MODULE 1

Topics: Vectors space, subspace, span

I. Vector spaces:

General setting: We need

$V =$ a set: its elements will be the vectors x, y, f, u, etc.

$F =$ a scalar field: its elements are the numbers α, β, etc.

$+$ a rule to add elements in V

$\bullet$ a rule to multiply elements in V with numbers in F.

V, F, $+$ and $\bullet$ can be quite general within the abstract framework of vector spaces.

IN THIS COURSE a vector in V is generally

i) an n-tuple of real or complex numbers

$$x = (x_1, \ldots, x_n)$$

or

ii) a function defined on a given set D.

In this case it is common to denote the vector by f.

The scalar field F is generally the set of real or complex numbers.

$+$ is the component wise addition of n-tuples of numbers

$$x + y = (x_1 + y_1, \ldots, x_n + y_n)$$

or the pointwise addition of functions

$$(f + g)(t) = f(t) + g(t)$$

$\bullet$ is the usual multiplication of an n-tuple with a scalar

$$x = (x_1, \ldots, x_n)$$

or the pointwise multiplication of a function

$$(\alpha f)(t) = \alpha f(t).$$

Hence nothing special or unusual is happening in this regard.

Definition: If for any $x, y \in V$ and any $\alpha \in F$

$$x + y \in V$$

$$\alpha x \in V$$

then V is a vector space (over F).

We say that V is closed under vector addition and scalar multiplication.

Examples:

i) All n-tuples of real numbers form the vector space $\mathbb{R}_n$ over the real numbers $\mathbb{R}$.

ii) All n-tuples of complex numbers form the vector space $\mathbb{C}_n$ over the complex numbers $\mathbb{C}$.

iii) All continuous real valued functions on a set D form a vector space over $\mathbb{R}$.

iv) All k-times continuously differentiable functions on a set D form a vector space over $\mathbb{R}$.

Convenient notation:

$C^k(a, b)$ denotes the vector space of k-times continuously differentiable functions defined on the open interval (a, b).

$C^k[a, b]$ denotes the vector space of k-times continuously differentiable functions defined on the closed interval $[a, b]$.

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v) All real valued n-tuples of the form

$$x = (1, x_2, \ldots, x_n)$$

do not form a vector space over $\mathbb{R}$ because $0 \bullet x$ is the zero vector and has a 0 and not a 1 in the first component.

vi) Finally, let V be the set of all $m \times n$ real matrices. Let $+$ denote the usual matrix addition and $\bullet$ the multiplication of a matrix with a scalar, then V is closed under vector addition and scalar multiplication. Hence V is a vector space and the vectors here are the $m \times n$ matrices.

When the vectors, scalars and functions are real valued then V is a real vector space. Most of our applications will involve real vector spaces.

When complex numbers and functions arise then V is called a complex vector space.

Subspaces:

Definition: Let M be a subset of V. If M itself is closed with respect to the vector addition and scalar multiplication defined for V then M is a subspace of V.

Examples (in all cases $F = R$):

i) $M = \{x : x = (x_1, 0, x_3)\}$ is a subspace of $\mathbb{R}_3$

ii) $M = \{f \in C^1[0,1] : f(0) = 0\}$ is a subspace of $C^0[0,1]$.

iii) $M = \{$all functions in $C^0[0,1]$ which you can integrate analytically$\}$ form a subspace of $C^0[0,1]$.

iv) $M = \{$all functions in $C^0[0,1]$ which you cannot integrate analytically$\}$ do not form a subspace of $C^0[0,1]$ because a subspace has to contain 0 (i.e., $f \equiv 0$) and you know how to integrate the zero function.

v) $\mathbb{R}_2$ is not a subspace of $\mathbb{R}_3$ because $\mathbb{R}_2$ is not a subset of $\mathbb{R}_3$. On the other hand, $M = \{x : x = (x_1, x_2, 0)\}$ is a subspace of $\mathbb{R}_3$.

vi) $M = \{$all polynomials of degree $< N\}$ is a subspace of $C^k(-\infty, \infty)$ for any integer $k > 0$.

vii) Let $\{x_1, x_2, x_3, \ldots, x_K\}$ be K given vectors in a vector space V.

Let M be the set of all linear combinations of these vectors, i.e., $m \in M$ if

$$m = \sum_{j=1}^{K} \alpha_j x_j \qquad \text{for } \{\alpha_j\} \subset F.$$

Then M is a subspace of V.

The previous example fits this setting if the vector x_j is identified with the function $f(t) = t^{j-1}$ for $j = 1, \ldots, K$ where $K = N + 1$.

The last example will now be discussed at greater length.

Definition: Let V be a vector space, let $\{x_1, \ldots, x_n\}$ be a set of n vectors in V. Then the span of these vectors is the subspace of all their linear combinations, i.e.,

$$\text{span}\{x_1, \ldots, x_n\} = \left\{ x : x = \sum_{j=1}^{n} \alpha_j x_j \right\} \qquad \text{for } \alpha_j \in F.$$

For example, if $\hat{e}_1 = (1,0,0)$, $\hat{e}_2 = (0,1,0)$ and $\hat{e}_3 = (0,0,1)$ then $\text{span}\{\hat{e}_1, \hat{e}_2, \hat{e}_3\} = \mathbb{R}_3$. If x is a given vector in $\mathbb{R}_3$ then $\text{span}\{x\}$ is the straight line through 0 with direction x. If $x_1 = (1,2,3)$ and $x_2 = (2,-1,4)$, then

$$x = \alpha_1 x_1 + \alpha_2 x_2$$

for arbitrary α_1 and α_2 is just the parametric representation of the plane

$$11x + 2y - 5z = 0$$

so that $\text{span}\{x_1, x_2\}$ is this plane in $\mathbb{R}_3$.

We note that if any x_k is a linear combination of the remaining vectors $\{x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n\}$ then

$$\text{span}\,\{x_i\}_{i=1}^{n} = \text{span}\,\{x_i\}_{i=1,\ i\neq k}^{n}\,.$$

Module 1 - Homework

1) In each case assume that $F = \mathbb{R}$. Prove or disprove that $\mathcal{M}$ is a subspace of V.

   i) $V = \mathbb{R}_n$

$$\mathcal{M} = \left\{ x : \sum_{j=1}^{n} x_j = 0 \right\}$$

   ii) $V = \mathbb{R}_n$

$$\mathcal{M} = \left\{ x : \sum_{j=1}^{k} j x_j = 0 \right\}$$

   for some given $k < n$.

   iii) $V = \mathbb{R}_3$

$$\mathcal{M} = \{ x : x_1 x_2 x_3 = 0 \}$$

   iv) $V = \mathbb{R}_3$

$$\mathcal{M} = \left\{ x : e^{x_1^2 + x_2^2 + x_3^2} = 1 \right\}$$

   v) $V = \mathbb{R}_3$

$$\mathcal{M} = \{ x : |x_1| = |x_2| \} .$$

2) Let $V = C^0[-\pi, \pi]$. Let $\mathcal{M}$ be the subspace given by

   $\mathcal{M} = \operatorname{span}\{1, \cos t, \cos 2t, \ldots, \cos Nt, \sin t, \sin 2t, \ldots, \sin Nt\}$.

   For given $f \in V$ define

$$Pf(t) = \sum_{j=0}^{N} \alpha_j \cos jt + \sum_{j=1}^{N} \beta_j \sin jt$$

   where

$$\alpha_j = \frac{\int_{-\pi}^{\pi} f(t) \cos jt \, dt}{\int_{-\pi}^{\pi} \cos^2 jt \, dt} , \qquad \beta_j = \frac{\int_{-\pi}^{\pi} f(t) \sin jt \, dt}{\int_{-\pi}^{\pi} \sin^2 jt \, dt}$$

   Compute $Pf(t)$ when

   i) $f(t) = t$

   ii) $f(t) = t^2$

   iii) $f(t) = \sin 5t$

   iv) $f(t) = e^t$.