MODULE 20

Topics: Functions of a complex variable, Cauchy-Riemann equations

We already have discussed a few functions of a complex variable $z = x + iy$, such as

$$f(z) = z$$

$$f(z) = \bar{z}$$

$$f(z) = |z|$$

$$f(z) = \operatorname{Arg} z.$$

We shall now take a broader look at complex valued functions of a complex variable. The common notation is

$$w = f(z) = u(x, y) + iv(x, y).$$

We think of f as taking a point $z = x+iy$ in the z-plane to a point $w = u+iv$ in the w-plane. Later on we shall worry about the image of sets in the z-plane under the transformation, as well as about the inverse of the mapping which takes sets in the w-plane to the z-plane.

Properties of f which do not link the real and imaginary parts are readily examined with the calculus for real valued functions. For example, if $\{z_n\}$ is a sequence of points in the z-plane such that

$$\lim_{n\to\infty} f(z_n) = A, \qquad \lim_{n\to\infty} g(z_n) = B$$

then

$$\lim_{n\to\infty} [f(z_n)g(z_n)] = AB$$

and

$$\lim_{n\to\infty} [f(z_n)/g(z_n)] = A/B, \quad \text{provided } B \neq 0.$$

Something essentially new is introduced when we talk about differentiation of f because as we shall see there is now a link between u and v.

Definition: The derivative of f at z is

$$f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

provided this limit exists and is independent of how Δz is chosen.

While the definition of the derivative is familiar from the calculus for real valued functions the independence of the limiting value of how $\Delta z \to 0$ means that we can try to find $f'(z)$ for choices like

$$\Delta z = \Delta x$$

$$\Delta z = i\Delta y$$

$$\Delta z = \Delta t^\alpha + i\Delta t^\beta \quad \text{where } \Delta t \to 0 \text{ and } \alpha, \beta > 0.$$

Applications:

1) Suppose $f(z) = z$ then

$$f'(z) = \lim_{\Delta z \to 0} \frac{z + \Delta z - z}{\Delta z} = 1.$$

2) Suppose $f(z) = \bar{z}$ then for

$$\Delta z = \Delta x : \frac{\overline{z + \Delta x - z}}{\Delta x} = 1; \qquad z = i\Delta y : \frac{\overline{z + iy - z}}{i\Delta y} = -1.$$

There cannot be the same limiting value for all Δz, so $f(z) = \bar{z}$ is not differentiable anywhere.

3) Suppose $f(z) = \text{Arg}(z)$, then for $z = re^{i\theta}$ and $\Delta z = re^{i(\theta+\Delta\theta)} - re^{i\theta}$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta\theta \to 0} \frac{\theta + \Delta\theta - \theta}{re^{i\theta}(e^{i\Delta\theta} - 1)} = \lim_{\Delta\theta \to 0} \frac{\Delta\theta}{re^{i\theta}[\cos\Delta\theta + i\sin\Delta\theta - 1]} = \frac{1}{iz}$$

as may be seen, e.g., from l'Hospital's rule by differentiating with respect to $\Delta\theta$. But if $\Delta z = (r + \Delta r)e^{i\theta} - re^{i\theta}$ then $f(z + \Delta z) - f(z) = 0$, so the limit depends on Δz and f is not differentiable.

If f and g are differentiable then a manipulation of limits as in the real case will yield the familiar formulas

$$\frac{d}{dz}(f(z) + g(z)) = f'(z) + g'(z), \qquad (f(z)g(z))' = f'(z)g(z) + f(z)g'(z)$$

$$\left(\frac{f(z)}{g(z)}\right)' = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}, \qquad (f(g(z))' = f'(g(z))g'(z).$$

Application:

$$\frac{d}{dz} z^n = nz^{n-1}.$$

We use induction:

$$(zz)' = z'z + zz' = 2z,$$

hence the formula is true for $n = 2$; suppose the formula is true for arbitrary k then

$$(z^{k+1})' = (z^k z)' = kz^{k-1}z + z^k z' = (k+1)z^k.$$

Hence the formula is true for $k + 1$ and thus for all integers n. It follows that polynomials $P_m(z)$, $Q_n(z)$ and rational functions $P_m(z)/Q_n(z)$ are differentiable.

The Cauchy-Riemann equations:

In general it is not feasible to take the limit of the difference quotient to determine whether f is differentiable and to find its derivative. A more mechanical method is based on the following observation. Suppose that f is differentiable then necessarily

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) + iv(x + \Delta x, y) - (u(x, y) + iv(x, y))}{\Delta x} = \frac{\partial u(x, y)}{\partial x} + i\frac{\partial v(x, y)}{\partial x}$$

and

$$f'(z) = \lim_{i\Delta y \to 0} \frac{u(x, y + \Delta y) + iv(x, y + \Delta y) - (u(x, y) + iv(x, y)}{i\Delta y} = -i\,\frac{\partial u(x, y)}{\partial y} + \frac{\partial v(x, y)}{\partial y}\,.$$

Since two complex numbers are the same only if the real and imaginary parts are the same we see that if f is differentiable at the point z then necessarily

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,.$$

These two equations are the Cauchy-Riemann equations and must be satisfied if f is differentiable at a point. Without proof we state the condition under which the Cauchy-Riemann equations guarantee that f is differentiable.

Theorem: *Suppose the partial derivatives of u and v exist and are continuous in a neighborhood of the point z. If they satisfy the Cauchy-Riemann equations at z then f is differentiable at z.*

Examples: $f(z) = x - iy$ was already shown not to be differentiable anywhere. We find that $u_x = 1$, $u_y = 0$, $v_x = 0$, $v_y = -1$. Hence the partial derivatives are continuous everywhere but satisfy the Cauchy-Riemann equation $u_x = v_y$ nowhere. $f(z) = |z| = \sqrt{x^2+y^2}$ has continuous non-zero derivatives u_x and u_y for all $z \neq 0$, but $v_x = v_y = 0$ so the Cauchy-Riemann equations hold nowhere.

The Cauchy-Riemann equation can be transformed into polar coordinates since

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

where $r = |z|$ and $\theta = \arg(z)$. However, instead of a formal transformation let us derive the Cauchy-Riemann equations in polar coordinates directly from the difference quotient. If we express z in polar coordinates then $z = re^{i\theta}$ and

$$f(z) = u(r,\theta) + iv(r,\theta).$$

If $\Delta z = (r+\Delta r)e^{i\theta} - re^{i\theta}$ then $\Delta z \to 0$ if and only if $\Delta r \to 0$ and

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{u(r+\Delta r,\theta)+iv(r+\Delta r,\theta)-i(u(r,\theta)+iv(r,\theta))}{\Delta r\exp(i\theta)}$$

and

$$\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{1}{\exp(i\theta)}\frac{\partial u(r,\theta)}{\partial r} + i\frac{\partial v(r,\theta)}{\partial r}\,.$$

Similarly, if $\Delta z = r\exp(i(\theta+\Delta\theta)) - r\exp(i\theta) = r\exp(i\theta)[\cos\Delta\theta - 1 + i\sin\Delta\theta]$ then

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{u(r,\theta+\Delta\theta)+iv(r,\theta+\Delta\theta)-i(u(r,\theta)+iv(r,\theta))}{r\exp(i\theta)[\cos\Delta\theta-1+i\sin\Delta\theta]}\,.$$

For $\Delta\theta = 0$ we obtain formally $0/0$, so by l'Hospital's rule with differentiation with respect to $\Delta\theta$ we obtain

$$\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{\frac{\partial u}{\partial\theta}+i\frac{\partial v}{\partial\theta}}{r\exp(i\theta)i}\,.$$

Again, these two limits must be the same so that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta}$$

and

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial\theta}\,.$$

The above theorem can be restated. If the partial derivatives with respect to r and θ exist and are continuous, and if the Cauchy-Riemann equations hold then again we may conclude that f is differentiable. Note that if $z = x + iy$ then we may take

$$f'(z) = u_x(x, y) + iv_x(x, y)$$

and if $z = r\exp(i\theta)$ then we may take

$$f'(z) = \exp(-i\theta)(u_r + iv_r).$$

The Cauchy-Riemann equations have a very important consequence. Suppose that $f(z) = u(x, y) + iv(x, y)$ is differentiable. And suppose also that u and v have continuous second partial derivatives. Then

$$u_{xx}(x, y) + u_{yy}(x, y) = 0$$

$$v_{xx}(x, y) + v_{yy}(x, y) = 0.$$

This property follows immediately from the Cauchy-Riemann equations. The order of differentiation for smooth functions can be reversed so that adding

$$u_{xx} = v_{yx}$$

$$u_{yy} = -v_{xy} = -v_{yx}$$

shows that $u_{xx} + u_{yy} = 0$. The result for v follows analogously. The equation

$$Lw(x, y) \equiv w_{xx} + w_{xx} = 0$$

is known as Laplace's equation and functions which satisfy this equation are called harmonic. Laplace's equation arises in the context of steady state heat transfer, potential flow of fluids and electrostatics. The fact that smooth u and v are harmonic means that we obtain a catalog of solutions for Laplace's equation from the differentiable complex functions. Moreover, we shall learn later that once differentiable complex functions are in fact infinitely differentiable so that continuous first partial derivatives satisfying the Cauchy-Riemann equations characterize harmonic functions u and v.

Finally, suppose that we have a harmonic function $u(x, y)$. Then u is the real part of a differentiable complex valued function $f(z)$ and we can use the Cauchy-Riemann equations to find the so-called harmonic conjugate $v(x, y)$. Suppose, for example, that

$$u(x, y) = x^3 - 3xy^2 + y.$$

It is straightforward to verify that u is harmonic. Then

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2$$

so that

$$v(x, y) = 3x^2 y - y^3 + g(x).$$

From the second Cauchy-Riemann equation follows

$$\frac{\partial v}{\partial x} = 6xy + g'(x) = -\frac{\partial u}{\partial y} = 6xy - 1$$

so that $g(x) = -x + c$ for an arbitrary constant c. Hence

$$f(z) = (x^3 - 3xy^2 + y) + i(3x^2 y - y^3 - x) + c.$$

We recognize that

$$f(z) = z^3 - iz + c$$

is another way of writing this function.

Since the real and imaginary parts of any differentiable complex valued function have to be harmonic, and since polynomials $P_n(z)$ are differentiable we see that the null space of the partial differential operator

$$Lw \equiv w_{xx}(x, y) + w_{yy}(x, y)$$

contains infinitely many linearly independent elements (e.g., $\operatorname{Re} z^n$ and $\operatorname{Im} z^n$ for any n).

The real and imaginary parts of a differentiable complex valued function of z have additional importance beyond solving Laplace's equation. They are used to define new orthogonal coordinate systems in the plane. Consider

$$f(z) = u(x, y) + iv(x, y)$$

where f is differentiable. Then

$$u(x,y) = \text{constant}$$

$$v(x,y) = \text{constant}$$

define curves in the x-y plane. For example

$$u(x,y) = u(x_0,y_0)$$

$$v(x,y) = v(x_0,y_0)$$

would be two curves passing through the point (x_0,y_0). Suppose that we can parametrize these curves so that for $R(t) = (x(t),y(t))$ and $T(s) = (x(s),y(s))$ we have $u(x(t),y(t)) = u(x_0,y_0)$ and $v(x(s),y(s)) = v(x_0,y_0)$. Then

$$\frac{d}{dt}u(x(t),y(t)) = ux' + uy' \equiv \langle \nabla u, R'(t)\rangle = 0$$

and

$$\frac{d}{dt}v(x(s),y(s)) = ux' + uy' \equiv \langle \nabla v, T'(s)\rangle = 0$$

which state that the tangents R' and T' to the curves $u =$ constant and $v =$ constant are orthogonal (in the Euclidean sense) to the gradients of u and v, i.e., that the vectors ∇u and ∇v are perpendicular to the level sets $u =$ constant and $v =$ constant. But from the Cauchy-Riemann equations we find

$$\langle \nabla u(x_0,y_0), \nabla v(x_0,y_0)\rangle = 0.$$

Hence the curves $u =$ constant and $v =$ constant cross at right angles. For example, if

$$f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$$

then the level sets through the point, say, $(2,1)$, are given by

$$x^2 - y^2 = 3$$

$$2xy = 4.$$

It is simple enough to graph these curves and observe that they cross at right angles. Hence the real and imaginary parts of smooth complex functions can be used to introduce new orthogonal coordinate systems. We shall conclude this discussion of the Cauchy-Riemann equations and their applications by considering the following problem. Suppose we need to find a root of a nonlinear equation

$$f(x) = 0.$$

In general this can only be done numerically. By far the most efficient method is Newton's method – if it works. We start with an initial guess x^0 (which generally must be close to the solution x^*) and iteratively generate a sequence $\{x^k\}$ of improved estimates. Specifically, given x^k we find x^{k+1} as the solution of the linear equation

$$Lx = 0$$

where

$$Lx \equiv f'(x^k)(x^{k+1} - x^k) + f(x^k).$$

Note that $y = Lx$ is just the equation of the tangent to f at x^k. It is known that Newton's method is equally applicable to vector systems. For example, if we are to solve

$$u(x, y) = 0$$

$$v(x, y) = 0$$

then we generate a sequence $\{(x^k, y^k)\}$ by solving for (x^{k+1}, y^{k+1}) the matrix equation

$$L\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} u_x(x^k, y^k) & u_y(x^k, y^k) \\ v_x(x^k, y^k) & v_y(x^k, y^k) \end{pmatrix} \begin{pmatrix} (x - x^k) \\ (y - y^k) \end{pmatrix} + \begin{pmatrix} u(x^k, y^k) \\ v(x^k, y^k) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consider now the problem of finding a root of

$$f(z) = 0$$

where f is a differentiable function of the complex variable z. This problem is, of course, resolvable by solving for the zeroes of u and v simultaneously as just outlined. The question now is: Can the problem also be solved as a scalar problem, albeit in complex arithmetic, by computing the solution z^{k+1} of the linear approximation

$$Lz \equiv f'(z^k)(z - z^k) + f(z^k) = 0.$$

Let us write $z^k = x^k + iy^k$ and $f'(z^k) = u_x(x^k, y^k) + iv_x(x^k, y^k)$ and isolate the real and imaginary parts of Lz. We find that $z^{k+1} = x^{k+1} + iy^{k+1}$ solves

$$\begin{pmatrix} u_x(x^k, y^k) & -v_x(x^k, y^k) \\ v_x(x^k, x^k) & u_x(x^k, y^k) \end{pmatrix} \begin{pmatrix} (x - x^k) \\ (y - y^k) \end{pmatrix} + \begin{pmatrix} u(x^k, y^k) \\ v(x^k, y^k) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

But by the Cauchy-Riemann equations $-v_x = u_y$ and $u_x = v_y$ so that the equation $Lz = 0$ is exactly the same as the real system $L\begin{pmatrix} x \\ y \end{pmatrix} = 0$. Thus Newton's method for a single scalar equation is valid for real as well as for complex variables.

Module 20 - Homework

1) Let $f(z) = e^{e^z}$, $z = re^{i\theta}$, $\theta = \text{Arg}(z)$.

   i) Verify the Cauchy-Riemann equations in Cartesian and polar coordinates. Where do they hold?

   ii) Find $f'(z)$.

2) i) Show that $g(x, y) = \sin x \cosh y$ is the real part of a differentiable function $f(z)$. Find $f(z)$.

   ii) Show that $h(x, y) = \sin x \cosh y$ is the imaginary part of a differentiable function $f(z)$. Find $f(z)$.

3) Let $x = r\cos\theta$ and $y = r\sin\theta$.

   i) Show that $\frac{\partial x}{\partial r} = \frac{\partial r}{\partial x}$.

   ii) Transform the Cauchy-Riemann equations

$$u_x = v_y$$

$$u_y = -v_x$$

   into polar coordinates with a formal change of variable.

4) Prove or disprove: Newton's method in complex arithmetic for

$$f(z) \equiv z^2 + 1 = 0$$

   cannot converge for any real initial guess z^0.

5) Let $f(z) = z^3$.

   i) Find the angle between the positive x-axis and the tangent to $\text{Re}\, f(z) = \text{Re}\, f(1+i)$ at $z = 1 + i$.

   ii) Find the tangent between the positive x-axis and the tangent to $\text{Im}\, f(z) = \text{Im}\, f(i + 1)$ at $z = 1 + i$. Verify that $\text{Re}\, f(z)$ and $\text{Im}\, f(z)$ cross at right angles at $z = 1 + i$.

6) Use the Cauchy-Riemann equations in polar coordinates to derive a single second order equation for u and v. The resulting equation is Laplace's equation in polar coordinates.

7) Let $f(z) = u + iv$ be a differentiable function such that u and v are harmonic. Let D be an open set in the z-plane and let D' be the image of D in the w-plane. Finally, suppose that

$$\phi(u, v)$$

is harmonic in D'. Use the chain rule to show that

$$\phi(u(x, y), v(x, y)) \quad \text{is harmonic in } D.$$