# MODULE 14

## Topics: Linear inhomogeneous equations

We recall that if $A(t)$ is an $n \times n$ matrix with differentiable entries $a_{ij}(t)$ then

$$A'(t) = \lim_{\Delta t \to 0} \frac{A(t+\Delta t) - A(t)}{\Delta t} = (a'_{ij}(t))$$

and that if $B(t)$ is an $m \times n$ matrix with differentiable entries then

$$(B(t)A(t))' = B'(t)A(t) + B(t)A'(t).$$

Moreover, if $A(t)$ is differentiable and invertible in a neighborhood of a point $t_0$ then it follows from

$$(A(t)A^{-1}(t))' = A'(t)A^{-1}(t) + A(t)(A^{-1}(t))' = I' = 0$$

that

$$A^{-1}(t)' = -A^{-1}(t)A'(t)A^{-1}(t)$$

which is the matrix analog of $(1/f(t))' = f'(t)/f^2(t)$. We shall use these relationships in discussing further the solution of

$$Lu \equiv u' - A(t)u = 0, \qquad \text{or equivalently, of } u'(t) = A(t)u.$$

As we have seen, the null space of $L$ has dimension $n$, and any $n$ functions which solve the equation and which are linearly independent at one point span the null space of $L$. For example, a convenient set of $n$ such functions may be found by integrating

$$u'_i = A(t)u, \qquad u_i(0) = \hat{e}_i, \qquad i = 1, \ldots, n.$$

It is clear that the matrix $U(t) = (u_1(t) \cdots u_n(t))$ satisfies

$$U'(t) = A(t)U(t),$$

and since $U(0) = I$ it is invertible in a neighborhood of $t = 0$. The next theorem shows that it is invertible for all $t$.

**Theorem:** *Let $\{u_i(t)\}$ be a basis of $\mathcal{N}(L)$. The $U(t) = (u_1(t) \cdots u_n(t))$ is invertible everywhere.*

**Proof.** We know that $U(t)\vec{\alpha} \equiv 0$ for all $t$ if and only if $\vec{\alpha} = 0$ because of the linear independence of the functions $\{u_i(t)\}$, but this does not rule out that $U(t_0)\vec{\beta} = 0$ with $\vec{\beta} \neq 0$ at some particular point $t_0$. So let us suppose that for some $t_0$ we have indeed $U(t_0)\vec{\beta} = 0$ for some $\vec{\beta} \neq 0$. But then the function $v(t) = U(t)\beta$ satisfies

$$v' = A(t)v, \qquad v(t_0) = 0.$$

By uniqueness this implies that $v(t) \equiv 0$ for all $t$ which contradicts the linear independence of the $\{u_i(t)\}$. Hence there cannot be a point $t_0$ where $U(t_0)$ is singular.

**Definition:** A non-singular matrix $U(t)$ which satisfies

$$U' = A(t)U$$

is called a fundamental matrix for the differential equation

$$Lu \equiv u' - A(t)u = 0.$$

We observe that $U$ is a fundamental matrix if and only if its columns span $\mathcal{N}(L)$. Let us define the $n \times n$ matrix

$$\phi(t,s) = U(t)U^{-1}(s)$$

then it follows that

$$\frac{\partial}{\partial t}\phi(t,s) = U'(t)U^{-1}(s) = A(t)\phi(t,s)$$

$$\phi(s,s) = I,$$

in other words, $\phi(t,s)$ is the fundamental matrix which is the identity for $t = s$. If we have computed a $U(t)$, say by imposing initial conditions $u_i(0) = \hat{e}_i$ on the columns of $U$, then the solution to the initial value problem

$$u' = A(t)u, \qquad u(t_0) = u_0$$

is simply

$$u(t) = \phi(t, t_0)u_0.$$

For completeness we observe that

$$\begin{aligned}\frac{\partial}{\partial s}\,\phi(t,s) &= U(t)U^{-1}(s)' = -U(t)[U^{-1}(s)U'(s)U^{-1}(s)]\\ &= -U(t)U^{-1}(s)A(s)U(s)U^{-1}(s) = -\phi(t,s)A(s).\end{aligned}$$

We shall now discuss the solution of

$$Lu \equiv u' - A(t)u = F(t)$$

which is usually written in the form

$$u' = A(t)u + F(t)$$

where $F(t) = (f_1, \ldots, f_n)$ is a given source term. Such an equation is called a linear inhomogeneous equation.

We already learned that for the matrix problem $Ax = b$ with $b \in R(A)$ the solution is not unique when $\dim \mathcal{N}(A) \geq 1$. For the differential equation we know that $\dim \mathcal{N}(L) = n \neq 0$ and we shall see that the solution of the equation cannot be unique unless additional conditions, such as initial or boundary conditions, are imposed which do make the solution unique.

Let $u_p(t)$ be ANY function which satisfies the equation

$$u' = A(t)u + F(t)$$

then the most general solution of this equation is

$$u(t) = u_c(t) + u_p(t)$$

where for any basis $\{u_i(t)\}$ of $\mathcal{N}(L)$ and arbitrary $\alpha \in \mathbb{R}_n$

$$u_c(t) = U(t)\alpha = (u_1\ u_2\ \cdots u_n)\alpha$$

is known as the complementary solution. $u_p(t)$ is called a particular integral of the equation. We note that if $u_p$ and $w_p$ are two particular integrals then $u_p - w_p \in \mathcal{N}(L)$ so that two particular integrals differ by an element of $\mathcal{N}(L)$.

The method of variation of parameters for the calculation of $u_p(t)$: Given the differential equation $u' = A(t)u + F(t)$, i.e., $Lu \equiv u' - A(t)u = F(t)$, and a basis $\{u_i(t)\}$ of $\mathcal{N}(L)$ we can find a particular integral of the form

$$u_p(t) = U(t)v(t).$$

We compute that

$$u_p' = U'(t)v(t) + U(t)v'(t) = A(t)U(t)v + U(t)v' = A(t)u_p + U(t)v'.$$

Substitution into the differential equation yields

$$A(t)u_p + U(t)v' = A(t)u_p + F(t)$$

so that $v(t)$ is to be found by integrating

$$v' = U^{-1}(t)F(t).$$

Hence

$$v(t) = \int_{t_0}^{t} U^{-1}(s)F(s)ds$$

where $t_0$ is a convenient lower limit. Note that changing the lower limit changes $v(t)$ by a constant and hence $u_p(t)$ by an element in $\mathcal{N}(L)$ as expected. Thus we have found a particular integral of the form

$$u_p(t) = U(t)v(t) = \int_{t_0}^{t} U(t)U^{-1}(s)F(s)ds = \int_{t_0}^{t} \phi(t,s)F(s)ds.$$

To illustrate this process consider the scalar equation

$$u' = a(t)u + F(t)$$

$$u(t_0) = u_0.$$

The null space is one-dimensional and is spanned by the solution of

$$U' = a(t)U$$

$$U(t_0) = 1$$

which by separation of variables we find to be

$$U(t) = e^{\int_{t_0}^{t} a(r)dr}.$$

(In all these calculations be super-careful not to confuse the independent variable $t$ with any dummy variable of integration!) Because we have a scalar $U(t)$ we find by inspection that

$$U^{-1}(s) = e^{-\int_{t_0}^{s} a(r)dr} \quad \text{and} \quad \phi(t,s) = e^{\int_{s}^{t} a(r)dr}.$$

The general solution is

$$u(t) = U(t)\alpha + \int_{t_0}^{t} \phi(t,s)F(s)ds.$$

The initial condition requires that

$$\alpha = u_0.$$

Note that had we not chosen a particular integral which vanishes at $t_0$ then the computation of $\alpha$ would have been more involved.

**A word of caution.** This scalar calculation has no matrix equivalent.

Given a matrix $A(t)$ we know that

$$A'(t) = (a'_{ij}(t)) \quad \text{and} \quad \int A(s)ds = \left(\int a_{ij}(s)ds\right)$$

so it is tempting to write a solution of

$$U' = A(t)U, \qquad U(t_0) = I$$

for an $n \times n$ non-constant matrix $A(t)$ as

$$U(t) = \exp\left(\int_{t_0}^{t} A(s)ds\right)$$

where as before the matrix exponential is defined by the Taylor series

$$e^{\int_{t_0}^{t} A(s)ds} = \sum_{n=0}^{\infty} \frac{\left(\int_{t_0}^{t} A(s)ds\right)^n}{n!}.$$

The series can be differentiated term by term, but unfortunately

$$\begin{aligned}\frac{d}{dt}\left(\int_{t_0}^{t} A(s)ds\right)\left(\int_{t_0}^{t} A(s)ds\right) &= A(t)\int_{t_0}^{t} A(s)ds + \int_{t_0}^{t} A(s)ds\, A(t) \\ &\neq 2A(t)\int_{t_0}^{t} A(s)ds\end{aligned}$$

because in general $A(t)$ and its integral do not commute. This implies that usually

$$\frac{d}{dt}\, e^{\int_{t_0}^{t} A(s)ds} \neq A(t) e^{\int_{t_0}^{t} A(s)ds}$$

i.e.,

$$U'(t) \neq A(t)U(t)$$

so that $\exp\left(\int_{t_0}^{t} A(s)ds\right)$ is NOT a fundamental matrix. This raises the question of just when can we actually find a basis of $\mathcal{N}(L)$ analytically which will be discussed in the next two modules.

Let us conclude this discussion by relating the computation of $u_p$ for a first order system to the calculation of a solution $u_p$ of the second order equation

$$Lu \equiv u'' + a_1(t)u' + a_0(t)u = f(t).$$

As we saw, if we set

$$v_1 = u$$

$$v_2 = u'$$

then the equivalent first order system is

$$v' = \begin{pmatrix} 0 & 1 \\ -a_0(t) & -a_1(t) \end{pmatrix} v + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

Let $\left\{\begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}, \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}\right\}$ be two linearly independent solutions of $v' = A(t)v$ then

$$\begin{aligned} u_1 &= v_{11} \qquad & u_2 &= v_{12} \\ u_1' &= v_{21} \qquad & u_2' &= v_{22} \end{aligned}$$

defines two linearly independent solutions $\{u_1, u_2\}$ in $\mathcal{N}(L)$. The particular integral $v_p$ of the first order system is

$$v_p(t) = V(t)z(t)$$

where $v(t) = (v_1(t)\ v_2(t))$ and

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = V^{-1}(t)\begin{pmatrix} 0 \\ f(t) \end{pmatrix} = \frac{1}{u_1u_2' - u_1'u_2}\begin{pmatrix} -u_2 f(t) \\ u_1 f(t) \end{pmatrix}.$$

The first component of $v_p(t)$ is a solution of $Lu = f(t)$. Hence we have the particular integral

$$u_p(t) = u_1(t)z_1(t) + u_2(t)z_2(t)$$

were $z_1$ and $z_2$ are found by integration. We point out that this form of a particular integral for the second order equation $Lu = f$ is usually derived by assuming a solution of the form

$$u_p(t) = u_1(t)z_1(t) + u_2(t)z_2(t)$$

where $z_1$ and $z_2$ are determined such that

$$u_1z_1' + u_2z_2' = 0$$

and

$$Lu_p = f(t).$$

When these two equations are solved for $z_1'$ and $z_2'$ exactly the same equations as those derived above via the first order system are found. Hence the method of variation of parameters exists for a first order system or a higher order scalar equation. Its dominant advantage is its applicability for all continuous functions $F(t)$ whatever their form and structure. Its major drawback is the necessity to carry out integrations which may not be possible in closed form.

**Example:** Find a particular integral for

$$Lu = u'' - u = \sin t$$

**Answer:** It is easy to verify that two linearly independent solutions of $Lu = 0$ are

$$u_1(t) = \cosh t, \qquad u_2(t) = \sinh t$$

It then follows that $u_1 u_2' - u_1' u_2 = 1$ and

$$z_1' = -\sinh t \sin t$$

$$z_2' = \cosh t \sin t.$$

Integrating from 0 to $t$ we obtain

$$z_1(t) = -(\cosh t \sin t - \cos t \sinh t)/2$$

$$z_2(t) = (1 - \cosh t \cos t + \sinh t \sin t)/2$$

so that

$$\begin{aligned} u_p(t) &= -(\cosh^2 t \sin t - \cosh t \sinh t \cos t - \sinh t + \sinh t \cosh t \cos t - \sinh^2 t \sin t)/2 \\ &= (-\sin t + \sinh t)/2. \end{aligned}$$

Since $\sinh t \in \mathcal{N}(L)$ we may take

$$u_p(t) = -(\sin t)/2.$$

## Module 14 Homework

1) Let

$$A(t) = \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix}.$$

   i) Prove or disprove: $2A'(t)A(t) = \frac{d}{dt} A^2(t)$ for all $t$.

   ii) Prove or disprove

$$\frac{d}{dt} e^{A(t)} = A(t)e^{A(t)} \qquad \text{for all } t.$$

2) Solve

$$u' = (\cos t)u + \sin t$$

$$u(3) = 4$$

3) Let

$$B(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$C(t) = (3I + B)t.$$

   i) Show that

$$e^{C(t)} = e^{3It} e^{Bt}.$$

   ii) Compute explicitly

$$e^{3It} \quad \text{and} \quad e^{Bt}$$

   by summing their Taylor series.

   iii) Solve

$$u' = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} u + \begin{pmatrix} 1 \\ t \end{pmatrix}$$

$$u(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

   iv) Find a scalar second order equation equivalent to the system of iii) and give its solution.

4) Consider

$$t^2 u'' - tu' + u = e^{-t^2}$$

$$u(1) = 1, \qquad u'(1) = 0.$$

Find $u_c(t)$ and $u_p(t)$ and solve the problem. It is acceptable to have an integral for $u_p$.