MODULE 3

Topics: Inner products

The inner product of two vectors:

The inner product of two vectors $x, y \in V$, denoted by $\langle x, y\rangle$ is (in general) a complex valued function which has the following four properties:

i) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

ii) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

iii) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ where z is any other vector in V

iv) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

We note from these properties that:

$$\langle 0, y\rangle = 0 \qquad \text{for all } y$$

$$\langle x, \alpha y\rangle = \overline{\langle \alpha y, x\rangle} = \overline{\alpha\langle y, x\rangle} = \overline{\alpha}\langle x, y\rangle$$

and

$$\langle x, y+z\rangle = \overline{\langle y+z, x\rangle} = \overline{\langle y, x\rangle + \langle z, x\rangle} = \langle x, y\rangle + \langle x, z\rangle$$

For real vector spaces the complex conjugation has no effect so that the inner product is linear in the first and second argument, i.e.

$$\langle x+\alpha y, z\rangle = \langle x, z\rangle + \alpha\langle y, z\rangle$$

$$\langle x, y+\alpha z\rangle = \langle x, y\rangle + \alpha\langle x, z\rangle.$$

For complex scalars we have what is sometimes called anti-linearity in the second argument.

$$\langle x, y+\alpha z\rangle = \langle x, y\rangle + \overline{\alpha}\langle x, z\rangle.$$

Examples:

1) Setting: $V = \mathbb{R}_n$ and $F = \mathbb{R}$
   i) $\langle x, y\rangle = \sum_{j=1}^{n} x_j y_j$

      This is the familiar dot product and we shall often use the more familiar notation $x \bullet y$ instead of $\langle x, y\rangle$.

   ii) $\langle x, y\rangle = \sum_{j=1}^{n} x_j y_j w_j$ where $w_i > 0$ for $i = 1, \dots, n$.

Note that this inner product can be written as

$$\langle x, y \rangle = Wx \bullet y$$

where W is the diagonal matrix $W = \mathrm{diag}\{w_1, \ldots, w_n\}$.

iii) $\langle x, y \rangle = Cx \bullet y$ where C is a positive definite symmetric matrix. The proof that this defines an inner product will be deferred until we have discussed matrices and their eigenvalues.

2) Setting: $V = \mathbb{C}_n$ and $F = \mathbb{C}$

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j \overline{y_j}$$

This is the complex dot product and likewise is more commonly denoted by $x \bullet y$. Note that the order of the arguments now matters because the components of y are conjugated.

3) Setting: $V = C^0[D]$, $F = \mathbb{C}$

$$\langle f, g \rangle = \int_D f(x)\overline{g(x)}w(x)dx$$

where the so-called weight function w is continuous and positive except at isolated points of the given (possibly multi-dimensional) domain D. For real valued functions the conjugation has no effect and $\langle f, g \rangle = \langle g, f \rangle$.

In general when checking whether a function defined on pairs of vectors is an inner product properties ii) and iii) are easy to establish. Properties i) and iv) may require more work. For example, let us define on $\mathbb{R}_2$ the function

$$\langle x, y \rangle = Ax \bullet y$$

where A is the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Then

$$\langle x, y \rangle = x_1y_1 + x_2y_1 + x_2y_2$$

while

$$\langle y, x \rangle = Ayx = y_1x_1 + y_2x_1 + x_2y_2.$$

Hence $\langle x,y\rangle \neq \langle y,x\rangle$ for all x and y (take, for example, $x=(1,0)$ and $y=(1,1)$) and property i) does not hold. Looking ahead, if for vectors $x,y \in \mathbb{R}_n$ we require that $\langle x,y\rangle = Ax \bullet y = \langle y,x\rangle = Ay \bullet x$ for a given real $n\times n$ matrix A then A must be a symmetric matrix, i.e., $A=A^T$. However, $A=A^T$ is not sufficient to make $\langle x,y\rangle$ an inner product. For example, if $A=\text{diag}\{1,-1\}$ then A is symmetric but $\langle x,x\rangle = Ax \bullet x = 0$ for the non-zero vector $x=(1,1)$. Thus, property iv) does not hold. It turns out that this matrix A is not positive definite.

As a final example consider the following function defined on $C^0[-2,2]$

$$\langle f,g\rangle = \int_{-2}^{2} f(t)g(t)w(t)dt$$

where $w(t)=\max\{0,t^2-1\}$. We see that properties i)–iii) hold but that for the non-zero function $f(t)=\max\{0,1-t^2\}$ we obtain $\langle f,f\rangle = 0$. Clearly, the weight function w may not be zero over any interval.

The following theorem shows that inner products satisfy an important inequality which is known as Schwarz's inequality.

Theorem: *Let V be a vector space over $\mathbb{C}$ with inner product $\langle x,y\rangle$. Then for any $x,y\in V$ we have*

$$|\langle x,y\rangle|^2 \leq \langle x,x\rangle\langle y,y\rangle.$$

Proof: Let x and y be arbitrary in V. If $y=0$ then the inequality is trivially true. Hence let us assume that $y\neq 0$. Next let us choose θ such that for $\hat{x}=e^{i\theta}x$ the inner product $\langle \hat{x},y\rangle$ is real. Then from the properties of the inner product we see that

$$g(\lambda)=\langle \hat{x}-\lambda y,\hat{x}-\lambda y\rangle = \langle \hat{x},\hat{x}\rangle - 2\lambda\langle\widehat{x,y}\rangle + \lambda^2\langle y,y\rangle \geq 0$$

for all real λ. In particular, the minimum of g is ≥ 0 This minimum is achieved at

$$\lambda_\theta = \langle \hat{x},y\rangle/\langle y,y\rangle$$

so that $g(\lambda_\theta)=\langle \hat{x},\hat{x}\rangle - \langle \hat{x},y\rangle^2/\langle y,y\rangle \geq 0$ from which we obtain $|\langle \hat{x},y\rangle| \leq \langle \hat{x},\hat{x}\rangle\langle y,y\rangle$. Finally, we observe that this inequality remains unchanged when $\hat{x}$ is replaced by x since the phase factor $e^{i\theta}$ drops out.

Two illustrations:

i) It is usually shown in a first course on vectors in $\mathbb{R}_2$ that

$$x \bullet y = \|x\|_2 \|y\|_2 \cos\theta$$

where θ is the angle between the vectors x and y. Since $|\cos\theta| \leq 1$ it follows that

$$|\langle x, y\rangle| \equiv |x \bullet y| \leq \|x\|_2 \|y\|_2.$$

ii) Let D be the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$. Let w be a continuous function positive function in $C^0[D]$. For any $f, g \in C^0[D]$ define the inner product

$$\langle f, g\rangle = \int_D f(x,y)g(x,y)w(x,y)dx\,dy$$

Then for $\epsilon > 0$ we obtain from Schwarz's inequality

$$\begin{aligned} |\langle f, g\rangle|^2 &= \langle \sqrt{\epsilon} f, g/\sqrt{\epsilon}\rangle < \langle \sqrt{\epsilon} f, \sqrt{\epsilon} f\rangle \langle g/\sqrt{\epsilon}, g/\sqrt{\epsilon}\rangle \\ &= \epsilon \int_D f(x,y)^2 w(x,y)dx\,dy \cdot \frac{1}{\epsilon} \int_D g(x,y)^2 w(x,y)dx\,dy \end{aligned}$$

Theorem: *Let V be a vector space with inner product $\langle\, , \,\rangle$. Then*

$$\|x\| = \langle x, x\rangle^{1/2}$$

is a norm on V.

Proof. Properties i) and ii) of the norm are a direct consequence of properties ii) and iv) of the inner product. To establish the triangle inequality we observe that

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x\rangle + 2\,Re\langle x, y\rangle + \langle y, y\rangle \leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2.$$

By Schwarz's inequality $|\langle x, y\rangle| \leq \|x\|\,\|y\|$ so that $\|x+y\|^2 \leq (\|x\|+\|y\|)^2$ which establishes the triangle inequality and hence that $\langle x, x\rangle^{1/2} = \|x\|$ is a norm on V. For example,

$$\|f\| = \left(\int_D f(x,y)^2 w(x,y)dx\,dy\right)^{1/2}$$

defines a norm on the vector space $C^0[D]$ provided w is a positive weight function.

Definition: Two vectors x, y in a vector space V with inner product $\langle x, y\rangle$ are orthogonal if $\langle x, y\rangle = 0$.

Examples:

i) In $\mathbb{R}_2$ with inner product $x \bullet y$ two vectors are orthogonal if they are perpendicular to each other because $x \bullet y = \|x\|\|y\|\cos\theta$.

ii) Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and define $\langle x, y\rangle = Ax \bullet y$. Take for granted for the time being that $\langle x, y\rangle$ is an inner product. Let $x = (1, 0)$, then $y = (1, -2)$ satisfies $\langle x, y\rangle = 0$ and hence is orthogonal to x. Note that orthogonality in this case says nothing about the angle between the vectors.

iii) Let $\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)dt$ be the inner product on $C^0(-\pi, \pi)$ then the functions $\{\cos nt, \sin mt\}$ are all mutually orthogonal.

iv) The functions t^{2k} and t^{2n+1} for any non-negative integers k and n are also mutually orthogonal with respect to the inner product of iii).

We have repeatedly introduced norms and inner products expressed in terms of integrals for continuous functions defined on some set D. However, integrals remain defined for much more general functions, for example, functions with certain discontinuities. In fact, even the notion of the integral can be extended beyond the concept of the Riemann integral familiar to us from calculus. We shall introduce and use routinely the following vector (i.e., function) space.

Definition: $L_2(D) = \{$all functions defined on D such that $\int_D |f(x)|^2 dx < \infty\}$. We observe that if $f \in L_2(D)$ then $\alpha f \in L_2(D)$. If $f, g \in L_2(D)$ then it follows from

$$\int_D |f(x) + g(x)|^2 dx = \int_D (|f(x)|^2 + 2|f(x)g(x)| + |g(x)|^2) dx$$

and the algebraic-geometric mean inequality

$$2|f(x)g(x)| \le |f(x)|^2 + |g(x)|^2$$

that

$$\int_D |f(x) + g(x)|^2 dx \le 2\left(\int_D |f(x)|^2 dx + \int_D |g(x)|^2 dx\right) < \infty$$

so that $L_2(D)$ is closed under vector addition and scalar multiplication and hence a vector space.

We observe that $C^k[D]$ for any $k > 0$ is a subspace of $L_2(D)$ provided that D is a bounded set. Finally, we note that

$$\langle f, g\rangle = \int_D f(x)\overline{g(x)}dx \quad \text{and} \quad \|f\| = \langle f, f\rangle^{1/2}$$

define the inner product and norm usually associated with $L_2(D)$.

In general, the functions will be real and the conjugation can be ignored. On occasion the inner product and norm are modified by including a weight function $w(x) > 0$ in the integral. In this course the integral will remain to be the Riemann integral. In a more abstract setting it should be the Lebesgue integral.

Module 3 - Homework

1) $V = \mathbb{C}_n$, $F = \mathbb{C}$, $\langle x, y\rangle = x \bullet y$.

   Let $x = (x_1, \ldots, x_n)$ where $x_j = j$;

   $y = (y_1, \ldots, y_n)$ where $y_j = (1+i)j$ and $i^2 = -1$.

   Compute $\langle x, y\rangle, \langle y, x\rangle, \langle x, x\rangle, \langle y, y\rangle$.

2) i) Let A be an $m \times n$ real matrix and A^T its transpose. Show that

$$Ax \bullet y = x \bullet A^T y \quad \text{for all } x \in \mathbb{R}_n \quad \text{and } y \in \mathbb{R}_m.$$

   ii) Let A be an $m\times n$ complex matrix and A^* its conjugate transpose (i.e., $A^* = \overline{A^T}$). Show that

$$Ax \bullet y = x \bullet A^* y \quad \text{for all } x \in \mathbb{C}_n \quad \text{and } y \in \mathbb{C}_m.$$

3) Let

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

   and

$$\langle x, y\rangle = Ax \bullet y$$

   for $x, y \in \mathbb{R}_2$. Show that $\langle\ ,\ \rangle$ defines an inner product on $\mathbb{R}_2$.

4) Prove or disprove:

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt$$

   is an inner product on $C^0[-1, 1]$.

5) Show that if w is a positive continuous function on $[a, b]$ then

$$\left|\int_a^b f(t)w(t)dt\right| \le \left[\int_a^b w(t)dt \int_a^b f(t)^2 w(t)dt\right]^{1/2}$$

   and in particular that

$$\int_a^b f(t)dt \le \sqrt{b-a}\sqrt{\int_a^b f(t)^2 dt}\,.$$