MODULE 2

Topics: Linear independence, basis and dimension

We have seen that if in a set of vectors one vector is a linear combination of the remaining vectors in the set then the span of the set is unchanged if that vector is deleted from the set. If no one vector can be expressed as a combination of the remaining ones then the vectors are said to be linearly independent. We make this concept formal with:

Definition: The vectors $\{x_1, x_2, \dots, x_n\} \in V$ are linearly independent if

$$\sum_{j=1}^{n} \alpha_j x_j = 0$$

has only the trivial solution $\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$.

We note that this condition says precisely that no one vector can be expressed as a combination of the remaining vectors. For were there a non-zero coefficient α_k then x_k can be expressed as a linear combination of the remaining vectors. In this case the vectors are said to be linearly dependent.

Examples:

1) Given k vectors $\{x_i\}_{i=1}^k$ with $x_i \in \mathbb{R}_n$ then they are linearly dependent if the linear system

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k = 0,$$

which can be written as

$$A\vec{\alpha} = 0,$$

has a nontrivial solution. Here A is the $n \times k$ matrix whose jth column is the vector x_j and $\vec{\alpha} = (\alpha_1, \dots, \alpha_k)$.

2) The $k+1$ vectors $\{t^j\}_{j=0}^k$ are linearly independent in $C^n(-\infty, \infty)$ for any n because if

$$\sum_{j=0}^{k} \alpha_j t^i \equiv 0 \qquad \text{for all } t$$

then evaluating the polynomial and its derivatives at $t = 0$ shows that all coefficients vanish.

3) The functions $\sin t, \cos t$ and $\cos(3-t)$ are linearly dependent in $C^k[a,b]$ for all k because they are k times continuously differentiable and

$$\cos(3-t) = \cos 3 \cos t + \sin 3 \sin t.$$

The first example shows that a check for linear independence in $\mathbb{R}_n$ or $\mathbb{C}_n$ reduces to solving a linear system of equations

$$A\vec{\alpha} = 0$$

which either has or does not have a nontrivial solution. The test for linear dependence in a function space seems more ad-hoc. Two consistent approaches to obtain a partial answer in this case are as follows.

Let $\{f_1, \ldots, f_n\}$ be n given functions in the (function) vector space $V = C^{n-1}(a,b)$. We consider the arbitrary linear combination

$$H(t) \equiv \alpha_1 f_1(t) + \alpha_2 f_2(t) + \cdots + \alpha_n f_n(t).$$

If $H(t) \equiv 0$ then the derivatives $H^{(j)}(t) \equiv 0$ for $j = 0, 1, \ldots, n-1$. We can write these equations in matrix form

$$W(t)\vec{\alpha} = \begin{pmatrix} f_1(t) & f_2(t) & f_n(t) \\ f_1'(t) & f_2'(t) & f_n'(t) \\ \cdots & \cdots & \cdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & f_n^{(n-1)}(t) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0.$$

If the matrix W is non-singular at one point t in the interval then necessarily $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n) = (0, \ldots, 0)$ and the functions are linearly independent. However, a singular W at a point (even zero everywhere on (a,b)) does not in general imply linear dependence. We note that in the context of ordinary differential equations the determinant of W is known as the Wronskian of the functions $\{f_i\}$. Hence if the Wronskian is not zero at a point then the functions are linearly independent.

The second approach is to evaluate $H(t)$ at n distinct points $\{t_i\}$. If $H(t_i) = 0$ for all i implies $\vec{\alpha} = 0$ then the functions $\{f_j\}$ are necessarily linearly independent. Written in matrix form we obtain

$$A\vec{\alpha} = 0$$

where $A_{ij} = f_j(t_i)$. Hence if A is not singular then we have linear independence. As in the other test, a singular A does not guarantee linear dependence.

Definition: Let $\{x_1, \ldots, x_n\}$ be a set of linearly independent vectors in the vector space V such that

$$\text{span}\{x_1, \ldots, x_n\} = V$$

Then $\{x_1, \ldots, x_n\}$ is a basis of V.

Definition: The number of elements in a basis of V is the dimension of V. If there are infinitely many linearly independent elements in V then V is infinite-dimensional.

Theorem: *Let $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ be bases of the vector space V then $m = n$, i.e., the dimension of the vector space is uniquely defined.*

Proof: Suppose that $m < n$. Since $\{x_j\}$ is a basis we have

$$y_j = \sum_{i=1}^{m} \alpha_{ij} x_i \qquad \text{for } j = 1, 2, \cdots, n.$$

Since the matrix $A = (\alpha_{ij})$ has fewer rows than columns, Gaussian elimination shows that there is a non-zero solution $\beta = (\beta_1, \cdots, \beta_n)$ of the non-square system

$$A\beta = 0.$$

But then

$$\sum_{j=1}^{n} \beta_j y_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij} \beta_j \right) x_i = 0$$

which contradicts the linear independence of $\{y_j\}$.

Examples:

i) If $\{x_1, \ldots, x_k\}$ are linearly independent then these vectors are a basis of $\text{span}\{x_1, \ldots, x_k\}$ which has dimension k. In particular, the unit vectors $\{\hat{e}_i\}$, $1 \leq i \leq n$, where $\hat{e}_i = (0, 0, \ldots, 1, 0, \ldots, 0)$ with a 1 in the ith coordinate, is a (particularly convenient) basis of $\mathbb{R}_n$ or $\mathbb{C}_n$.

ii) The vectors $x = (1, 2, 3)$ and $x = (2, -1, 4)$ are linearly independent because one is not a scalar multiple of the other; hence they form a basis for the plane $11x + 2y - 5z = 0$, i.e. for the subspace of all vectors in $\mathbb{R}_3$ whose components satisfy the equation of the plane.

iii) The vectors $x_i = t^i$, $i = 0, \ldots, N$ form a basis for the subspace of all polynomials of degree $\leq N$ in $C^k(-\infty, \infty)$ for arbitrary k. Since N can be any integer, the space $C^k(-\infty, \infty)$ contains countably many linearly independent elements and hence has infinite dimension.

iv) Any set of n linearly independent vectors in $\mathbb{R}_n$ is a basis of $\mathbb{R}_n$.

The norm of a vector:

The norm of a vector x, denoted by $\|x\|$, is a real valued function which describes the "size" of the vector. To be admissible as a norm we require the following properties:

i) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$.

ii) $\|\alpha x\| = |\alpha|\|x\|$, $\alpha \in F$.

iii) $\|x + y\| \leq \|x\| + \|y\|$ (the triangle inequality).

Certain norms are more useful than others. Below are examples of some commonly used norms:

Examples:

1) Setting: $V = \mathbb{R}_n$ and $F = \mathbb{R}$ or $V = \mathbb{C}_n$ and $F = \mathbb{C}$

   i) $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$

   ii) $\|x\|_2 = \left(\sum_{j=1}^{n} |x_j|^2 \right)^{1/2}$

   iii) $\|x\|_1 = \sum_{j=1}^{n} |x_j|$

   iv) Let C be any non-singular $n \times n$ matrix then

$$\|x\|_C = \|Cx\|_\infty$$

2) Setting: $V = C^0[a, b]$, $F = R$

   v) $\|f\|_\infty = \max_{a \leq t \leq b} \|f(t)\|$

   vi) $\|f\|_2 = \left(\int_a^b |f(t)|^2 dt \right)^{1/2}$

   vii) $\|f\|_1 = \int_a^b |f(t)| dt$

To show that 1-i) satisfies the conditions for a norm consider:

$$\|x\|_\infty > 0 \quad \text{for } x \neq 0 \qquad \text{and} \qquad \|0\|_\infty = 0 \text{ by inspection}$$

$$
\begin{aligned}
\|\alpha x\|_\infty &= \max_i |\alpha x_i| = |\alpha| \max_i |x_i| = |\alpha| \|x\|_\infty \\
\|x + y\|_\infty &= \max_i |x_i + y_i| \le \max_i \{|x_i| + |y_i|\} \\
&\le \max_i |x_i| + \max_i |y_i| = \|x\|_\infty + \|y\|_\infty .
\end{aligned}
$$

The verification of the norm properties for 1-iii) and 1-iv) as well as for 2-v) and 2-vii) is also straightforward. However, the triangle inequalities for 1-ii) and 2-vi) are not obvious and will only be considered after we have introduced inner products.

Examples:

i) $\|x\|_2$ for $x \in \mathbb{R}_3$ is just the Euclidean length of x.

ii) $\|x\|_1$ is nicknamed the taxicab (or Manhattan norm) of a vector in $\mathbb{R}_2$.

iii) $\|f\|_2$ is related to the root mean square of f defined by $\mathrm{rms}(f) = \|f\|_2/\sqrt{b-a}$ which is used to describe, e.g., alternating electric current. To give an illustration:

The voltage of a 110V household current is modeled by

$$E(t) = E_0 \cos(\omega t - \alpha).$$

Let us consider this function as an element of $C^0[0, T]$ where $T = 2\pi/\omega$ is one period. Then

$$\|E\|_2 = \left(\|E\|_2/\sqrt{T}\right) \sqrt{T} = 110\sqrt{T}$$

where the term in parentheses is the root mean square of the voltage i.e., 110V. Since also, by direct computation, $\|E\|_2 = E_0\sqrt{T/2}$ we see that $E_0 = \sqrt{2}\,110$ and hence that the peak voltage is given by

$$\|E\|_\infty = E_0 = \sqrt{2}\,110.$$

Module 2 - Homework

1) Let $x_1 = (i, 3, 1, i)$, $x_2 = (1, 1, 2, 2)$, $x_3 = (-1, i, A, 2)$. Prove or disprove: There is a (complex) number A such that the vectors $\{x_1, x_2, x_3\}$ are linearly dependent.
2) Plot the set of vectors in $\mathbb{R}_2$ for which
   i) $\|x\|_1 = 1$
   ii) $\|x\|_2 = 1$
   iii) $\|x\|_\infty = 1$.
3) For a given vector $x = (x_1, \ldots, x_n) \in \mathbb{C}_n$ with $|x_i| \leq 1$ show that

$$f(p) \equiv \|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p}$$

   is a decreasing function of p for $p \in [1, \infty)$. Show that this result is consistent with your plots of homework problem 2.
4) Find an element x in $\text{span}\{1, t\} \subset C^0(0, 1)$ such that

$$\|x\|_1 = 1$$
$$\|x\|_2 = 1.$$

   Is this element uniquely determined or are there many such elements?
5) Show that the functions $e^{\alpha t}$ and $e^{\beta t}$ are linearly independent in $C^0(-5, 5)$ for $\alpha \neq \beta$.
6) Prove or disprove: The functions $f_1 = 1 + t$, $f_2 = 1 + t + t^2$ and $f_3 = 1 - t^2$ are linearly dependent in $C^0(-\infty, \infty)$.
7) Let $f(t) = \max\{0, t^3\}$ and $g(t) = \min\{0, t^3\}$.
   i) Show that these functions are linearly independent in $C^0(-\infty, \infty)$.
   ii) Show that the Wronskian of these two functions is always zero.
8) Let $f(t) = t(1 - t)$ and $g(t) = t^2(1 - t)$. Let $H(t)$ denote a linear combination of f and g. Show that $H(0) = H(1) = 0$ but that the functions are linearly independent.