CHAPTER 5

American Options

American options written on an asset S differ from the corresponding European options considered so far in that they can be exercised at any time prior to their expiration. This additional right can become important to the holder of the option. Consider the extreme case where the owner of an American put observes that the underlying asset has become worthless ($S = 0$) at some time $t < T$. Surely the option should be exercised right away to obtain the strike price K at time t rather than later at time T. What is true for $S = 0$ should also be true for S near zero. Hence it is reasonable to assume that immediate exercise of a put is advantageous whenever the asset price falls below a certain threshold $S(t)$. Similarly, if an asset should pay continuously a dividend in proportion to its value then there will be a threshold $S(t)$ beyond which a call should be exercised so that the owner can profit from the dividend income. In both cases $S(t)$ is known as the early exercise boundary.

A derivation of the equations for pricing an American option may be found in (). We shall only summarize the formulation here.

American put: At time t let $S(t)$ denote the position of the early exercise boundary. Then the price $P(S,t)$ of the put can be described by

$$P(S,t) = K - S, \qquad 0 < S < S(t)$$

$$LP(S,t) = 0 \qquad S(t) < S < \infty$$

$$P(S(t),t) = K - S(t)$$

$$\frac{\partial P}{\partial S} S(t),t) = -1$$

$$P(S,t) \to 0 \qquad \text{as } S \to \infty$$

$$P(S,T) = \max\{K - S, 0\}$$

$$S(T) = K.$$

American call: At time t let $S(t)$ denote the early exercise boundary. Then the price $C(S,t)$ of the call is given by

$$C(S,t) = S - K, \qquad S(t) < S < \infty$$

$$LC(S,t) = 0 \qquad 0 < S < S(t)$$

$$
\begin{aligned}
&C(S(t),t) = S(t) - K \\
&\frac{\partial C}{\partial S}\, S(t),t) = 1 \\
&C(0,t) = 0 \\
&C(S,T) = \max\{S-K,0\} \\
&S(T) = K.
\end{aligned}
$$

Here

$$
LV = \frac{1}{2}\,\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\rho)S\,\frac{\partial V}{\partial S} - rV + \frac{\partial V}{\partial t}
$$

where ρ denotes a dividend yield of the underlying asset. It has little influence on the pricing of put but is essential for the pricing of American calls as we shall see a little later.

Both of these problems are known as free boundary problems because the boundary $S(t)$ is one of the unknowns of the problem. We mention that these problems are also called obstacle problems and that they can be formulated as variational inequalities. Moreover, there is an alternate formulation of these equations as a complementarity problem which for a put takes on the form

$$
\begin{gathered}
LP(S,t) \le 0 \\
(LP)(P-(K-S)) = 0, \qquad S > 0 \\
P(S,T) = \max\{K-S,0\}.
\end{gathered}
$$

There is an extensive mathematical theory for such formulations which guarantees the existence of a unique solution. However, the American option problem is nonlinear and cannot be solved in closed form. Approximate or numerical solutions are called for.

Before we begin to adapt the numerical techniques of Chapter 4 to American options let us briefly comment on the role of the dividend yield ρ in the Black-Scholes equation. Consider a put. We may assume that P and its derivatives are smooth and have continuous limits as $S \to S(t)$ and that $S(t)$ is differentiable for $t < T$. Differentiating

$$
P(S(t),t) = K - S(t)
$$

with respect to t we see that

$$
\frac{\partial P}{\partial S}\,(S(t),t)S'(t) + \frac{\partial P}{\partial t}\,(S(t),t) = -S'(t).
$$

$\partial P/\partial S\,(S(t),t) = -1$ implies

$$\frac{\partial P}{\partial t}\,(S(t),t) = 0.$$

The Black-Scholes equation yields

$$\frac{1}{2}\,\sigma^2 S(t)^2 \frac{\partial^2 P}{\partial S^2} = rK - \rho S(t) > 0$$

because $S(t) < K$ and $r > \rho$ for realistic data. Hence $P(S,t)$ is convex upward at $S = S(T)$ which insures that $P(S,t)$ will lift off the intrinsic value $K - S$ as S increases from $S(t)$. If this condition did not hold then the model equations for the put could not be correct because $P(S,t)$ would fall below $\max\{K-S, 0\}$ and thus allow a risk-free profit by buying a put P and the underlying at S and immediately selling it for $K > P + S$.

A similar argument can be applied to the American call. Again it follows from the boundary conditions that

$$\frac{\partial C}{\partial t}\,(S(t),t) = 0$$

and from the Black-Scholes equation that

$$\frac{1}{2}\,\sigma^2 S(t)^2 \frac{\partial^2 C}{\partial S^2} = \rho S(t) - rK.$$

In order for the call not to fall below its intrinsic value $S - K$ near $S(t)$ we need that

$$\frac{\partial^2 C}{\partial S^2}\,(S(t),t) \geq 0.$$

This requires that

$$\rho S(t) \geq rK > 0.$$

This condition cannot be satisfied if $\rho = 0$. In this case there cannot be an early exercise boundary so that an American call without dividend yield will be held to maturity and hence behave like a European call. If $\rho > 0$ then early exercise is consistent with the Black-Scholes equation only if

$$\rho S(t) \geq rK.$$

Again, since we may assume that $\rho < r$ we see that $\lim_{t \to T} S(t) \neq K = S(T)$. In other words, the early exercise boundary is discontinuous at $t = T$. However, for $t < T$ we may again assume that it is smooth.

Let us now turn to the numerical solution of a put. As before, we scale out the strike price K by setting

$$x = \frac{S}{K}\,, \quad u = \frac{P}{K}\,, \quad s(t) = \frac{S(t)}{K}$$

then it is straightforward to verify that u satisfies the problem

$$\frac{1}{2}\,\sigma^2 x^2 u_{xx} + (r-\rho)x u_x - ru + u_t = 0 \tag{5.1}$$

$$u(s(t),t) = 1 - x$$

$$u_x(s(t),t) = -1$$

$$u(X,t) = 0$$

$$u(x,T) = \max\{1-x, 0\}$$

where as before we have approximated the condition at ∞ by an up and out barrier.

It now becomes a matter of preference of whether one would like to use the above formulation or transform to a constant coefficient form of the differential equation. If we use

$$y = \ln x$$

then the put is also described by

$$\frac{1}{2}\,\sigma^2 u_{yy} + (r-\rho-\sigma^2/2)u_y - ru + u_t = 0 \tag{5.2}$$

$$u(s(t),t) = 1 - e^{s(t)}$$

$$u_y(s(t),t) = -e^{s(t)}$$

$$u(\ln X,t) = 0$$

$$u(y,T) = \max\{1-e^y, 0\}.$$

There is little doubt that the latter formulation is easier to solve numerically. This is especially true for the American put because the early exercise boundary $S(t)$ is known to lie above the steady state solution S_∞ which is readily computed as

$$S_\infty = \frac{\gamma K}{1+\gamma}$$

where $\gamma = A + \sqrt{A^2 + 2r\sigma^2}$ with $A = (r - \rho - \sigma^2/2)$. For reasonable data this implies that

$$-\ln 3 < y < \ln 3$$

so that (5.2) needs to be solved only over a small y-interval.

On the other hand, the logarithmic change of variable depends crucially on the specific form of the Black-Scholes equation. For any model involving asset dependent volatilities or interest rates the transformation will no longer work while in the x coordinate system one can allow general coefficients in the Black-Scholes equation. For definiteness we shall deal with (5.1). Analogous algorithms for (5.2) will be easy to deduce from the general case.

The explicit Euler method: The method introduced in the last chapter for European puts is trivial to modify for an American put. We simply compute u_j^{n-1} as before and then make the substitution

$$u_j^{n-1} \leftarrow \max\{1 - x_j, u_j^{n-1}\}$$

whenever the numerical value of u_j^{n-1} should fall below the intrinsic value of the put. Of course, the stability condition discussed in Chapter 4 must be satisfied in order to obtain bounded solutions.

The early exercise boundary does not explicitly appear in this formulation. It lies somewhere between the mesh points x_j and x_{j+1} where $u_j^{n-1} = 1 - x_j$ and $u_{j+1}^{n-1} > 1 - x_{j+1}$. The explicit method will be seen to be closely related to the binomial method. It does a fair job of pricing the option simply and efficiently but is in general not too effective in determining the early exercise boundary.

The implicit Euler method: As for the European option we have to solve the implicit equations () for all mesh points to the right of the early exercise boundary. One may be tempted to apply the highly efficient Thomas algorithm to the matrix problem

$$Au^n = b^n \tag{5.1}$$

as outlined in Chapter 4 for European options, and then "postprocess" the solution by setting

$$u_j^n \leftarrow \max\{1 - x_j, u_j^n\}$$

to make sure that the option value does not fall below its intrinsic value. However, it is known that this approach can fail because the resulting numerical solution does not always satisfy the discrete version of the complementarity form of the problem. A correct, if somewhat more time consuming method, is a modification of the SOR method for the system given in Chapter 4 which is known as the projected SOR method. The only change is that () is replaced by

$$w_j^{k+1} = \max\{1 - x_j, w_j^k + (\tilde{w}_j - w_j^k)\}$$

where w^k is the kth iterate for the solution of (5.1). It is known that this method converges to the correct discrete solution u^n of the American option at time t_n as $k \to \infty$. After the SOR iteration has converged we know that the early exercise boundary is somewhere between the two mesh points where $u_j^n = 1 - x_j$ and $u_{j+1}^n > 1 - x_{j+1}$. Both for the explicit and implicit method the delta and gamma of the option must be determined from the discrete data. One can use difference formulas or one can differentiate smooth interpolants of u to obtain u_x and u_{xx}.

The Riccati transformation method: Finally, let us consider the modification of the sweep method due to the free boundary $s(t)$. We observe that the Riccati transformation and the defining equations for R and w do not depend on the boundary condition at the left hand side. Thus at a given time level the transformation

$$u(x) = R(x)v(x) + w(x)$$

has to hold for all $x < X$. In particular, it has to hold at the early exercise boundary $x = s$ where

$$u(s) = 1 - s \quad \text{and} \quad v(s) = -1.$$

It follows that the early exercise boundary must be a root of

$$\varphi(x) \equiv (1 - x) - R(x)(-1) - w(x) = 0.$$

Hence as the values of $R(x_j)$ and $w(x_j)$ are found with the trapezoidal rule as $j = M, M - 1, M - 2, \ldots$ we evaluate the function $\varphi(x_j)$. If it changes sign between x_k and x_{k+1} for some k we know that s lies between these two points. s may be placed by the zero crossing of the linear interpolant to $\varphi(x)$. For the computation of v over $[s, X]$ with the trapezoidal rule one can set

$$v = -1, \qquad u = 1 - x$$

at the mesh point x_k or x_{k+1} whichever is nearer to s. For $x < s$ we simply set $u_j = 1 - x_j$. (A more sophisticated cubic interpolation and an integration of v over one fractional Δx-step are some modifications of this method if better accuracy in placing the early exercise boundary is desired.) Note that in this approach $v(x)$ is the delta of the option at time t_n and that the right hand side defining v' is the gamma.