

MODULE 18

Topics: Eigenvalues and eigenvectors for differential operators

In analogy to the matrix eigenvalue problem $Ax = \lambda x$ we shall consider the eigenvalue problem

$$Lu = \mu u$$

where μ is a real or complex number and u is a non-zero function (i.e., u is not identically equal to zero). μ will be called an eigenvalue of L and $u(t)$ is the corresponding eigenvector = eigenfunction. As in the case of inverses we shall find that the existence of eigenvalues depends crucially on the domain of L . For example, suppose

$$Lu \equiv u'$$

is defined on $\mathcal{V} = \{u(t) = (u_1(t), \dots, u_n(t)) : u_i(t) \in C^1[a, b]\}$ then for any constant $\mu \in (-\infty, \infty)$ the equation

$$Lu = \mu u$$

has the non-zero solution

$$u(t) = xe^{\mu t}$$

where x is any non-zero vector in \mathbb{R}_n . Hence every number μ is an eigenvalue and x is a corresponding eigenfunction. On the other hand, if the same problem is considered in the subspace

$$\mathcal{M} = \{u \in \mathcal{V} : u(t_0) = 0\}$$

then it follows from the uniqueness of the solution of

$$u' = \mu u$$

$$u(t_0) = 0$$

that only the zero solution is possible. Hence there is no eigenvalue $\mu \in (-\infty, \infty)$. A more complicated example is provided by the following

Problem: Find all nontrivial solutions of

$$Lu = u' - Au = \mu u$$

$$u(0) = u(1)$$

where A is an $n \times n$ matrix. Note that the problem can be restated as finding all eigenvalues and eigenvectors of the linear operator

$$Lu \equiv u' - Au$$

on the subspace of vector valued functions which satisfy $u(0) = u(1)$.

Answer: Any such solution solves the linear equation

$$u' - (A + \mu I)u = 0.$$

Our discussion of constant coefficient systems says that $u(t)$ should be of the form

$$u(t) = xe^{\lambda t}$$

where x is a constant vector. Substitution into the equation yields

$$[\lambda x - (A + \mu)x]e^{\lambda t} = 0.$$

This implies that

$$[A - (\lambda - \mu)I]x = 0.$$

A non-trivial solution can exist only if $(\lambda - \mu)$ is an eigenvalue of A and x is a corresponding eigenvector. Let $\{\rho_1, \dots, \rho_n\}$ be the eigenvalues of A with corresponding eigenvectors $\{x_1, \dots, x_n\}$. Then for each ρ_j we obtain a vector valued function $x_j e^{(\rho_j + \mu)t}$ which solves

$$u' - Au = \mu u.$$

The additional constraint on the solution is that

$$u(0) = u(1).$$

This requires

$$e^{(\rho_j + \mu)} = 1$$

and hence that

$$(\rho_j + \mu) = 2m\pi i n$$

where $i^2 = -1$ and m is any integer. Thus the eigenvalues are

$$\mu_{j,m} = 2m\pi i - \rho_j$$

with corresponding eigenfunctions

$$u_{j,m} = x_j e^{2m\pi i t}.$$

Incidentally, the condition $u(0) = u(1)$ is not so strange. If the first order system is equivalent to an n th order scalar equation then this condition characterizes smooth periodic functions with period 1.

All further discussion of eigenvalue problems for differential operators will be restricted to second order equations of the form

$$Lu \equiv (a(t)u')' + q(t)u = \mu p(t)u$$

defined on the interval $[0, T]$ with various boundary conditions at $t = 0$ and $t = T$. We shall assume that all coefficients are real and continuous in t . In addition we shall require that

$$a(t) > 0 \quad \text{on } [0, T]$$

$$p(t) > 0 \quad \text{on } [0, T] \text{ except possibly at isolated points}$$

where p may be zero.

The function $p(t)$ multiplying the eigenvalue may look unusual but it does arise naturally in many applications. Its presence can greatly complicate the calculation of eigenfunctions and eigenvalues but it has little influence on the general eigenvalue theory. As we have seen time and again, a differential operator is completely specified only when we are given the domain on which it is to be defined. The domain of L given above will be a subspace \mathcal{M} of $C^2[0, T]$. In applications a number of different subspaces are common, but here we shall restrict ourselves to

$$\mathcal{M} = \{u \in C^2[0, T] : u(0) = u(T) = 0\}.$$

We now can obtain a number of results which follow from the specific form of the operator.

Theorem: *The eigenvalues of*

$$Lu \equiv (au')' + q(t)u = \mu p(t)u$$

$$u(0) = u(T) = 0$$

are real and the corresponding eigenfunctions may be chosen to be real.

Proof. As in the case of a matrix eigenvalue problem we don't know a priori whether an eigenvalue will turn out to be complex and require a complex eigenvector. Suppose that μ is an eigenvalue. Let u be the corresponding eigenvector. Then

$$\int_0^T \bar{u}Lu \, dt = \mu \int_0^T p(t)u\bar{u} \, dt.$$

The integral on the right is real and positive. Integration by parts shows that

$$\begin{aligned} \int_0^T \bar{u}Lu \, dt &= \int_0^T [(au')'\bar{u} + q(t)u\bar{u}] \, dt = \int_0^T [(au'\bar{u})' - au'\bar{u}' + q(t)u\bar{u}] \, dt \\ &= (au'\bar{u})\Big|_0^T - \int_0^T [au'\bar{u}' + q(t)u\bar{u}] \, dt = \int_0^T -[au'\bar{u}' - q(t)u\bar{u}] \, dt \end{aligned}$$

is real. Hence μ is real and u may be taken to be real since for a complex function both the real and the imaginary part would have to satisfy the eigenvalue equation.

Theorem: *Let μ_m and μ_n be distinct eigenvalues and ϕ_m and ϕ_n the corresponding eigenfunctions. Then*

$$\langle \phi_m, \phi_n \rangle = \int_0^T \phi_m(t)\phi_n(t)p(t) \, dt = 0$$

i.e., the functions ϕ_m and ϕ_n are orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_0^T f(t)g(t)p(t) \, dt.$$

Proof: It follows from

$$L\phi_m = \mu_m\phi_m p(t)$$

$$L\phi_n = \mu_n\phi_n p(t)$$

that

$$\int_0^T [\phi_n L\phi_m - \phi_m L\phi_n] \, dt = (\mu_m - \mu_n) \int_0^T \phi_m(t)\phi_n(t)p(t) \, dt.$$

But

$$\int_0^T [\phi_n L\phi_m - \phi_m L\phi_n] dt = \int_0^T [(a\phi'_m \phi_n)' - (a\phi'_n \phi_m)'] dt = 0$$

because of the boundary conditions. Hence $\langle \phi_m, \phi_n \rangle = 0$ for $m \neq n$.

Theorem: If μ is an eigenvalue with eigenfunction ϕ then

$$\mu < \frac{\int_0^T q(t)\phi^2(t) dt}{\int_0^T p(t)\phi^2(t) dt}.$$

Proof: We observe that $\phi' \neq 0$ because if $\phi' \equiv 0$ then ϕ would be constant and $\phi(0) = 0$ would imply that $\phi \equiv 0$ which is not possible for an eigenfunction. It then follows from

$$\int_0^T (\mu p(t) - q(t))\phi^2(t) dt = \int_0^T (a\phi')'\phi dt = - \int_0^T a\phi'\phi' dt$$

that necessarily

$$\mu < \frac{\int_0^T q(t)\phi^2(t) dt}{\int_0^T p(t)\phi^2(t) dt}.$$

In particular, this implies that the eigenvalue is strictly negative whenever $q \leq 0$.

An example and an application: Consider the eigenvalue problem

$$Lu(x) \equiv u''(x) = \mu u(x)$$

$$u(0) = u(b) = 0$$

where in view of the subsequent application we have chosen x as our independent variable. In this case $a(x) = p(x) = 1$ and $q(x) = 0$. The above theorems assure that eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the usual L_2 inner product on $[0, b]$, and that the eigenvalues are strictly negative. It is convenient to set

$$-\mu = \lambda^2, \quad \lambda \neq 0$$

so that the problem to be solved is

$$u''(x) + \lambda^2 u(x) = 0$$

$$u(0) = u(b) = 0.$$

This is a constant coefficient second order equation with solution

$$u(x) = c_1 \cos \lambda x + c_2 \sin \lambda x.$$

The boundary conditions can be expressed as

$$\begin{pmatrix} 1 & 0 \\ \cos b & \sin b \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

A non-trivial solution is possible only if the coefficient matrix is singular. It is singular whenever its determinant is zero. Hence λ has to be found such that

$$\sin \lambda b = 0.$$

This implies that

$$\lambda_n = \frac{n\pi}{b}, \quad n = \pm 1, \pm 2, \dots$$

A vector (c_1, c_2) which satisfies the singular system is seen to be $(0, 1)$. Hence the eigenvalues and eigenfunctions for this problem are

$$\mu_n = -\lambda_n^2, \quad \lambda = \left(\frac{n\pi}{b}\right), \quad \phi_n(x) = \sin \lambda_n x, \quad n = 1, 2, \dots$$

A direct computation verifies that

$$\langle \phi_n, \phi_m \rangle = 0 \quad \text{for } m \neq n.$$

An application: The so-called wave equation and boundary and initial conditions

$$u_{xx}(x, t) - \frac{1}{c^2} u_{tt} = F(x, t)$$

$$u(0, t) = u(b, t) = 0$$

$$u(x, 0) = u_0(x)$$

$$u_t(x, 0) = u_1(x)$$

describe the displacement $u(x, t)$ of a string at point x and time t from its equilibrium position. The string is held fixed at $x = 0$, $x = b$ and has the initial displacement $u_0(x)$ and initial velocity $u_1(x)$. The source term $F(x, t)$ is given.

The problem as stated is in general too difficult to solve analytically and has to be approximated. It becomes manageable if we think of t as a parameter and project all data functions as functions of x into the finite dimensional subspace

$$\mathcal{M} = \text{span}\{\phi_1, \dots, \phi_N\}.$$

Here ϕ_n is the n th eigenfunction of $Lu \equiv u''(x)$, i.e.,

$$\phi_n(x) = \sin \lambda_n x, \quad \lambda_n = \frac{n\pi}{b} \quad \text{and} \quad \mu_n = -\lambda_n^2.$$

The projection is the orthogonal projection with respect to the inner product for which the $\{\phi_n\}$ are orthogonal, i.e., the inner product

$$\langle f, g \rangle = \int_0^b f(x)g(x)dx.$$

As we learned early on, the projection onto \mathcal{M} with an orthogonal basis is

$$PF(x, t) = \sum_{j=1}^N \beta_j(t) \phi_j(x)$$

where

$$\beta_j(t) = \frac{\langle F(x, t), \phi_j(x) \rangle}{\langle \phi_j(x), \phi_j(x) \rangle}.$$

Remember, t is considered a parameter and β_j will have to depend on t because F in general will change with t . The other data functions have the simpler projections

$$Pu_0(x) = \sum_{j=1}^N c_j \phi_j(x) \quad \text{with} \quad c_j = \frac{\langle u_0, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle}$$

and

$$Pu_1(x) = \sum_{j=1}^N d_j \phi_j(x) \quad \text{with} \quad d_j = \frac{\langle u_1, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle}.$$

We now try to solve

$$u_{Nxx} - \frac{1}{c^2} u_{Ntt} = PF(x, t)$$

$$u_N(0, t) = u_N(b, t) = 0$$

$$u_N(x, 0) = Pu_0(x)$$

$$u_{N_t}(x, 0) = Pu_1(x).$$

The reason that this problem is manageable is that it has a solution $u_N(x, t)$ which for fixed t also belongs to \mathcal{M} . Thus we look for a solution of the form

$$u_N(x, t) = \sum_{j=1}^N \alpha_j(t) \phi_j(x).$$

We substitute into the wave equation for u_N , use that $\phi_i''(x) = -\lambda_i^2 \phi_i(x)$ and collect terms. We find that

$$\sum_{j=1}^N \left[-\lambda_j^2 \alpha_j(t) - \frac{1}{c^2} \alpha_j''(t) - \beta_j(t) \right] \phi_j(x) = 0.$$

But as orthogonal functions the $\{\phi_i\}$ are linearly independent so that this equation can only hold if the coefficient of each ϕ_j vanishes. Thus the function $\alpha_i(t)$ must satisfy the linear constant coefficient second order differential equation

$$\alpha_i''(t) + c^2 \lambda_i^2 \alpha_i(t) = -c^2 \beta_i(t).$$

Moreover, the initial conditions $u_N = Pu_0$ and $u_{N_t} = Pu_1$ at $t = 0$ require that

$$\alpha_i(0) = c_i$$

and

$$\alpha_i'(0) = d_i.$$

It follows that

$$\alpha_i(t) = \gamma_i \cos \lambda_i ct + \delta_i \sin \lambda_i ct + \alpha_{ip}(t)$$

where the coefficients $\{\gamma_i, \delta_i\}$ can only be determined after the particular integral $\alpha_{ip}(t)$ is known.

Problems of this type arise routinely in connection with diffusion, electrostatics and wave motion. They often can be solved on finite dimensional subspaces defined by eigenfunctions of the spatial operator of the partial differential equation. For a concrete application of the above development for a driven string we refer to the problem section of this module.

Module 18 - Homework

1) Consider

$$Lu \equiv u''$$

defined on $\mathcal{M} = \{u \in C^2[0, 1] : u(0) = 0, u'(1) = 0\}$. Modify the proofs of this module to prove that:

- i) The eigenvalues of L must be real.
- ii) The eigenvalues of L must be strictly negative.
- iii) The eigenfunctions corresponding to distinct eigenvalues must be orthogonal with respect to the usual $L_2[0, 1]$ inner product.
- iv) Which results change when the space \mathcal{M} is changed to

$$\mathcal{M} = \{u \in C^2[0, 1] : u'(0) = u'(1) = 0\}?$$

2) Compute the eigenvalues and eigenvectors of

$$Lu \equiv u'' = \mu u$$

$$u(0) = u(1), \quad u'(0) = u'(1).$$

3) Compute the eigenvalues of

$$Lu \equiv u'' = \mu u$$

$$u(0) = u'(0), \quad u(1) = -u'(1).$$

(It suffices to set up the equation which has to be solved to find the eigenvalues. You will not be able to find them in closed form.)

4) Consider the problem

$$Lu \equiv u'' - u' = \mu u$$

$$u(0) = u(1) = 0.$$

- i) Compute the eigenvalues and eigenvectors of L .

This operator L is not of the form required by the theorems of this module. But it can be brought into the correct form as outlined in the following instructions:

ii) Find a function $\phi(t)$ such that

$$[e^{\phi(t)}u'(t)]' = [u'' - u']e^{\phi(t)} = \mu ue^{\phi(t)}.$$

iii) What orthogonality is predicted by the theorem of the module for eigenfunctions corresponding to distinct eigenvalues?

iv) Verify by direct computation that the eigenfunctions are orthogonal in the correct inner product.

5) Consider the problem of a vibrating string which is oscillated at $x = 0$. The model is:

$$u_{xx} - u_{tt} = 0$$

$$u(0, t) = F_0 \cos \delta t$$

$$u(1, t) = 0$$

$$u(x, 0) = F_0(1 - x)$$

$$u_t(x, 0) = 0.$$

Find an approximate solution. When does resonance occur?

Hint: Reformulate the problem for $w(x, t) = u(x, t) - F_0(1 - x) \cos \delta t$ and apply the ideas outlined in the module to find an approximation $w_N(x, t)$.