

MODULE 19

Topics: The number system and the complex numbers

The development of the number system:

1. Positive integers (natural in counting)

We can add and multiply.

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2. All integers

We can add, multiply and subtract.

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3. Ordered pairs of integers (a, b) , $b \neq 0$

Rules for addition and multiplication:

$$(a, b) + (c, d) = (ad + bc, bd)$$

$$(a, b)(c, d) = (ac, bd)$$

These ordered pairs make up the rational numbers and are usually written in the form

$$(a, b) = \frac{a}{b}$$

Now we can add, subtract, multiply and divide.

We can enumerate the rational numbers, that is, we can put all rational numbers into a 1-1 correspondence with the positive integers. For example, if we write the array

$$\begin{array}{ccccccc} 0/1 & 0/2 & 0/3 & 0/4 & 0/5 & 0/6 & 0/7 \\ 1/1 & 1/2 & 1/3 & 1/4 & 1/5 & 1/6 & \dots \\ 2/1 & 2/2 & 2/3 & 2/4 & 2/5 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m/1 & m/2 & \dots & \dots & \dots & \dots & \dots \end{array}$$

then every rational number p/q appears in this array, and appears only once (where we distinguish between $1/1$, $2/2$ etc.). If we identify the rational number p/q with the number n of entries which lie in the triangle with corners $(0/1)$, $(0/p+q-1)$, $(p+q-1/1)$ plus the number of entries on the line from $0/(p+q)$ to p/q , then we have a unique function which maps p/q to n and any n to a unique p/q . Hence there are as many rational numbers as

there are positive integers. We say that the rational numbers are countable. Similarly, the rational numbers in any finite interval are countable since there are infinitely many but not more than all rational numbers.

Integers = $(a, 1)$ are embedded in rational numbers.

But, not all numbers are rational numbers.

There is no p/q such that $(p/q)^2 = 2$, for if there were a p/q which we may assume to have no common factors, then $p^2 = 2q^2$. $2q^2$ is even which implies that p is even, so that $p = 2p'$. But then $2p'^2 = q^2$ which makes q even contrary to the assumption that p and q have no common factor. Hence $\sqrt{2}$ is not rational.

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4. Real numbers = limits of all sequences of rational numbers
 = rational and irrational numbers

Between any two rational numbers there is an irrational number. For example, if a and b are rational then $a(1 - 1/\sqrt{2}) + b/\sqrt{2}$ is irrational and between a and b .

Between any two irrational numbers there is a rational number because we can approximate any irrational number by a rational number from above or below.

Theorem: *All the rational numbers on the interval $[0, 1]$ can be covered with open intervals such that the sum of the length of these intervals is arbitrary small.*

Proof: We know we can enumerate the rationals, i.e., each rational can be labeled with an integer n . Let I_n be an interval of length $\ell(I_n) = \epsilon/2^n$ centered at the n th rational. Then all rationals will be covered by open intervals and

$$\sum_{n=1}^{\infty} \ell(I_n) = \epsilon \sum_{n=1}^{\infty} 2^{-n} = \epsilon.$$

Thus the rationals take up no space in the interval $[0, 1]$ so the irrationals must fill up the interval. Yet between two of one kind there is one of the other kind. This should convince you that the reals are very complicated.

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5. Complex numbers:

Ordered pairs of real numbers and rules:

$$(a, b) + (c, d) = (a + c, b + d)$$

$$(a, b)(c, d) = (ac - bd, ad + bc)$$

$$(ac - bd, ad + bc) = (1, 0) \rightarrow (c, d) = (a/(a^2 + b^2), -b/(a^2 + b^2))$$

real numbers: $(a, 0)$ are embedded in complex numbers.

Common notation

$$(a, b) = a + ib \quad \text{where } i^2 = -1.$$

Geometric representation:

$$(a, b) = \text{vector in } \mathbb{R}_2.$$

Addition: This is the same as vector addition in \mathbb{R}_2 (component-wise addition).

Multiplication: No analog for vectors in \mathbb{R}_2 .

Two elementary functions of a complex variable:

Complex variable: $z = x + iy$

i) $f(z) = \bar{z} = x - iy$

Properties: $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$

$$\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2$$

ii) $f(z) = |z| = \sqrt{x^2 + y^2}$ (Euclidean length of the vector $(x, y) \in \mathbb{E}_2$).

Properties: $|z_1 z_2| = |z_1| |z_2|$

$$z\bar{z} = |z|^2.$$

Triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

This result follows immediately from the triangle inequality for vectors in \mathbb{E}_2 .

Reverse triangle inequality:

$$||z_1| - |z_2|| \leq |z_1 + z_2|.$$

This result follows from

$$|z_1| = |z_1 - z_2 + z_2| \leq |z_1 - z_2| + |z_2|.$$

Definition: $e^{i\theta} = \cos \theta + i \sin \theta$ for any real θ .

From the trigonometric addition formulas follows: $e^{i(\theta+\phi)} = e^{i\theta} e^{i\phi}$.

The polar form of a complex number:

$$z = |z|e^{i\theta}.$$

Definition: $|z|$ is the magnitude (modulus) of z

θ is the argument of $z = \arg(z)$

$\arg(z)$ is not uniquely defined because adding any multiple of 2π to θ does not change the point in the complex plane. We say: $\arg(z)$ is “a multiple valued function.”

We can choose a “branch” of this multiple valued function by restricting the argument to a specific interval of length 2π . This makes the function single valued. The most common branch is the principal value branch defined next.

Definition: The principal value of the argument of z , denoted by $\text{Arg } z$, is the value of the argument restricted to $(-\pi, \pi]$.

For example, $\text{Arg } -1 = \pi$, $\lim_{n \rightarrow \infty} \text{Arg}(-1 - i/n) = -\pi$. $\text{Arg } 0$ is not defined. Away from the negative x -axis $\text{Arg } z$ is a nice continuous function, but it jumps from π to $-\pi$ as we cross from the half plane $y > 0$ to $y < 0$ across the negative real axis.

Geometric interpretation of complex multiplication:

If $z_1 = |z_1|e^{i\theta_1}$ and $z_2 = |z_2|e^{i\theta_2}$ then

$$z_1 z_2 = |z_1| |z_2| e^{i(\theta_1 + \theta_2)}.$$

In words: Magnitudes multiply, arguments add.

The fact that arguments add on multiplication allows us to take roots of complex numbers.

Definition: $w = \sqrt[n]{z} \equiv (z)^{1/n}$ for a given complex number z means that we are looking for all complex numbers w such that

$$w^n = z.$$

Calculation of roots of z :

Given a complex number z we express it in polar form

$$z = |z|e^{i(\theta + 2\pi k)}$$

where k is any integer and where θ is a conveniently chosen angle for the ray through z (maybe an angle $\in (-\pi, \pi]$ i.e., $\theta = \text{Arg } z$). Then

$$w_k = \sqrt[n]{z} = \sqrt[n]{|z|} e^{i\left(\frac{\theta+2\pi k}{n}\right)}, \quad k = 0, 1, 2, \dots, n-1$$

yields n complex numbers $\{w_k\}$ with the property that $(w_k)^n = z$. It is readily verified that for any other integer k we obtain no new points in the complex plane. For example, if $k = -1$ then

$$e^{i\left(\frac{\theta-2\pi}{n}\right)} = e^{i\left(\frac{\theta-2\pi}{n}+2\pi\right)} = e^{i\left(\frac{\theta+2(n-1)\pi}{n}\right)}$$

so that

$$w_{-1} = w_{n-1}.$$

Example: $-1 = e^{i(\pi+2k\pi)}$ so that $w = \sqrt{-1}$ has the solutions

$$w_0 = e^{i\frac{\pi}{2}} = i$$

$$w_1 = e^{i\frac{3\pi}{2}} = -i$$

Note: Complex numbers are not ordered.

$$z_1 < z_2 \quad \text{makes no sense.}$$

Sets in the complex plane:

$S = \{z : |z - a| = r\}$ is a circle about a of radius r ;

$z = a + re^{i\theta}$ defines the points on S .

$S = \{z : |z - a| < R\}$ is the open disk around a of radius R

where $z = a + re^{i\theta}$ with $r < R$ defines the points in S .

Module 19 - Homework

- 1) Use the ordered pair notation and the rules for computing with ordered pairs to show that for every non-zero (a, b) there is a unique (c, d) such that $(a, b)(c, d) = (1, 0)$.
- 2) Put into polar form

$$z = \frac{-1 + \sqrt{3}i}{2 + 2i}$$

- 3) Compute $(-1)^{1/5}$.
- 4) Compute the following limits or show that they do not exist:

$$\lim_{n \rightarrow \infty} \operatorname{Arg} \left(7 + \frac{(-1)^n}{n} i \right)$$

$$\lim_{n \rightarrow \infty} \operatorname{Arg} \left(-7 + \frac{(-1)^n}{n} i \right)$$

$$\lim_{n \rightarrow \infty} \operatorname{Arg} \left(-7 - \frac{1}{n} i \right)$$

$$\lim_{n \rightarrow \infty} \operatorname{Arg} \left(-7 + \frac{(-1)^n}{n!} i \right)$$

$$\lim_{n \rightarrow \infty} \operatorname{Arg} \left(7 + \frac{(-1)^n}{n} \right)$$

- 5) Find the value of $\arg(-1 - i)$ which belongs to the interval $(121, 121 + 2\pi]$.
- 6) Plot $S = \{z : |z - 1| = \operatorname{Re} z + 1\}$
- 7) Given z with $|z| \neq R$ and $R > 0$ find z' and q with $z' \neq z$ such that

$$|z - w| = q|z' - w| \quad \text{for all } w \text{ such that } |w| = R.$$

Where is z' if $|z| < R$, where is z' if $|z| > R$?

- 8) i) Let D be a complex number. Let d_0 and d_1 be the two square roots of D . Show that $d_0 = -d_1$.
- ii) Show that the quadratic formula for $az^2 + bz + c = 0$ holds for complex a, b, c .