MODULE 10

Topics: Some examples and applications

1) In a small chemical plant three tanks are connected with each other with an inflow and an outflow pipe.

   i) Characterize all admissible flow rates for which the volume in each tank will remain constant.

   ii) Suppose in this closed system the volume of each tank changes at a prescribed rate. Characterize the admissible volume changes.

Answer: Let $c(i, j)$ be the flow rate from tank i to tank j. Then a mass balance requires that

$$c(1,2) + c(1,3) = c(2,1) + c(3,1)$$

$$c(2,1) + c(2,3) = c(2,1) + c(3,2)$$

$$c(3,1) + c(3,2) = c(1.3) + c(2,3).$$

Let $x_1 = c(1,2)$, $x_2 = c(1,3)$, $x_3 = c(2,1)$, $x_4 = c(2,3)$, $x_5 = c(3,1)$ and $x_6 = c(3,2)$ then the mass balance equations can be rewritten as

$$Ax = 0$$

where

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & -1 \end{pmatrix}.$$

If we carry out Gaussian elimination we find that

$$U = \begin{pmatrix} 1 & 1 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $\text{rank}(A) = 2$, $R(A) = \text{span}\{(1,1,0),(1,0,1)\}$ and $\dim \mathcal{N}(A) = 4$. A basis of the null space is found from $Ux = 0$ as

$u_1 = (1,0,1,0,0,0)$ (tank 1 and 2 exchange fluid)

$u_2 = (0,1,0,0,1,0)$ (tank 1 and 3 exchange fluid)

$u_3 = (0,0,0,1,0,1)$ (tank 2 and 3 exchange fluid)

$u_4 = (1,0,0,1,1,0)$ (the three tanks are connected in series).

Any flow schedule in the span$\{u_i\}$ is an admissible flow schedule.

ii) The mass balance equations become

$$Ax = b$$

where b is the prescribed change of fluid in each tank. In order to solve the system we need that

$$b \in \text{span}\{(1,1,0),(1,0,1)\}$$

Hence $b = \alpha(-1,1,1)$ would not be allowed for any $\alpha \neq 0$. Of course, the model breaks down when a tank becomes empty or overflows.

2) Eigenvalues are usually obtainable only through a numerical calculation, but on occasion it is possible to obtain some useful a-priori estimates of what they might be. Suppose that

$$Au = \lambda u.$$

Since u is not the zero vector we can normalize u. We shall write

$$y = \frac{u}{\|u\|_\infty}$$

so that $|y_k| = 1$ for some k and $|y_j| \leq 1$ for all j. If we now look at the kth equation of $Ay = \lambda y$ we obtain

$$(a_{kk} - \lambda)y_k = \sum_{\substack{j=1 \\ j \neq k}}^{n} a_{kj} y_j$$

so that for each eigenvalue there is a k such that

$$|a_{kk} - \lambda| \leq \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}|.$$

Hence the eigenvalues have to lie in a union of disks given by

$$\bigcup_{i=1}^{n} \left\{ z : |a_{ii} - z| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \right\}.$$

For example, suppose that A is strictly diagonally dominant so that

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \qquad \text{for each } i$$

Then none of these circles contains the origin. Hence $\lambda = 0$ cannot be an eigenvalue which implies that

$$Ax = 0$$

cannot have a non-zero solution (which otherwise would be an eigenvector corresponding to $\lambda = 0$). Hence if A is strictly diagonally dominant then A is invertible.

3) Let L denote the linear transformation in $\mathbb{R}_2$ which describes a reflection in $\mathbb{R}_2$ about the line $x_2 = x_1$. Find the matrix of A and its eigenvalues and eigenvectors.

Answer: We know that a linear transformation from $\mathbb{R}_2$ to $\mathbb{R}_2$ has a matrix representation

$$Lx \equiv Ax$$

where the ith column of A is the image of the ith unit vector. It follows in this case that

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

From the geometry we observe that the vector $u_1 = (1, 1)$ stays unchanged and the vector $u_2 = (1, -1)$ goes into $(-1, 1) = -u_2$. Hence without calculation

$$Au_1 = u_1 \quad \text{and} \quad Au_2 = -u_2$$

so that $\lambda = 1$ with eigenvector u_1 and $\lambda = -1$ with eigenvector u_2. It is straightforward to verify these results algebraically.

4) Find the matrix for the orthogonal projection in $\mathbb{E}_3$ onto the plane

$$x_1 + x_2 + x_3 = 0$$

and determine its eigenvalues and eigenvectors geometrically.

Answer: In order to write down the matrix we need to find the images of the three unit vectors $\{\hat{e}_i\}$. We can find these images once we have a basis for the subspace onto which we project. Since the subspace is a plane in $\mathbb{E}_3$ any two linearly independent vectors in this plane will serve as a basis. By inspection we see that

$$u_1 = (1, -1, 0) \quad \text{and} \quad u_2 = (1, 1, -2)$$

form an orthogonal basis for the plane which simplifies the calculation of P. It follows that

$$P\hat{e}_i = \frac{\langle \hat{e}_i, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 + \frac{\langle \hat{e}_i, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2$$

so that

$$A = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}.$$

A quick check shows that at least all images $P\hat{e}_i$ belong to the plane which is necessary but not sufficient for the correctness of the derivation of A. Without calculation we recognize that $Pu_1 = u_1$ and $Pu_2 = u_2$ so that $\lambda = 1$ must be an eigenvalue which occurs twice with corresponding orthogonal eigenvectors. We also note that any vector orthogonal to the plane is mapped to the origin. Hence $P(1, 1, 1) = 0$ or, if you prefer,

$$P(u_1 \times u_2) = A(u_1 \times u_2) = 0$$

so that $\lambda = 0$ is also an eigenvalue with eigenvector $(1, 1, 1)$. Again, these results can be verified algebraically.

5) Suppose we have a rotation in $\mathbb{R}_3$ around the x_1-axis in the clockwise direction (looking along the positive x_1-axis toward the origin) followed by a rotation through $\pi/4$ clockwise around the x_3-axis. Find the matrix for the combined rotation and the axis of rotation.

Answer: Let A be the matrix for the first rotation. Then

$$A\hat{e}_1 = \hat{e}_1$$

$$A\hat{e}_2 = -\hat{e}_3$$

$$A\hat{e}_3 = \hat{e}_2.$$

Hence

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Let B be the matrix for the second rotation, then

$$B\hat{e}_1 = 1/\sqrt{2}\hat{e}_1 - 1/\sqrt{2}\hat{e}_2$$

$$B\hat{e}_2 = 1/\sqrt{2}\hat{e}_1 + 1/\sqrt{2}\hat{e}_2$$

$$B\hat{e}_3 = \hat{e}_3$$

so that

$$B = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The combined rotation is

$$C = BA = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -1 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors of C are obtained from the computer.

In[1] $:= m = \{\{1/\text{Sqrt}[2],0,1/\text{Sqrt}[2]\}, \{-1/\text{Sqrt}[2],0,1/\text{Sqrt}[2]\}, \{0,-1,0\}\}$

Out[1] $= \left\{\left\{\frac{1}{\text{Sqrt}[2]}, 0, \frac{1}{\text{Sqrt}[2]}\right\}, \left\{-\left(\frac{1}{\text{Sqrt}[2]}\right), 0, \frac{1}{\text{Sqrt}[2]}\right\} \{0,-1,0\}\right\}$

In[2] := Eigenvalues[m]

Out[2] $= \left\{1, \frac{-2+\text{Sqrt}[2] - I\ \text{Sqrt}[10 + 4\ \text{Sqrt}[2]]}{4},\right.$

$> \left.\frac{-2+\text{Sqrt}[2] + \text{I Sqrt}[10 + 4\ \text{Sqrt}[2]]}{4}\right\}$

In[3] := Eigenvectors[m]

Out[3] $= \left\{\{1+\text{Sqrt}[2], -1, 1\}, \left\{\right.\right.$

$> \frac{-\text{Sqrt}[2] \text{ - I Sqrt}[10 + 4\ \text{Sqrt}[2]] + \text{I Sqrt}[2\ (10 + 4\ \text{Sqrt}[2])]}{4}$

$> \left.\frac{2-\text{Sqrt}[2] + \text{I Sqrt}[10 + 4\ \text{Sqrt}[2]]}{4}, 1\right\}$

$> \left\{\frac{-\text{Sqrt}[2] + \text{I Sqrt}[10 + 4\ \text{Sqrt}[2]] \text{ - I Sqrt}[2\ (10 + 4\ \text{Sqrt}[2])]}{4},\right.$

$> \frac{2 - \text{Sqrt[2] - I Sqrt[10 + 4 Sqrt[2]]}}{4}, 1\}\}$

The relevant information is the axis of rotation which is the eigenvector

$$u_1 = \left(1 + \sqrt{2}, -1, 1\right)$$

corresponding to the eigenvalue $\lambda = 1$.

6) Find the matrix for the rotation about about an axis of rotation parallel to the vector $\vec{u}_1$ through an angle θ counterclockwise when looking along $\vec{u}_1$ toward $\vec{0}$.

Answer: The rotation is easy to describe in a right handed orthogonal coordinate system $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ of $\mathbb{E}_3$ where $\vec{u}_2$ and $\vec{u}_3$ are orthogonal vectors in the plane perpendicular to $\vec{u}_1$. Let the given vector $\vec{u}_1$ be $\vec{u}_1 = (u_1, u_2, u_3)$. Then a vector perpendicular to $\vec{u}_1$ is the vector $\vec{u}_2 = (u_2, -u_1, 0)$. A right handed coordinate system is obtained if we set $\vec{u}_3 = \vec{u}_1 \times \vec{u}_2 = (u_1u_3, +u_2u_3, -u_1^2 - u_2^2)$. Let us normalize the vectors and choose

$$v_i = \vec{u}_i / \|\vec{u}_i\|_2 \qquad \text{for } i = 1, 2, 3.$$

The set $\{v_i\}$ will play much the same role as the set of unit vectors $\{\hat{e}_i\}$. (We are dropping the arrows indicating vectors because the components will no longer appear explicitly.)

Let R denote the rotation operator. Then

$$Rv_1 = v_1$$

because v_1 is the axis of rotation and does not change.

$$Rv_2 = \alpha_2 v_2 + \beta_2 v_3$$

because the image of v_2 remains in the plane spanned by $\{v_2, v_3\}$. Moreover

$$\langle Rv_2, v_2\rangle = \alpha_2\langle v_2, v_2\rangle + \beta_2\langle v_3, v_2\rangle = \|Rv_2\|\|v_2\|\cos\theta = \cos\theta$$

which together with

$$\langle v_2, v_2\rangle = \langle Rv_2, Rv_2\rangle = \alpha_2^2 + \beta_2^2 = 1$$

determines α_2 and β_2. Similarly we find

$$Rv_3 = \alpha_3 v_2 + \beta_3 v_3.$$

Since $\{v_1, v_2, v_3\}$ forms on orthonormal basis of $\mathbb{E}_3$ there are constants $\{\gamma_{1i}, \gamma_{2i}, \gamma_{3i}\}$ such that

$$\hat{e}_i = \gamma_{1i} v_1 + \gamma_{2i} v_2 + \gamma_{3i} v_3 = (v_1 \ v_2 \ v_3) \begin{pmatrix} \gamma_{1i} \\ \gamma_{2i} \\ \gamma_{3i} \end{pmatrix}$$

It follows that

$$I = (e_1 \ e_2 \ e_3) = (v_1 \ v_2 \ v_3) \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix}$$

so that the last matrix is $(v_1 \ v_2 \ v_3)^{-1}$. Now consider the image of the unit vectors $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$. We have

$$Re_i = \gamma_{1i} Rv_1 + \gamma_{2i} Rv_2 + \gamma_{3i} Rv_3$$

so that

$$\begin{aligned} Re_i &= \gamma_{1i} v_1 + \gamma_{2i}(\alpha_2 v_2 + \beta_2 v_3) + \gamma_{3i}(\alpha_3 v_2 + \beta_3 v_3) \\ &= V \begin{pmatrix} \gamma_{1i} \\ \gamma_{2i}\alpha_2 + \gamma_{3i}\alpha_3 \\ \gamma_{2i}\beta_2 + \gamma_{3i}\beta_3 \end{pmatrix} = V \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha_2 & \alpha_3 \\ 0 & \beta_2 & \beta_3 \end{pmatrix} \begin{pmatrix} \gamma_{1i} \\ \gamma_{2i} \\ \gamma_{3i} \end{pmatrix} \end{aligned}$$

where $V = (v_1 \ v_2 \ v_3)$. Hence the matrix describing the rotation can be written as

$$(Re_1 \ Re_2 \ Re_3) = V \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha_2 & \alpha_3 \\ 0 & \beta_2 & \beta_3 \end{pmatrix} V^{-1}$$

Finally we observe from

$$\langle v_i, v_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

that $(v_1 \ v_2 \ v_3)^T (v_1 \ v_2 \ v_3) = I$ so that $V^{-1} = V^T$. Hence the rotation matrix can be simplified to

$$R = V \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha_2 & \alpha_3 \\ 0 & \beta_2 & \beta_3 \end{pmatrix} V^T.$$

The matrix V^T maps the units vectors $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ onto the basis $\{v_1, v_2, v_3\}$, the next matrix tells us how these basis vectors transform, and the matrix V maps the $\{v_1, v_2, v_3\}$ basis back to the $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ basis.

Module 10 - Homework

1) Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

Prove or disprove: A describes a rotation in $\mathbb{R}_3$.

2) Find the matrix P for the projection in $\mathbb{E}_3$ onto the subspace $M = \text{span}\{(1,1,1),(1,2,1)\}$.

Find the eigenvalues $\{\lambda_i\}$ of A. Determine the dimension and a basis of the null space of $A - \lambda_i I$ for each i. If your basis is not orthogonal find an orthogonal basis of the null spaces.

3) Find the matrix for a rotation about the axis $\text{span}\{(1,1,1)\}$ through $\pi/2$ radians in the counterclockwise direction when looking from $(1,1,1)$ toward $(0,0,0)$.