos\theta\ k + 2 \\ &= -(4-b^2)(k-k_3)(k-\bar{k}_3) \end{aligned}$$

where

$$k_3 = \frac{-b\cos\theta + \sqrt{b^2\cos^2\theta + 2(4-b^2)(1+\sin\theta)}}{4-b^2} > 0$$

and $\bar{k}_3 < 0$. Thus,

$$c_2^2 - c_0^2 = -(4-b^2)^2(k-k_2)(k-\bar{k}_2)(k-k_3)(k-\bar{k}_3)$$

and the important sign of $c_2^2 - c_0^2$ which determines which of the two terms in the numerator in

$$\sin t = \frac{-c_1c_0 \pm \sqrt{c_1^2c_0^2 + (c_2^2+c_1^2)(c_2^2-c_0^2)}}{c_2^2+c_1^2}$$

is dominant is determined by the relation of the curvature k to k_2 and k_3.

Assuming $k_{\max}$ remains greater than k_0, as seems to be the case for $b = 3/4$, there is a second special configuration arising when k_1 and $k_{\max}$ have the same value at a unique angle $\theta_1 = \theta_1(b)$ with $0 < \theta_1 < \pi/2$. In fact, equating k_1 and $k_{\max}$ we find

$$\theta_1 = \cos^{-1}\left(\frac{4b}{4+b^2}\right).$$

When $\theta = \theta_1$ the initial value of $\sin t_{\max}$ is zero corresponding to a point of tangency for the circle $\Gamma_{\max}$ of radius $r_{\min} = 1/k_{\max}$ with $\partial B_2(\mathbf{0})$ at the point $(2,0)$. This is our second special configuration.

D.1.5 $\theta = \theta_1$; tangency at $(2, 0)$.

Recall that on the right of FigureFigure D.9) the curves $k = k_1$ and $k = k_0$ represent values for which the first term $-c_1c_0$ in the numerator of

$$\sin t = \frac{-c_1c_0 \pm c_2\sqrt{c_2^2 + c_1^2 - c_0^2}}{c_2^2 + c_1^2}.$$

changes sign. The curve $k = k_{\max}$ represents values where the quantity under the square root in the numerator vanishes and for which there is exactly one possible value of $\sin t$. The special configuration considered here may be viewed as that in which there is a unique value $\sin t = \sin t_{\max}$ and that value is $\sin t_{\max} = 0$. We have seen above that this holds precisely when

$$\theta = \theta_1(b) = \cos^{-1}\left(\frac{4b}{4+b^2}\right) \qquad \text{and} \qquad k = k_1(\theta_1) = k_{\max}(\theta_1) = \frac{4}{4+b^2}.$$

These values can also be derived directly using the elementary geometry of the configuration identified as that in which the unique point of tangency $(0, b) + (1/k)(\sin\theta, -\cos\theta) + (1/k)(\cos t, \sin t)$ coincides with $(2, 0)$ as illustrated in Figure D.10.

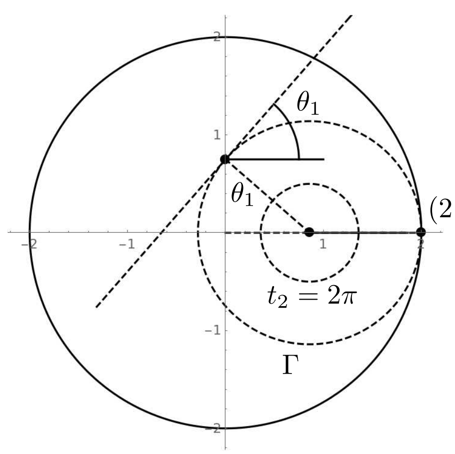

Figure D.10: Special configuration with $t_{1\max} = 0$ and point of initial tangency at $(2, 0)$.

Specifically, the radius $r = 1/k = 1/k_{\max}$ should satisfy

$$\sqrt{\frac{1}{k^2} - b^2} = 2 - \frac{1}{k}$$

so that

$$-b^2 = 4 - \frac{4}{k} \qquad \text{and} \qquad \frac{1}{k_{\max}} = \frac{4+b^2}{4} = 1 + \left(\frac{b}{2}\right)^2.$$

Using this value, the illustration also suggests the particular angle $\theta = \theta_1(b)$ must satisfy

$$\cos\theta_1 = \frac{b}{1/k_{\max}} = \frac{4b}{4+b^2}$$

as obtained in the previous section.

In this instance, as noted above, the unique value of $\sin t_{\max}$ is $\sin t_{\max} = 0$, and it is quite natural to take $t_{1\max} = 0$ and $t_{2\max} = t_{1\max} + 2\pi = 2\pi$. In order to understand the values of $\sin t$ for $k < k_{\max}$, we should look also at

$$\begin{aligned} k_0 &= k_0(\theta_1) \\ &= \frac{-b\cos\theta_1 + \sqrt{b^2\cos^2\theta_1 + 2(4-b^2)}}{4-b^2} \\ &= \frac{-4b^2 + \sqrt{16b^4 + 2(16-b^4)(4+b^2)}}{16-b^4} \\ &= \frac{-4b^2 + \sqrt{2(64+16b^2+4b^4-b^6)}}{16-b^4}. \end{aligned}$$

We claim this quantity satisfies $k_0(\theta_1) < k_{\max}(\theta_1) = k_1(\theta_1)$. That is,

$$\frac{-4b^2 + \sqrt{2(64+16b^2+4b^4-b^6)}}{16-b^4} < \frac{4}{4+b^2}$$

for $0 < b < 2$. We recall that both the quantity on the left an the quantity under the square root are positive. Therefore, the claim holds if

$$\sqrt{2(64+16b^2+4b^4-b^6)} < 16 \qquad \text{or} \qquad 64+16b^2+4b^4-b^6 < 128.$$

Equivalently, it is enough to show/note

$$b^6 - 4b^4 - 16b^2 + 64 = (16-b^4)(4-b^2) > 0 \qquad \text{for} \qquad 0 < b < 2.$$

Thus, in this case we have $0 < k_0 < k_{\max} = k_1$. For $k_0 < k < k_{\max}$ the quantity $-c_1c_0 = -2b(4-b^2)(k-k_1)(k-k_0)(k-\bar{k}_0)$ is negative.

When $\theta = \theta_1$ where

$$\cos\theta_1 = \frac{4b}{4+b^2} \qquad \text{and} \qquad \sin\theta_1 = \sqrt{1 - \frac{16b^2}{(4+b^2)^2}} = \frac{4-b^2}{4+b^2},$$

then

$$\begin{aligned} k_2 &= \frac{-b\cos\theta + \sqrt{b^2\cos^2\theta + 2(4-b^2)(1-\sin\theta)}}{4-b^2} \\ &= \frac{-4b^2 + \sqrt{16b^4 + 2(4-b^2)(4+b^2)(2b^2)}}{16-b^4} \\ &= \frac{-4b^2 + 2b\sqrt{4b^2 + (4-b^2)(4+b^2)}}{16-b^4} \\ &= \frac{-4b^2 + 2b\sqrt{16+4b^2-b^4}}{16-b^4}. \end{aligned}$$

and

$$\begin{aligned} k_3 &= \frac{-b\cos\theta + \sqrt{b^2\cos^2\theta + 2(4-b^2)(1+\sin\theta)}}{4-b^2} \\ &= \frac{-4b^2 + \sqrt{16b^4 + 2(4-b^2)(4+b^2)(8)}}{16-b^4} \\ &= \frac{-4b^2 + 16}{16-b^4} \\ &= \frac{4}{4+b^4}. \end{aligned}$$

From the latter expression, we see $k_3 = k_1 = k_{\max}$ in this case. The value of k_2 also clearly satisfies $0 < k_2 < k_3$. This tells us that for k with

$$\max\{k_0, k_2\} < k < k_{\max} = k_1 = k_3$$

the quantity $c_2^2 - c_0^2 = -4(4-b^2)^2(k-k_2)(k-k_3)(k-\bar{k}_2)(k-\bar{k}_3)$ is positive, so there are two roots for $\sin t$, one positive and one negative. Thus, on this same interval, we should be able to take

$$t_1 = \sin^{-1}\left(\frac{-c_1c_0 + \sqrt{c_1^2c_0^2 + (c_2^2+c_1^2)(c_2^2-c_0^2)}}{c_2^2+c_1^2}\right) \tag{D.13}$$

$$= \sin^{-1}\left(\frac{-c_1c_0 + c_2\sqrt{c_2^2+c_1^2-c_0^2}}{c_2^2+c_1^2}\right) \tag{D.14}$$

taking the positive value and

$$t_2 = 2\pi + \sin^{-1}\left(\frac{-c_1c_0 - \sqrt{c_1^2c_0^2 + (c_2^2 + c_1^2)(c_2^2 - c_0^2)}}{c_2^2 + c_1^2}\right) \tag{D.15}$$

$$= 2\pi + \sin^{-1}\left(\frac{-c_1c_0 - c_2\sqrt{c_2^2 + c_1^2 - c_0^2}}{c_2^2 + c_1^2}\right) \tag{D.16}$$

taking the negative value as illustrated in Figure D.11.

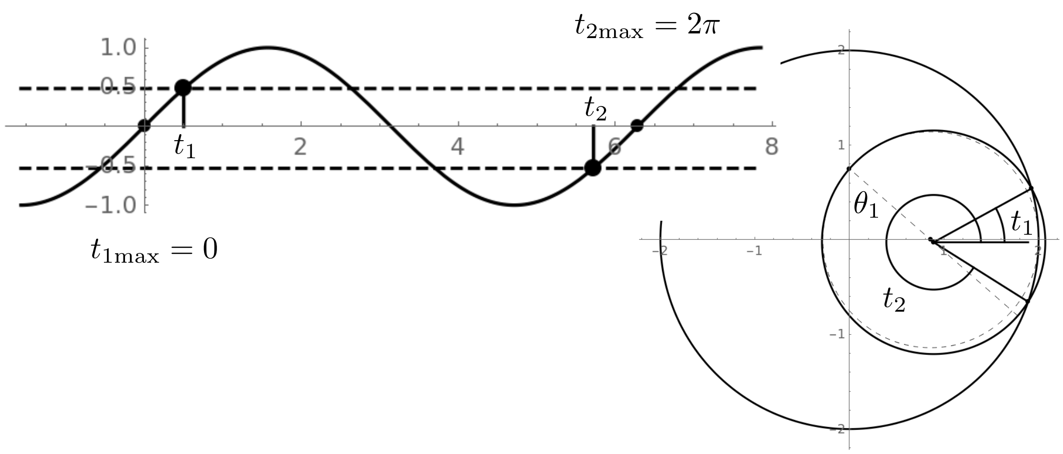

Figure D.11: Special configuration with $t_{1\max} = 0$ and point of initial tangency at $(2, 0)$. The illustration shows the circle and intersection points corresponding to $k = k_{\max} - 0.03$.

Let us go ahead and try to compare k_2 and k_0 which in this case take the forms

$$k_2 = \frac{-4b^2 + 2b\sqrt{16 + 4b^2 - b^4}}{16 - b^4}$$

and

$$k_0 = \frac{-4b^2 + \sqrt{2(64 + 16b^2 + 4b^4 - b^6)}}{16 - b^4}.$$

Notice that in the limit as $b \searrow 0$ there clearly holds $k_2 < k_0$. Let us attempt to show $k_2 < k_0$ when $\theta = \theta_1$ in all cases. This is equivalent to the inequality

$$4b^2(16+4b^2-b^4) < 2(64+16b^2+4b^4-b^6) \qquad \text{or} \qquad b^6-4b^4-16b^2+64 > 0.$$

In fact, $b^6 - 4b^4 - 16b^2 + 64 = (b^2 - 4)^2(b^2 + 4) \geq 0$ with strict inequality for $0 \leq b < 2$. This is precisely the same polynomial we encountered when showing $k_0(\theta_1) < k_{\max}(\theta_1)$ for all b above.

Judging from Figure D.12 which shows numerical plots of k_0, k_1, k_2, k_3, and $k_{\max}$ when $b = 3/4$ one might guess the formulas (D.13-D.16) should hold and give valid intersection points for $k_0 \leq k < k_{\max}$. This does **not** turn out to be the case.

Exercise D.6. Let the real roots of the equation for $\sin t$ be given as

$$\sigma_2 = \sigma_2(k, b, \theta) \leq \sigma_1 = \sigma_1(k, b, \theta) \qquad \text{for} \qquad 0 \leq k \leq k_{\max}(b, \theta)$$

so that

$$\sigma_1 = \frac{-c_1 c_0 + c_2\sqrt{c_2^2 + c_1^2 - c_0^2}}{c_2^2 + c_1^2}$$

and

$$\sigma_2 = \frac{-c_1 c_0 - c_2\sqrt{c_2^2 + c_1^2 - c_0^2}}{c_2^2 + c_1^2}.$$

Show that for $b = 3/4$ and $\theta = \theta_1(3/4) = \cos^{-1}(48/73)$, there exists a value

$$k_{-1} = \frac{4(-33 + \sqrt{9119})}{365}$$

for which the following hold

(a) $k_0 < k_{-1} < k_{\max}$.

(b) $0 < \sigma_1(k) < 1$ for $k_{-1} \leq k < k_{\max}$.

(c) $-1 < \sigma_2(k) < 0$ for $k_{-1} < k < k_{\max}$.

(d) $\sigma_2(k_{-1}) = -1$.

According to the assertions (and consequences) of Exercise D.6, the values k_0, k_1, k_2, k_3, and $k_{\max}$ are not (at least not always) adequate to determine the forms/formulas for the angles t_1 and t_2. I believe in this case with $k = 3/4$ and $\theta = \theta_1(3/4)$ for example, one should use

$$t_2 = \pi - \sin^{-1}(\sigma_2)$$

for $k < k_{-1}$ instead of the angle formula $t_2 = 2\pi + \sin^{-1}(\sigma_2)$ which appears to work for $k_{-1} \leq k \leq k_{\max}$. It would be nice to show the following:

Conjecture 1. For fixed b and θ, the functions $\sigma_1 = \sigma_1(k)$ and $\sigma_2 = \sigma_2(k)$ are smooth for $0 < k < k_{\max}$, say

$$\sigma_j \in C^\infty(0, k_{\max}) \qquad \text{for} \qquad j = 1, 2,$$

with $|\sigma_j(k)| \le 1$ for $0 < k < k_{\max}$ and $j = 1, 2$ and

$$\frac{d^2\sigma_j}{dk^2} \neq 0 \qquad \text{for} \qquad 0 < k < k_{\max} \qquad \text{and} \qquad j = 1, 2.$$

At the moment, I do not know how to prove most of this. I've just checked it numerically in the case $b = 3/4$. In particular, if the conjecture holds and is applied to the assertions of Exercise D.6 implies

$$\frac{d\sigma_2}{dk}(k_{-1}) = 0 \qquad \text{and} \qquad \frac{d^2\sigma_2}{dk^2}(k_{-1}) > 0$$

when $b = 3/4$ and $\theta = \theta_1$.

D.1.6 The general case

Let us now undertake a more systematic treatment of the quadratic equation

$$(c_1^2 + c_2^2)\sin^2 t + 2c_0c_1 \sin t + c_0^2 - c_2^2 = 0 \tag{D.17}$$

featured in (D.4) and (D.12) and having been treated in special cases above where (as above)

$$c_0 = (4 - b^2)k^2 + 2b\cos\theta\ k - 2, \tag{D.18}$$
$$c_1 = 2(\cos\theta - bk), \qquad \text{and} \tag{D.19}$$
$$c_2 = 2\sin\theta. \tag{D.20}$$

and

$$\sin t = \frac{-c_1c_0 \pm \sqrt{c_1^2c_0^2 + (c_2^2 + c_1^2)(c_2^2 - c_0^2)}}{c_2^2 + c_1^2} \tag{D.21}$$
$$= \frac{-c_1c_0 \pm c_2\sqrt{c_2^2 + c_1^2 - c_0^2}}{c_2^2 + c_1^2}. \tag{D.22}$$

As noted above

$$c_0 = (4 - b^2)(k - k_0)(k - \bar{k}_0)$$

where

$$\bar{k}_0 = -\frac{b\cos\theta + \sqrt{b^2\cos^2\theta + 2(4-b^2)}}{4-b^2} < 0$$

and

$$k_0 = \frac{-b\cos\theta + \sqrt{b^2\cos^2\theta + 2(4-b^2)}}{4-b^2} > 0.$$

Also,

$$c_1 = -2b(k-k_1); \qquad k_1 = \frac{\cos\theta}{b}.$$

It follows that

$$-c_1c_0 = -2b(4-b^2)(k-k_1)(k-k_0)(k-\bar{k}_0).$$

Note that this is the first term in the numerator in the expressions (D.21) and (D.22). We recall also that the next important calculation gives an expression for the quantity under the square root in (D.22):

$$c_2^2 + c_1^2 - c_0^2 = -(4-b^2)^2k^2(k-k_{\max})(k-\bar{k}_{\max})$$

where

$$k_{\max} = \frac{2(2-b\cos\theta)}{4-b^2} > 0 \qquad \text{and} \qquad \bar{k}_{\max} = -\frac{2(2+b\cos\theta)}{4-b^2} < 0.$$

The quantity $k_{\max} = k_{\max}(\theta)$ is crucial because it allows one to isolate the parameter space of interest $\{(\theta, k) : 0 < k < k_{\max}\}$ for b fixed with $0 < b < 2$ as indicated in Figure D.9.

Lemma D.2. $0 < k_0 < k_{max}$ for $0 \leq \theta \leq \pi$

Proof: The claimed inequality holds if

$$\sqrt{b^2\cos^2\theta + 2(4-b^2)} < 4 - b\cos\theta.$$

This is equivalent to

$$2(4-b^2) < 16 - 8b\cos\theta \qquad \text{or} \qquad b^2 - 4b\cos\theta + 4 > 0.$$

The last inequality is true because

$$b^2 - 4b\cos\theta + 4 = (b-2)^2 + 4b(1-\cos\theta) > 0. \qquad \square$$

Lemma D.3. For each fixed b with $0 < b < 2$, there is a unique angle

$$\theta_1 = \cos^{-1}\left(\frac{4b}{4+b^2}\right)$$

with $0 < \theta_1 < \pi$ for which $k_1(\theta_1) = k_{\max}(\theta_1)$. Furthermore, $0 < \theta_1 < \pi/2$ and

$$k_1(\theta) > k_{\max}(\theta) \qquad \text{for} \qquad 0 \le \theta < \theta_1$$

and

$$k_1(\theta) < k_{\max}(\theta) \qquad \text{for} \qquad \theta_1 < \theta \le \pi.$$

Proof: The condition $k_1 = k_{\max}$ holds if and only if

$$2b(2 - b\cos\theta) = (4 - b^2)\cos\theta \qquad \text{or} \qquad (4 + b^2)\cos\theta = 4b.$$

Noting that $\cos^{-1}$ is well-defined on $[0, \pi]$ and $4b/(4 + b^2) > 0$, all assertions of the result follow. $\square$

We next consider the product $c_2^2 - c_0^2 = (c_2 + c_0)(c_2 - c_0)$ appearing under the square root in the first expression (D.21) for $\sin t$. We have

$$\begin{aligned}
c_2 + c_0 &= 2\sin\theta + (4 - b^2)k^2 + 2b\cos\theta\ k - 2 \\
&= (4 - b^2)\left(k^2 + \frac{2b\cos\theta}{4 - b^2}\ k\right) - 2(1 - \sin\theta) \\
&= (4 - b^2)\left(k + \frac{b\cos\theta}{4 - b^2}\right)^2 - \frac{b^2\cos^2\theta}{4 - b^2} - 2(1 - \sin\theta) \\
&= (4 - b^2)\left(k + \frac{b\cos\theta}{4 - b^2}\right)^2 - \frac{b^2\cos^2\theta + 2(4 - b^2)(1 - \sin\theta)}{4 - b^2} \\
&= (4 - b^2)\left[\left(k + \frac{b\cos\theta}{4 - b^2}\right)^2 - \frac{b^2\cos^2\theta + 2(4 - b^2)(1 - \sin\theta)}{(4 - b^2)^2}\right] \\
&= (4 - b^2)(k - k_2)(k - \bar{k}_2)
\end{aligned}$$

where

$$k_2 = \frac{-b\cos\theta + \sqrt{b^2\cos^2\theta + 2(4 - b^2)(1 - \sin\theta)}}{4 - b^2} \ge 0$$

and

$$\bar{k}_2 = -\frac{b\cos\theta + \sqrt{b^2\cos^2\theta + 2(4 - b^2)(1 - \sin\theta)}}{4 - b^2} \le 0.$$

Lemma D.4. Most importantly, $0 \le k_2 \le k_0 < k_{\max}$ with

(i) strict inequality $k_2 > 0$ if and only if $\theta \ne \pi/2$, and

(ii) strict inequality $k_2 < k_0$ if and only if $k \ne 0, \pi$.

Second, strict inequality $\bar{k}_2 < 0$ holds if and only if $k \ne \pi/2$.

Proof: It is clear $\bar{k}_2 \le 0 \le k_2$ with equality holding for θ with $0 \le \theta \le \pi$ only if $1 - \sin\theta = 0$. That is, only if $\theta = \pi/2$. However, equality $\bar{k}_2 = 0 = k_2$ does hold when $\theta = \pi/2$.

Comparison of k_2 to k_0 gives

$$\begin{aligned} k_2 &= \frac{-b\cos\theta + \sqrt{b^2\cos^2\theta + 2(4-b^2)(1-\sin\theta)}}{4-b^2} \\ &\le \frac{-b\cos\theta + \sqrt{b^2\cos^2\theta + 2(4-b^2)}}{4-b^2} \\ &= k_0 \end{aligned}$$

with equality $k_2 = k_0$ if and only if $1 - \sin\theta = 1$. On the interval of interest $0 \le \theta \le \pi$, this happens precisely for $\theta = 0, \pi$. $\square$

Similarly, we find

$$\begin{aligned} c_2 - c_0 &= 2\sin\theta - (4-b^2)k^2 - 2b\cos\theta\ k + 2 \\ &= -(4-b^2)\left(k^2 + \frac{2b\cos\theta}{4-b^2}\ k\right) + 2(1+\sin\theta) \\ &= -(4-b^2)\left(k + \frac{b\cos\theta}{4-b^2}\right)^2 + \frac{b^2\cos^2\theta}{4-b^2} + 2(1+\sin\theta) \\ &= -(4-b^2)\left(k + \frac{b\cos\theta}{4-b^2}\right)^2 + \frac{b^2\cos^2\theta + 2(4-b^2)(1+\sin\theta)}{4-b^2} \\ &= -(4-b^2)\left[\left(k + \frac{b\cos\theta}{4-b^2}\right)^2 - \frac{b^2\cos^2\theta + 2(4-b^2)(1+\sin\theta)}{(4-b^2)^2}\right] \\ &= -(4-b^2)(k-k_3)(k-\bar{k}_3) \end{aligned}$$

where

$$k_3 = \frac{-b\cos\theta + \sqrt{b^2\cos^2\theta + 2(4-b^2)(1+\sin\theta)}}{4-b^2} > 0$$

and

$$\bar{k}_2 = -\frac{b\cos\theta + \sqrt{b^2\cos^2\theta + 2(4-b^2)(1+\sin\theta)}}{4-b^2} < 0.$$

Lemma D.5. There holds $0 < k_0 \le k_3 \le k_{\max}$ with

(i) strict inequality $k_0 < k_3$ if and only if $\theta \ne 0, \pi$, and

(ii) strict inequality $k_3 < k_{\max}$ if and only if $\theta \ne \theta_1$.

Proof: Comparison of k_0 to k_3 gives

$$\begin{aligned} k_0 &= \frac{-b\cos\theta + \sqrt{b^2\cos^2\theta + 2(4-b^2)}}{4-b^2} \\ &\le \frac{-b\cos\theta + \sqrt{b^2\cos^2\theta + 2(4-b^2)(1+\sin\theta)}}{4-b^2} \\ &= k_3 \end{aligned}$$

with equality $k_0 = k_3$ if and only if $1 + \sin\theta = 1$. On the interval of interest $0 \le \theta \le \pi$, this happens precisely for $\theta = 0, \pi$.

Comparison of k_3 to $k_{\max}$ gives

$$\begin{aligned} k_3 &= \frac{-b\cos\theta + \sqrt{b^2\cos^2\theta + 2(4-b^2)(1+\sin\theta)}}{4-b^2} \\ &\le \frac{2(2-b\cos\theta)}{4-b^2} \\ &= k_{\max} \end{aligned} \tag{D.23}$$

with the inequality equivalent to

$$\sqrt{b^2\cos^2\theta + 2(4-b^2)(1+\sin\theta)} \le 4 - b\cos\theta$$

or

$$2(4-b^2)(1+\sin\theta) \le 16 - 8b\cos\theta \qquad \text{or} \qquad (4-b^2)\sin\theta + 4b\cos\theta \le 4 + b^2.$$

We note that $(4-b^2)^2 + 16b^2 = 16 + 8b^2 + b^4 = (4+b^2)^2$. Therefore, the last inequality may be written as

$$\cos(\theta - \psi) \le 1$$

where

$$\begin{cases} \cos\psi = \dfrac{4b}{4+b^2} \\[2ex] \sin\psi = \dfrac{4-b^2}{4+b^2}. \end{cases}$$

First of all, this establishes the inequality $k_3 \leq k_{\max}$ as written in (D.23). Furthermore, we can clearly take $\psi = \theta_1$, so equality

$$\cos(\theta - \theta_1) = 1$$

implies $\theta = \theta_1 + 2m\pi$ for some $m \in \mathbb{Z}$. The only possible choice of m for which θ satisfies $0 < \theta < \pi$ is $m = 0$, and we see equality holds if and only if $\theta = \theta_1$. $\square$

With the information above, we can write

$$(c_2^2 + c_1^2)(c_2^2 - c_0^2) = -(4-b^2)^2(c_2^2 + c_0^2)(k - k_3)(k - k_2)(k - \bar{k}_3)(k - \bar{k}_2)$$

and understand some of the basic properties of the roots giving the values of $\sin t$. In particular, we have established the relative ordering of the values of k_0, k_1, k_2, k_3, and $k_{\max}$ indicated in Figure D.12.

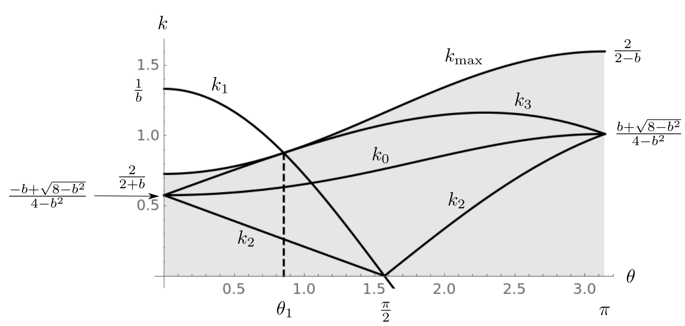

Figure D.12: Typical ordering of important values of k as a function of θ. The case $b = 3/4$ is shown, but the arrangement/ordering is typical for $0 < b < 2$.

From this perspective the situation in the special case $\theta = \theta_1$ is clearly seen to be singular.