On Surfaces of Revolution whose Mean Curvature is Constant

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May 14, 2002

When one seeks a surface of given area enclosing a maximal volume, one finds that the equation this surface must satisfy is the second order partial differential equation

$$(1)\qquad \frac{\partial^2 u}{\partial x^2}\left(1+\left(\frac{\partial u}{\partial y}\right)^2\right) - 2\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\left(1+\left(\frac{\partial u}{\partial x}\right)^2\right) + \underset{\substack{\uparrow \\ \text{(constant mean curvature) } \frac{1}{2a}}}{\frac{1}{a}}\left(1+|\nabla u|^2\right)^{3/2} = 0;$$

$$r(1+q^2) - 2pqs + t(1+p^2) + \frac{1}{a}(1+p^2+q^2)^{3/2} = 0$$

the question is therefore reduced to integrating this equation, which has not yet been done in general. In the particular case where $a = \infty$, equation (1) becomes that of the minimal surface which was integrated by Monge; but the complicated form of the integral he gave, from which one derives no benefit, suggests that if ever one could completely integrate equation (1), the integral would be of no use.

If one adds to equation (1) the condition that the surface be one of revolution, the difficulty entirely disappears, and not only can one find the general equation of the surface, one can also give a very simple geometric definition of its meridian curve. This is what I propose to show in this paper.

I first observe that equation (1) expresses that the sum of the principal curvatures is constant and equals $\frac{1}{a}$. Now one knows that at each point of a surface the sum of the curvatures of two normal sections which are perpendicular to each other, is equal to the sum of principal curvatures. One can therefore define the *mean curvature* of a surface at a point as half the sum of the principal curvatures at this point; in this way the surface represented by equation (1) is that where the mean curvature is constant and equal to $\frac{1}{2a}$, and it's this surface that we propose to find in the particular case where it is one of the revolution.

The radii of principal curvature at a point of a surface of revolution are, as one knows, the radius of curvature of the meridian curve at this point, and the portion of the normal to the surface included between the point and the axis; it follows therefore that if one references the meridian curve of the desired surface to a pair of rectangular coordinate axes, of which one, the x-axis, is the axis of the surface, that curve will be determined by the condition

$$\frac{1}{\rho} + \frac{1}{N} = \frac{1}{a}, \tag{2}$$

where ρ represents its radius of curvature and N the portion of its normal (see text). Let x and y be the coordinates of an arbitrary point on the curve, x' and y' the coordinates of the center of the osculating circle at that point, or equivalently the coordinates of the corresponding point on the evolute, and s' the arclength of the evolute starting from a fixed origin up to the point (x', y'). Now, according to the properties of evolutes,

$$\rho = b - s'$$

b is a constant which depends on the origin of the arc s'. Therefore the portion of the tangent to evolute at the point (x', y') included between the point and the x-axis is equal to $y' \frac{ds'}{dy'}$; adding ρ to y, we get the value of N which is

$$N = y' \frac{ds'}{dy'} + b - s';$$

by means of these values, equation (2) becomes

$$\frac{1}{b - s'} + \frac{1}{y' \frac{ds'}{dy'} + b - s'} = \frac{1}{a},$$

an equation which will help us determine the evolute of the desired curve. One obtains first, by integration,

$$y'^2 = \alpha(b - s')(2a - b + s'),$$

where α is an arbitrary constant. Upon resolving for s' and differentiating, one finds

$$\frac{dy'}{ds'} = -\frac{\alpha\sqrt{a^2 - \frac{y'^2}{\alpha}}}{y'};$$

and as one has

$$\frac{dx'}{ds'} = \sqrt{1 - \left(\frac{dy'}{ds'}\right)^2},$$

one deduces

$$\frac{dx'}{ds'} = \frac{\sqrt{y'^2(1+\alpha) - a^2\alpha^2}}{y'}.$$

Upon inspection of the values of $\frac{dy'}{ds'}$ and $\frac{dx'}{ds'}$, one sees that the constant α can take all the possible positive values, and that, in this case, the values of y' must be included between the limits $\frac{a\alpha}{\sqrt{1+\alpha}}$ and $\alpha\sqrt{a}$; but that if α is negative, it must be included between 0 and -1, and, in this case, y' can take all the values greater than $-\frac{a\alpha}{\sqrt{1+\alpha}}$, which indicates that the evolute has infinite branches.

Let φ be the angle that the tangent to the desired meridian curve at the point (x, y) makes with the x-axis; one will have $\frac{dx'}{ds'} = \sin\varphi$, $\frac{dy'}{ds'} = -\cos\varphi$, $dx' = -dy'\tan\varphi$. It follows (that)

$$y' = \frac{a\alpha}{\sqrt{1+\alpha-\sin^2\varphi}}, \tag{3}$$

and

$$x' = -\int dy'\tan\varphi = -y'\tan\varphi + \int \frac{y'd\varphi}{\cos^2\varphi};$$

(as well) or really, replacing y' by its value,

$$x' = \beta - \frac{a\alpha\tan\varphi}{\sqrt{1+\alpha-\sin^2\varphi}} + \int_0^\varphi \frac{a\alpha\, d\varphi}{\cos^2\varphi\sqrt{1+\alpha-\sin^2\varphi}}, \tag{4}$$

where β is a new arbitrary constant.

These two equations (3) and (4) represent the evolute. One can eliminate the angle φ, and one obtains thereby the equation of the curve; one can also express the integral which enters into the value of x' by means of elliptic functions; but it is preferable, for that which follows, to leave (to) these equations (in) the form which we have come (one comes) to give (of them).

We have determined the values of the coordinates x', y' of an arbitrary point of the evolute as a function of the auxiliary variable φ, it is easy from this to deduce the values of the coordinates x, y, at the corresponding point of the desired curve, as a function of the same variable; as a result, one has

$$y = y' + \rho \frac{dy'}{ds'} ,$$

$$x = x' + \rho \frac{dx'}{ds'} .$$

Therefore one has also, by that which proceeded,

$$\rho = b - s' = a - \sqrt{a^2 - \frac{y'^2}{\alpha}} ;$$

and if one replaces $\frac{dy'}{ds'}$ by $-\cos\varphi$, $\frac{dx'}{ds'}$, by $\sin\varphi$, x' and y' by their values (pulled) from equations (3) and (4), one finds the following equations for representing the desired meridian curve:

$$(5) \qquad \begin{cases} y &= -a\cos\varphi + a\sqrt{1+\alpha-\sin^2\varphi}, \\ x &= \beta + a\sin\varphi - a\tan\varphi\sqrt{1+\alpha-\sin^2\varphi} + \int_0^\varphi \frac{a\alpha d\varphi}{\cos^2\varphi\sqrt{1+\alpha-\sin^2\varphi}} . \end{cases}$$

The question is therefore now completely resolved from the analytic point of view; but we can proceed further along to interpret geometrically the result we have reached.

The integral which enters into the value of x is the same form as the one which is present in the investigation of the arclength of an ellipse or a hyperbola; one is therefore naturally led to the idea that the curve represented by the equations (5) can well be a kind of cycloid generated by a point in the plane of an ellipse or hyperboloid which is rolling upon a straight (line). As it was desired to verify this premise, I have recognized that indeed the meridian curve one finds is that which is generated by the focus of an ellipse or a hyperbola which rolls upon the x-axis. To demonstrate this, it suffices to seek directly the equation of this curve, and from doing this it is seen that it is identical with the equations (5).

Imagine therefore a hyperbola, for example, which rolls upon the x-axis, and consider this curve in an arbitrary position. Let s be the arclength of the curve (hyperbola) included between the vertex and the point of contact with the x-axis, r the radius vector which joins the point of contact to the focus above the origin of the arc s, and φ the angle that the radius vector makes with the normal to the curve at the point of contact: one will have for the coordinates of the focus.

$$x = \beta + s - r\sin\varphi$$
$$y = r\cos\varphi.$$

Now if x_1 and y_1 are the coordinates of the point of contact taken relative to the axes of the hyperbola, and if one represents by a transverse semi-axis, and by e the eccentricity of that curve, one will have

$$y_1 = \frac{a(e^2-1)}{e}\tan\varphi, \qquad x_1 = \frac{a}{e\cos\varphi}\sqrt{e^2-\sin^2\varphi}, \qquad r = ex_1 - a;$$

and in consequence

$$r\cos\varphi = -a\cos\varphi + a\sqrt{e^2-\sin^2\varphi},$$
$$r\sin\varphi = -a\sin\varphi + a\tan\varphi\sqrt{e^2-\sin^2\varphi}.$$

On the other hand, one has

$$ds = \sqrt{dx_1^2+dy_1^2} = \frac{a(e^2-1)d\varphi}{\cos^2\varphi\sqrt{e^2-\sin^2\varphi}},$$

from whence

$$s = \int_0^\varphi \frac{a(e^2-1)d\varphi}{\cos^2\varphi\sqrt{e^2-\sin^2\varphi}}.$$

One has therefore at last the following equations for representing the curve generated by the focus of the hyperbola as it rolls upon the x-axis:

$$y = -a\cos\varphi + a\sqrt{e^2-\sin^2\varphi},$$
$$x = \beta + a\sin\varphi - a\tan\varphi\sqrt{e^2-\sin^2\varphi} + \int_0^\varphi \frac{a(e^2-1)d\varphi}{\cos^2\varphi\sqrt{e^2-\sin^2\varphi}}.$$

These equations evidently coincide with the equations (5) if one puts

$$e = \sqrt{1+\alpha},$$

which can always be done when α is positive. Thus, in this case, the equations (5) represent the curve described by the focus of a hyperbola rolling upon the x-axis; a is the transverse semi-axis of this hyperbola, and $\sqrt{1+\alpha}$ is the eccentricity.

A similar calculation shows that, in the case that α is negative, the equations (5) represent the curve described by the focus of an ellipse rolling upon the x-axis; a is then the semi-major axis of the ellipse, and $\sqrt{1+\alpha}$ is its eccentricity.

One can therefore conclude (from) of everything that preceded, the following theorem:

For finding the meridian curve of a surface of revolution of which the mean curvature is constant and equal to $\frac{1}{2a}$*, it must be done by rolling upon the axis of the surface an ellipse or a hyperbola such that the major axis or the transverse axis is equal to* $2a$*, and the focus describes the desired curve.*

If the mean curvature of a surface of revolution of which one seeks the meridian curve is zero, one has $2a = \infty$; (and) then the meridian curve will be generated by the focus of a parabola rolling upon the axis of the surface. But one knows that this curve is a catenoid: one recovers therefore in this way an already known theorem, that if one rolls a parabola upon a straight line, the focus of this parabola describes a catenoid.