MODULE 17

Topics: The inverse of linear operators and Green's functions

When we discussed the matrix problem

$$Lx = Ax = b$$

we learned that the existence of a solution depends on whether $b \in R(A)$ and its uniqueness on whether $\dim \mathcal{N}(A) = 0$. If the null space was not trivial then the solution was of the form

$$x = x_c + x_p$$

where $x_c = X\alpha = (x_1 \cdots x_k)\alpha$ with $\{x_i\}$ a basis of $\mathcal{N}(A)$ and where x_p was any solution of $Lx = b$. Both X and x_p were generally found with Gaussian elimination. Special constraints were then imposed to pin down x_c; for example, one could find a minimum norm solution of $Lx = b$. Consider now

$$Lu = F(t)$$

where either

$$Lu \equiv u' - A(t)u$$

or

$$Lu = \sum_{j=0}^{n} a_j(t)u^{(j)}.$$

We have learned that the equation has a solution for all continuous functions $F(t)$ and that it is of the form

$$u(t) = U(t)\alpha + u_p(t)$$

where $U(t)$ is either an $n \times n$ or a $1 \times n$ matrix whose columns are a basis of $\mathcal{N}(L)$. Moreover, for constant coefficient equations we have a mechanism for actually computing $u(t)$ which may be thought of as the counterpart to Gaussian elimination for algebraic equations. Boundary, initial conditions or other constraints can now be imposed to determine α. As an illustration consider the following problem: Suppose the motion of a mechanical system is described by the following mathematical model

$$u'' + u = 2e^{-t} \qquad \text{for } t \in [0, 2\pi].$$

What initial conditions $u(0)$, $u'(0)$ should we choose so that the solution $u(t)$ is closest in the mean square sense to the motion $w(t) = t(2\pi - t)$ for $t \in [0, 2\pi]$?

Answer: This constant coefficient equation has the general solution

$$u(t) = \alpha_1 \cos t + \alpha_2 \sin t + e^{-t}.$$

The problem now is to determine $\{\alpha_1, \alpha_2\}$ such that

$$E(\alpha_1, \alpha_2) = \|u(t) - w(t)\|_2^2 = \int_0^{2\pi} (u(t) - w(t)^2)dt$$

is minimized. We can either use calculus and solve $\partial E/\partial \alpha_1 = \partial E/\partial \alpha_2 = 0$ or recall that the closest element in $\text{span}\{\cos t, \sin t\}$ to

$$g(t) = w(t) - e^{-t}$$

is the orthogonal projection of $g(t)$ with respect to the usual $L_2[0, 2\pi]$ inner product. Since $\{\cos t, \sin t\}$ are orthogonal in $L_2[0, 2\pi]$ we find immediately

$$Pg(t) = \frac{\langle g(t), \cos t\rangle}{\langle \cos t, \cos t\rangle} \cos t + \frac{\langle g(t), \sin t\rangle}{\langle \sin t, \sin t\rangle} \sin t.$$

We find (from the computer) that

$$Pg(t) = -4.159 \cos t - .1589 \sin t$$

so that the best trajectory is

$$u(t) = Pg(t) + e^{-t}$$

which yields initial conditions

$$u(0) = -3.159$$

$$u'(0) = -1.1589.$$

Note this problem is equivalent to finding the solution of $Ax = b$ which is closest to a given vector y when $\dim \mathcal{N}(A) \geq 1$.

Let us now broaden our discussion of linear differential operators by introducing the concept of the inverse for a differential operator L, i.e., the analog to A^{-1} for a matrix whose null space is $\{0\}$.

In order to talk about L^{-1} we need to be a bit more precise about the domain of L. We observe that if

$$Lu \equiv u' - A(t)u$$

is to be considered for $t \in [a, b]$ then L is defined on the vector space

$$\mathcal{V} = \left\{ \vec{u}(t) = \begin{pmatrix} u_1(t) \\ \cdots \\ u_n(t) \end{pmatrix} : u_i(t) \in C^1[a, b] \right\}.$$

(Note again that here we distinguish between the vector valued function $\vec{u}(t)$ and its components $\{u_i(t)\}$, each of which is a scalar function.) If for $t \in [a, b]$

$$Lu \equiv \sum_{j=0}^{n} a_j(t) u^{(j)}$$

the L is defined on $\mathcal{V} = C^n[a, b]$. In either case L cannot have an inverse because we already know that $\dim \mathcal{N}(L) = n$ while $\dim \mathcal{N}(L) = 0$ is necessary for the inverse to exist. We will have to restrict L to some subspace $\mathcal{M}$ such that for $u \in \mathcal{M}$

$$Lu = 0 \Rightarrow u = 0.$$

This is not hard to do. For the first order system $Lu \equiv u' - A(t)u$ let

$$\mathcal{M} = \{\vec{u}(t) = (u_1(t), \ldots, u_n(t) : u_i(t_0) = 0,\ i = 1, \ldots, n\}$$

for some $t_0 \in [a, b]$. M is clearly closed under vector addition because $u(t_0) + v(t_0) = 0$ and $\alpha u(t_0) = 0$ for u, $v \in \mathcal{M}$ and α a scalar. Then

$$Lu = 0 \quad \text{and} \quad u \in \mathcal{M}$$

implies that $u \equiv 0$ because the initial value problem

$$u' = A(t)u, \qquad u(t_0) = 0$$

has only the zero solution. We can actually write the inverse of L in this case. Let $U(t)$ be any fundamental matrix of $u' = A(t)u$ then

$$Lu = F(t)$$

has the inverse

$$u(t) = L^{-1}F = \int_{t_0}^{t} \phi(t,s)F(s)ds$$

where as before $\phi(t,s) = U(t)U^{-1}(s)$.

The corresponding result for the nth order scalar equation is obtained if we choose

$$\mathcal{M} = \left\{ u : u(t_0) = u'(t_0) = \cdots = u^{(n-1)}(t_0) = 0 \right\}$$

because the initial value problem

$$Lu = 0$$

$$u(t_0) = u'(t_0) = \cdots = u^{(n-1)}(t_0) = 0$$

likewise only has the zero solution. The inverse of L can be expressed as

$$u(t) = L^{-1}f(t) = U(t)v(t)$$

where $U(t) = (u_1(t) \cdots u_n(t))$ is a $1 \times n$ matrix whose columns are the basis functions of $\mathcal{N}(L)$ in $C^n[a,b]$, and where v is the n-dimensional vector valued function

$$v(t) = \int_{t_0}^{t} W^{-1}(s) \begin{pmatrix} 0 \\ \vdots \\ \frac{f(s)}{a_n(s)} \end{pmatrix} ds$$

and where W is the matrix whose determinant is the Wronskian.

In general, the inverse of L on a subspace associated with initial value problems is not particularly useful. More interesting are subspaces associated with boundary value problems. Here we shall consider only the following special case;

$$Lu \equiv a_2(t)u'' + a_1(t)u' + a_0(t)u = f(t)$$

$$u(a) = 0, \qquad u(b) = 0.$$

The associated subspace of $C^2[a,b]$ is

$$\mathcal{M} = \{u \in C^2[a,b] : u(a) = u(b) = 0\}.$$

It is clear that $\mathcal{M}$ is a subspace since sums and multiples of functions vanishing at a and b likewise vanish at a and b. In order to have an inverse for L we need that $Lu = 0$ has only the zero solution in $\mathcal{M}$. Sometimes this has to be decided by calculation, sometime it is guaranteed by the structure of the problem. For example, consider

$$Lu = u'' + u = 0$$

$$u(0) = 0, \qquad u(b) = 0.$$

Any solution of this problem will be an element of $\mathcal{N}(L) \subset \mathcal{M}$. This problem we can solve in closed form. We know that $u(t)$ also is an element of $\mathcal{N}(L)$ in the bigger space $C^2[0, b]$ which has dimension 2 and a basis $\{\cos t, \sin t\}$. Hence

$$u(t) = c_1 \cos t + c_2 \sin t.$$

The boundary conditions require that $c_1 = 0$ and $c_2 \sin b = 0$. If $b \neq n\pi$ for an integer n then $c_2 = 0$ and $\mathcal{N}(L) = \{0\}$. Hence L has an inverse and

$$Lu = f(t) \text{ has the solution } u(t) = (L^{-1}f)(t).$$

By the end of this module we shall also be able to compute this inverse. As an example of a class of problems where the structure of L guarantees an inverse we consider

$$Lu = (a(t)u')' - c(t)u = f(t)$$

$$u(a) = u(b) = 0.$$

The problem may be thought of as $Lu = f(t)$ defined on $\mathcal{M} = \{u \in C[a, b] : u(a) = u(b) = 0\}$.

Theorem: *If $a(t) > 0$ and $c(t) \geq 0$ on $[a, b]$ then $Lu = 0$ has only the trivial solution $u(t) = 0$ in $\mathcal{M}$ (i.e., the boundary value problem has only the zero solution).*

Proof: Assume that u is a solution of $Lu = 0$, $u(a) = u(b) = 0$, then

$$\int_a^b u(t)Lu(t)dt = \int_a^b (a(t)u')'u - c(t)u^2)dt = 0.$$

Integration by parts shows that

$$a(t)u'(t)u(t)\Big|_a^b - \int_a^b (a(t)u'^2 + c(t)u^2)dt = 0.$$

Since $a(t) > 0$ and $c(t) \geq 0$ this implies that $u' \equiv 0$; $u(a) = 0$ then assures that $u \equiv 0$ for all $t \in [a, b]$.

We note that this theorem does not apply to our first example $Lu = u'' + u$ where the trivial null space was found by actually solving $Lu = 0$ subject to the boundary data. On the other hand, the theorem assures that

$$Lu \equiv u'' - tu = 0 \qquad \text{on } 0 < a < t < b$$

$$u(a) = u(b) = 0$$

only has the zero solution. A closed form solution in terms of elementary functions does not exist for this L.

The computation of the inverse:

We consider

$$Lu \equiv a_2(t)u'' + a_1(t)u' + a_0(t)u = f(t)$$

$$u(a) = u(b) = 0.$$

As before we assume that $a_2(t) \neq 0$ on $[a, b]$.

Our goal is to find a mapping which takes a given f to the solution u, i.e.,

$$u(t) = (L^{-1})f(t).$$

The process is mechanical although we may not be able to carry out the requisite calculations in analytic form.

Let s be an arbitrary but fixed point in the interval (a, b). Then we compute two functions $G_2(t, s)$ and $G_1(t, s)$ which, as functions of t, satisfy:

$$LG_2(t, s) = 0, \qquad G_2(a, s) = 0, \qquad a < t < s$$

$$LG_1(t, s) = 0, \qquad G_2(b, s) = 0, \qquad s < t < b.$$

At $t = s$ the two functions are patched together such that

$$G_2(s,s) = G_1(s,s)$$

and

$$\frac{\partial}{\partial t} G_1(s,s) - \frac{\partial}{\partial t} G_2(s,s) = \frac{1}{a_2(s)} \, .$$

Definition: The Green's function for

$$Lu = f$$

$$u(a) = u(b) = 0$$

is the function

$$G(t,s) = \begin{cases} G_1(t,s) & a < s < t \\ G_2(t,s) & t < s < b. \end{cases}$$

Thus, $G(t,s)$ is defined on the square $[a,b] \times [a,b]$. It is continuous in t on the whole square, and for a given s it is a solution of $Lu(t) = 0$ on (a,s) and (s,b). However, at $t = s$ the first derivative with respect to t will show a jump.

The significance of the Green's function is that it essentially defines the inverse of L because the boundary value problem has the solution

$$u(t) = \int_a^b G(t,s) f(s) ds.$$

This is of course not obvious but can be verified as follows. We write

$$u(t) = \int_a^b G(t,s) f(s) ds = \int_a^t G_1(t,s) f(s) ds + \int_t^b G_2(t,s) f(s) ds$$

and differentiate.

$$u'(t) = G_1(t,t) f(t) + \int_a^t \frac{\partial}{\partial t} G_1(t,s) f(s) ds - G_2(t,t) f(t) + \int_t^b \frac{\partial}{\partial t} G_2(t,s) f(s) ds.$$

Since $G_1(s,s) = G_2(s,s)$ for all s two terms on the right cancel and we have

$$u'(t) = \int_a^t \frac{\partial}{\partial t} G_1(t,s) f(s) ds + \int_t^b \frac{\partial}{\partial t} G_2(t,s) f(s) ds.$$

Differentiating again we obtain

$$u''(t) = \frac{\partial}{\partial t} G_1(t,t)f(t) + \int_a^t \frac{\partial^2}{\partial t^2} G_1(t,s)f(s)ds - \frac{\partial}{\partial t} G_2(t,t)f(t) + \int_t^b \frac{\partial^2}{\partial t^2} G_2(t,s)f(s)ds.$$

When we substitute into the differential equation we find

$$\begin{aligned} Lu(t) = &\int_a^t LG_1(t,s)f(s)ds + \int_t^b LG_2(t,s)f(s)ds \\ &+ a_2(t)\left[\frac{\partial}{\partial t} G_1(t,t) - \frac{\partial}{\partial t} G_2(t,t)\right] f(t) = f(t). \end{aligned}$$

Furthermore

$$u(a) = \int_a^b G_2(a,s)f(s)ds = 0 \quad \text{and} \quad u(b) = \int_a^b G_1(b,s)f(s)ds = 0$$

by construction. Thus $u(t)$ satisfies the differential equation and the boundary conditions regardless of what f is, and we can write

$$u = L^{-1}f$$

where L^{-1} is a linear "integral" operator defined by

$$(L^{-1}f)(t) = \int_a^b G(t,s)f(s)ds$$

for every function $f \in C^0[a,b]$. Following are some examples for solving boundary value problems with a Green's function.

Problem: Find the Green's function for the problem

$$Lu \equiv u'' = f(t)$$

$$u(0) = u(1) = 0.$$

According to our recipe we need to solve

$$LG_2(t,s) = 0 \quad \text{on } (0,s)$$

$$G_2(0,s) = 0$$

and

$$LG_1(t,s) = 0 \quad \text{on } (s,1)$$

$$G_1(1,s) = 0.$$

We find

$$G_2(t,s) = at$$

$$G_1(t,s) = b(1-t)$$

where a and b will be functions of s yet to be determined. The two so-called interface conditions

$$G_1(s,s) = G_2(s,s)$$

$$\frac{\partial}{\partial t} G_1(s,s) - \frac{\partial}{\partial t} G_2(s,s) = 1$$

lead to

$$b(1-s) = as$$

$$-b - a = 1.$$

We solve for a and b and find

$$G(t,s) = \begin{cases} s(t-1) & 0 < s < t \\ t(s-1) & t < s < 1 \end{cases}$$

so that

$$u(t) = \int_0^1 G(t,s) f(s) ds.$$

Suppose we wanted to solve this simple problem directly, without use of a Green's function. We then proceed the usual way. The solution is

$$u(t) = c_1 1 + c_2 t + u_p(t)$$

where $\{1, t\}$ is a basis of $\mathcal{N}(L) \subset C^2[0,1]$ and where $u_p(t)$ is any solution of $u'' = f(t)$. We see that

$$u_p(t) = \int_0^t \int_0^r f(s) ds\, dr$$

will do because $u'(t) = \int_0^t f(s)ds$ and $u''(t) = f(t)$. It remains to find c_1 and c_2 from the boundary conditions. Since $u_p(0) = 0$ we find from $u(0) = 0$ that $c_1 = 0$. From $u(1) = 0$ we find that

$$c_2 = -\int_0^1 \int_0^r f(s)ds\, dr.$$

Thus

$$u(t) = -\left[\int_0^1 \int_0^r t f(s)ds\, dr - \int_0^t \int_0^r f(s)ds\, dr\right].$$

If we reverse the order of integration we find

$$\int_0^t \int_0^r f(s)ds\, dr = \int_0^t \int_s^t f(s)dr\, ds = \int_0^t (t-s)f(s)ds.$$

Hence

$$\begin{aligned} u(t) &= -t\int_0^1 (1-s)f(s)ds - \int_0^t (t-s)f(s)ds \\ &= \int_t^1 t(s-1)f(s)ds + \int_0^t s(t-1)f(s)ds \\ &= \int_0^1 G(t,s)f(s)ds \end{aligned}$$

as we would expect.

Problem: Solve

$$Lu \equiv u'' + u = f(t)$$

$$u(0) = A, \qquad u(T) = B$$

with a Green's function. We begin by noting that the boundary data are not satisfied by the zero function so L is not defined on a vector subspace. Hence we need to modify the problem. Let $v(t) = A\,\frac{T-t}{T} + B\,\frac{t}{T}$ and define $w(t) = u(t) - v(t)$. Then w satisfies

$$Lw = w'' + u = Lu - Lv = f(t) - v(t) \equiv g(t)$$

$$w(0) = w(b) = 0.$$

Now we can find w with the Green's function. We verify that

$$G_2(t,s) = a\sin t, \qquad G_1(t,s) = b\sin(t-T)$$

are solutions of $Lu = 0$ which satisfy the conditions $G_1(0, s) = G_2(T, s) = 0$. The interface conditions are

$$\begin{pmatrix} \sin s & -\sin(s-T) \\ -\cos s & \cos(s-T) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We solve for a and b and find

$$w(t) = \int_0^T G(t,s)g(s)ds$$

where

$$G(t,s) = \begin{cases} \dfrac{\sin s \sin(t-T)}{\sin T} & 0 < s < t \\ \dfrac{\sin t \sin(s-T)}{\sin T} & t < s < T. \end{cases}$$

We observe that the Green's function does not exist for $T = n\pi$. But as we showed earlier, the problem

$$Lu = u'' + u = 0$$

has the nontrivial solution $u(t) = \sin t$ in $\mathcal{M} = \{u \in C^2[0, n\pi],\ u(0) = u(n\pi) = 0\}$. Hence we cannot expect the inverse to exist in this case.

What is not clear at this point is why one would bother to compute the Green's function since if we can find G_1 and G_2 then we very likely can also write down the general solution of the problem and simply compute its constants from the boundary conditions. However, consider the following nonlinear boundary value problem:

$$u'' = F(t, u)$$

$$u(0) = 0, \qquad u(T) = 0.$$

It is clear from our discussion that any solution of this problem is also a solution of of the integral equation

$$u(t) = \int_0^T G(t,s)F(s,u(s))ds$$

where G is the known Green's function for $Lu = u''$, $u(0) = u(T) = 0$. This function was computed above and allows us to replace the nonlinear differential equation by a nonlinear integral equation. Both for the analysis of the problem, i.e., answering questions on existence and uniqueness of a solution, and for the numerical solution of the problem an integral equation is often preferable to the corresponding differential equation.

Module 17 - Homework

1) i) Integrate explicitly

$$\int_0^t \int_r^t e^{-s^2} ds\, dr.$$

ii) Let $u(t) = \int_1^{t^2} \cos(t - s^2) ds$. Compute $u'(t)$ and $u''(t)$.

2) Compute the Green's function for

$$Lu = t^2 u'' - tu' + u = f(t)$$

$$u(1) = 0, \qquad u(2) = 0,$$

or show that it cannot be done.

(Observe that $u(t) \equiv t \in \mathcal{N}(L)$.)

3) Solve with a Green's function only the problem

$$u''(t) = 0$$

$$u(1) = 5, \qquad u(2) = 2.$$

4) Show that the null space of $Lu \equiv u''$ defined on

$$\mathcal{M} = \{u \in C^2[0, 1] : u(0) = u'(0),\ u(1) = 0\}$$

consists of the zero function only. Then mimic the derivation of the Green's function in the module but impose the boundary condition $u(0) = u'(0)$ instead of $u(0) = 0$. What is the final Green's function?