MODULE 21

Topics: e^z, $\sin z$, $\cos z$, $\log(z)$ and z^α

Now that we have clarified the meaning of differentiability of a complex valued function of a complex variable we may begin to extend the elementary functions of the real calculus to the complex plane. Such an extension is not necessarily unique. For example, both

$$f(z) = z \quad \text{and} \quad f(z) = \bar{z}$$

may be thought of as the extension of the identity function $f(x) = x$ to the complex plane. However, we shall always require that the complex valued function be differentiable at all x where f as a function of x is differentiable. It turns out that this makes the extension unique. For example, $f(z) = \bar{z}$ would not be an admissible extension of $f(x) = x$ to the complex plane.

The complex exponential:

Definition:

$$e^z = e^x e^{iy} = e^x[\cos y + i \sin y].$$

We observe that if $z = x$ then $e^z = e^x$. In addition

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = e^x \sin y$$

so that the partial derivatives are continuous everywhere and satisfy the Cauchy Riemann equations. Hence

$$e^z = e^x e^{iy} = e^x \cos y + i e^x \sin y$$

is the correct extension of the exponential to the whole complex plane.

Properties of the complex exponential:

i) $\frac{d}{dx} e^z = u_x + i v_x = e^z$ which we see by inspection

ii) $e^{a+b} = e^a e^b$ for any complex numbers a and b

This result is not obvious and can be shown as follows:

Define $f(z) = e^z e^{c-z}$ for constant c then by the product and chain rule $f'(z) = e^z e^{c-z} - e^z e^{c-z} = 0$. This implies that $u_x = u_y = v_x = v_y = 0$ so that $f(z)$ is constant. Thus $f(z) = f(c)$ or $e^z e^{c-z} = e^c$. The result follows if we set $c = a + b$ and $z = b$.

Note that this last result was shown without recourse to any trigonometric addition formulas. It implies

$$\begin{aligned} e^{i(\theta+\phi)} &= \cos(\theta+\phi) + i\sin(\theta+\phi) \\ &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi). \end{aligned}$$

Equating real and imaginary parts we obtain

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\sin(\theta+\phi) = \cos\theta\sin\phi + \sin\theta\cos\phi.$$

Similarly,

$$\left(e^{i\theta}\right)^n = \left(e^{in\theta}\right)$$

so that

$$(\cos\theta + i\sin\phi)^b = \cos n\theta + i\sin n\theta.$$

By equating again the real and imaginary parts we see that $\cos n\theta$ and $\sin n\theta$ can always be expressed as functions of $\cos\theta$ and $\sin\theta$.

The geometry of $w = e^z$:

$$u + iv = e^x[\cos y + i\sin y]$$

can be interpreted in the vector sense

$$(u, v) = e^x(\cos y, \sin y)$$

and implies the following:

The lines $y =$ constant in the z-plane are mapped to rays in the w-plane, but we cannot get to the origin in the w-plane. The lines $x =$ constant in the z-plane are mapped to circles of radius e^x in the w-plane.

Complex trigonometric functions:

If we define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

then it follows from the definition of the complex exponential that $\cos z$ and $\sin z$ are the standard trigonometric functions when $z = x$. Moreover, since the exponential now is known to be differentiable it follows from the chain rule that $\cos z$ and $\sin z$ are differentiable. Moreover,

$$\frac{d}{dz}\cos z = \frac{ie^{iz} - e^{-iz}}{2} = -\sin z$$

and similarly,

$$(\sin z)' = \cos z.$$

A little bit of algebra shows that

$$\sin^2 z + \cos^2 z = 1$$

just as in the real case.

Analogously we define the hyperbolic cosine and sine by:

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2}$$

which implies that

$$(\cosh z)' = \sinh z \quad \text{and} \quad (\sinh z)' = \cosh z.$$

Algebra again shows that for all z

$$\cosh^2 z - \sinh^2 z = 1.$$

However, in the complex plane the distinction between the trigonometric and hyperbolic functions is blurred because

$$\cos iy = \cosh y$$

$$\sin iy = i \sinh y.$$

In particular, this shows that the trigonometric functions are not bounded in the complex plane. For example, we can always find a z such that, e.g.,

$$\sin z = 7 \quad \text{and} \quad \cos z = -3 + i.$$

The logarithm of a complex variable:

In contrast to the exponential and the trigonometric functions the logarithm is considerably more complicated.

Definition: $w = \log z$ ($\equiv \ln z$) denotes all those numbers w which satisfy

$$e^w = z.$$

The definition simply states that the log is the inverse of the exponential. To compute w we express z in polar form

$$z = re^{i\theta} \quad \text{where } r = |z| \text{ and } \theta = \arg(z).$$

Since $w = u + iv$ we find

$$e^{u+iv} = e^u e^{iv} = re^{i\theta}.$$

This implies that $e^u = r$ and $v = \arg(z)$. Since $e^u > 0$ it follows that $\log z$ cannot be defined for $z = 0$. But for $z \neq 0 \arg(z)$ is not uniquely defined but has infinitely many values which differ by multiples of 2π. Hence $\log z$ is a multiple valued function

$$\log z = u + iv = \text{Log } r + i \arg(z)$$

where Log r denotes the real log of a positive number. It is a consequence of the definition of $\log z$ that

$$e^{\log z} = e^{\text{Log } r + i\arg(z)} = re^{i\arg(z)} = z$$

$$\log e^z = \log e^x e^{iy} = \text{Log}|e^x| + iy + 2\pi ki = z + 2\pi ki.$$

Hence the nice symmetry of one function being the inverse of the other is lost when we define the log in the above form. Moreover, the log no longer is a function in the usual sense and writing

$$\log(z + \Delta z) - \log z$$

would not make sense because $\log z$ is a whole set of numbers. Yet we wanted to extend the real logarithm differentiably to the complex plane, if possible. We had learned before that the multiple valued function $\arg(z)$ can be made a standard function if we pick a branch where $\arg(z)$ is constrained to lie in a prescribed interval of length 2π. We shall agree to choose the principal branch $\text{Arg}(z)$ and define accordingly the principal branch of the logarithm by

$$\text{Log } z = \text{Log}|z| + i\,\text{Arg}(z) = \text{Log } r + i\theta$$

where $\theta \in (-\pi, \pi]$. We observe that this notation is consistent with our earlier usage since

$$\text{Log } x = \text{Log } |x| + i0 \quad \text{for } x \text{ real and positive.}$$

It it easy to see that $\text{Log } z$ is defined for all $z \neq 0$. Moreover, away from the negative axis $x < 0$ the function is differentiable because the Cauchy-Riemann equations in polar coordinates

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}$$

and

$$\frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial u}{\partial \theta} = 0$$

hold. Across the negative axis $\text{Log } z$ shows a jump, e.g.,

$$\lim_{n\to\infty} \text{Log}(-5 - i/n) = \text{Log } 5 - i\pi$$

$$\lim_{n\to\infty} \text{Log}(-5 + i/n) = \text{Log } 5 + i\pi.$$

But away from the negative axis the function is differentiable and satisfies

$$\frac{d}{dz}\text{Log } z = \frac{1}{e^{i\theta}}\left[\frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r}\right] = \frac{1}{z}\,.$$

The complex power function z^α:

For real $x > 0$ and any α the function x^α is shorthand notation for

$$x^\alpha = e^{\alpha \text{Log } x}$$

which is perfectly well defined and straightforward to handle. Analogously, we have

Definition: For $z \neq 0$ and any α

$$z^\alpha = e^{\alpha \log z}.$$

Note that this definition implies that in general z^α is multiple valued because $\log z$ is multiple valued. For example, we already have discussed the roots of z. Recall, for $z = |z|e^{i\theta}$

$$w = z^{1/n}$$

has the n values

$$w_k = |z|^{1/n} e^{i\left(\frac{\theta+2\pi k}{n}\right)}, \qquad k = 0, 1, 2, \ldots, n-1.$$

According to the above definition we have

$$\begin{aligned} z^{1/n} &= e^{\frac{1}{n}\log z = \frac{1}{n}[\mathrm{Log}\ |z| + i\arg(z)]} \\ &= |z|^{1/n} e^{\frac{-i\arg(z)}{n}} = |z|^{1/n} e^{i\left(\frac{\theta+2\pi k}{n}\right)} \end{aligned}$$

where θ is any one particular value of $\arg(z)$. For example

$$i^i = e^{i\log i} = e^{i[\mathrm{Log}\ 1 + i(\pi/2+2\pi k)]} = e^{-(1/2+2\pi k)}$$

for $k = 0, \pm 1, \pm 2, \ldots$ Again, as a multiple valued function we cannot talk about derivatives for the complex power function; we would again have to restrict ourselves to a single valued branch of the logarithm, like the principal value branch. However, we shall not pursue this topic any further.

Module 21 - Homework

1) Plot the image of the unit square $(0,1) \times (0,1)$ in the z-plane under the transformation $w = e^z$.

2) Show that the function

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

   i) maps a circle $|z| = r$, $r \neq 1$ onto an ellipse in the w-plane.

   ii) maps the unit circle $|z| = 1$ onto the real interval $[-1, 1]$ in the w-plane.

3) Show that $\sin z = 0$ if and only if $z = n\pi$ for any integer n.

   Show that $\cos z = 0$ if and only if $z = \pi/2 + n\pi$ for any integer n.

4) Evaluate: $\log i$, $\mathrm{Log}(3+i)$, $\mathrm{Log}(3-i)$, z^0 for any $z \neq 0$, $(1+i)^{(1-i)}$.

5) Find all z such that $\sin z = 7 - 3i$.

6) Find all z such that $\cosh z = .5$.

7) Prove or disprove:

$$\log z_1 z_2 = \log z_1 + \log z_2$$

$$\mathrm{Log}\ z_1 z_2 = \mathrm{Log}\ z_1 + \mathrm{Log}\ z_2$$