Trig Formulas by Complex Analysis

■ Introduction

This is bassed on the well known formula due to Euler:

$$e^{i x} = \cos(x) + i \sin(x).$$

Therefore

$$\begin{aligned} e^{i n x} &= (e^{i \, \mathrm{nx}})^n \\ &= (\cos(x) + i \sin(x))^n \end{aligned}$$

Expanding out the power, and equating real and imaginary parts gives the results.

■ Formulas for Cos(nx)

```
In[127]:= FormulaForCosNX[n_] :=
          ( ExpandAll[ (Cos[x] + I Sin[x])^n])  /.
             {Complex[0, _] -> 0}
```

```
In[129]:= FormulaForCosNX[2]
```

Out[129]= $\cos^2(x) - \sin^2(x)$

```
In[130]:= FormulaForCosNX[3]
```

Out[130]= $\cos^3(x) - 3\cos(x)\sin^2(x)$

```
In[131]:= FormulaForCosNX[10]
```

Out[131]= $\cos^{10}(x) - 45\sin^2(x)\cos^8(x) + 210\sin^4(x)\cos^6(x) - 210\sin^6(x)\cos^4(x) + 45\sin^8(x)\cos^2(x) - \sin^{10}(x)$

```
In[132]:= FormulaForCosNX[100]
```

Out[132]=

$$\begin{aligned}
&\cos^{100}(x) - 4950 \sin^2(x) \cos^{98}(x) + \\
&3921225 \sin^4(x) \cos^{96}(x) - 1192052400 \sin^6(x) \cos^{94}(x) + 186087894300 \sin^8(x) \cos^{92}(x) - \\
&17310309456440 \sin^{10}(x) \cos^{90}(x) + 1050421051106700 \sin^{12}(x) \cos^{88}(x) - 44186942677323600 \sin^{14}(x) \cos^{86}(x) + \\
&1345860629046814650 \sin^{16}(x) \cos^{84}(x) - 30664510802988208300 \sin^{18}(x) \cos^{82}(x) + \\
&535983370403809682970 \sin^{20}(x) \cos^{80}(x) - 7332066885177656269200 \sin^{22}(x) \cos^{78}(x) + \\
&79776075565900368755100 \sin^{24}(x) \cos^{76}(x) - 699574816500972464467800 \sin^{26}(x) \cos^{74}(x) + \\
&4998813702034726525205100 \sin^{28}(x) \cos^{72}(x) - 29372339821610944823963760 \sin^{30}(x) \cos^{70}(x) + \\
&143012501349174257560226775 \sin^{32}(x) \cos^{68}(x) - 580717429720889409486981450 \sin^{34}(x) \cos^{66}(x) + \\
&1977204582144932989443770175 \sin^{36}(x) \cos^{64}(x) - 5670048986634686922786117600 \sin^{38}(x) \cos^{62}(x) + \\
&13746234145802811501267369720 \sin^{40}(x) \cos^{60}(x) - 28258808871162574166368460400 \sin^{42}(x) \cos^{58}(x) + \\
&49378235797073715747364762200 \sin^{44}(x) \cos^{56}(x) - 73470998190814997343905056800 \sin^{46}(x) \cos^{54}(x) + \\
&93206558875049876949581681100 \sin^{48}(x) \cos^{52}(x) - 100891344545564193334812497256 \sin^{50}(x) \cos^{50}(x) + \\
&93206558875049876949581681100 \sin^{52}(x) \cos^{48}(x) - 73470998190814997343905056800 \sin^{54}(x) \cos^{46}(x) + \\
&49378235797073715747364762200 \sin^{56}(x) \cos^{44}(x) - 28258808871162574166368460400 \sin^{58}(x) \cos^{42}(x) + \\
&13746234145802811501267369720 \sin^{60}(x) \cos^{40}(x) - 5670048986634686922786117600 \sin^{62}(x) \cos^{38}(x) + \\
&1977204582144932989443770175 \sin^{64}(x) \cos^{36}(x) - 580717429720889409486981450 \sin^{66}(x) \cos^{34}(x) + \\
&143012501349174257560226775 \sin^{68}(x) \cos^{32}(x) - 29372339821610944823963760 \sin^{70}(x) \cos^{30}(x) + \\
&4998813702034726525205100 \sin^{72}(x) \cos^{28}(x) - 699574816500972464467800 \sin^{74}(x) \cos^{26}(x) + \\
&79776075565900368755100 \sin^{76}(x) \cos^{24}(x) - 7332066885177656269200 \sin^{78}(x) \cos^{22}(x) + \\
&535983370403809682970 \sin^{80}(x) \cos^{20}(x) - 30664510802988208300 \sin^{82}(x) \cos^{18}(x) + \\
&1345860629046814650 \sin^{84}(x) \cos^{16}(x) - 44186942677323600 \sin^{86}(x) \cos^{14}(x) + 1050421051106700 \sin^{88}(x) \cos^{12}(x) - \\
&17310309456440 \sin^{90}(x) \cos^{10}(x) + 186087894300 \sin^{92}(x) \cos^8(x) - 1192052400 \sin^{94}(x) \cos^6(x) + \\
&3921225 \sin^{96}(x) \cos^4(x) - 4950 \sin^{98}(x) \cos^2(x) + \sin^{100}(x)
\end{aligned}$$

■ Formulas For Sin(nx)

```
In[136]:= FormulaForSinNX[n_] :=
          ( ExpandAll[(-I) (Cos[x] + I Sin[x])^n])  /.
             {Complex[0, _] -> 0}
```

```
In[137]:= FormulaForSinNX[2]
```

Out[137]= $2 \cos(x) \sin(x)$

```
In[138]:= FormulaForSinNX[3]
```

Out[138]= $3 \cos^2(x) \sin(x) - \sin^3(x)$

```
In[139]:= FormulaForSinNX[10]
```

Out[139]= $10 \sin(x) \cos^9(x) - 120 \sin^3(x) \cos^7(x) + 252 \sin^5(x) \cos^5(x) - 120 \sin^7(x) \cos^3(x) + 10 \sin^9(x) \cos(x)$

In[140]:= `FormulaForSinNX[100]`

Out[140]=

$$
\begin{aligned}
&100 \sin(x) \cos^{99}(x) - 161700 \sin^{3}(x) \cos^{97}(x) + 75287520 \sin^{5}(x) \cos^{95}(x) - 16007560800 \sin^{7}(x) \cos^{93}(x) + \\
&1902231808400 \sin^{9}(x) \cos^{91}(x) - 141629804643600 \sin^{11}(x) \cos^{89}(x) + 7110542499799200 \sin^{13}(x) \cos^{87}(x) - \\
&253338471349988640 \sin^{15}(x) \cos^{85}(x) + 6650134872937201800 \sin^{17}(x) \cos^{83}(x) - \\
&132341572939212267400 \sin^{19}(x) \cos^{81}(x) + 2041841411062132125600 \sin^{21}(x) \cos^{79}(x) - \\
&24865270306254660391200 \sin^{23}(x) \cos^{77}(x) + 242519269720337121015504 \sin^{25}(x) \cos^{75}(x) - \\
&1917353200780443050763600 \sin^{27}(x) \cos^{73}(x) + 12410847811948286545336800 \sin^{29}(x) \cos^{71}(x) - \\
&66324638306863423796047200 \sin^{31}(x) \cos^{69}(x) + 294692427022540894366527900 \sin^{33}(x) \cos^{67}(x) - \\
&1095067153187962886461165020 \sin^{35}(x) \cos^{65}(x) + 3420029547493938143902737600 \sin^{37}(x) \cos^{63}(x) - \\
&9013924030034630492634340800 \sin^{39}(x) \cos^{61}(x) + 20116440213369968050635175200 \sin^{41}(x) \cos^{59}(x) - \\
&38116532895986727945334202400 \sin^{43}(x) \cos^{57}(x) + 61448471214136179596720592960 \sin^{45}(x) \cos^{55}(x) - \\
&84413487283064039501507937600 \sin^{47}(x) \cos^{53}(x) + 98913082887808032681188722800 \sin^{49}(x) \cos^{51}(x) - \\
&98913082887808032681188722800 \sin^{51}(x) \cos^{49}(x) + 84413487283064039501507937600 \sin^{53}(x) \cos^{47}(x) - \\
&61448471214136179596720592960 \sin^{55}(x) \cos^{45}(x) + 38116532895986727945334202400 \sin^{57}(x) \cos^{43}(x) - \\
&20116440213369968050635175200 \sin^{59}(x) \cos^{41}(x) + 9013924030034630492634340800 \sin^{61}(x) \cos^{39}(x) - \\
&3420029547493938143902737600 \sin^{63}(x) \cos^{37}(x) + 1095067153187962886461165020 \sin^{65}(x) \cos^{35}(x) - \\
&294692427022540894366527900 \sin^{67}(x) \cos^{33}(x) + 66324638306863423796047200 \sin^{69}(x) \cos^{31}(x) - \\
&12410847811948286545336800 \sin^{71}(x) \cos^{29}(x) + 1917353200780443050763600 \sin^{73}(x) \cos^{27}(x) - \\
&242519269720337121015504 \sin^{75}(x) \cos^{25}(x) + 24865270306254660391200 \sin^{77}(x) \cos^{23}(x) - \\
&2041841411062132125600 \sin^{79}(x) \cos^{21}(x) + 132341572939212267400 \sin^{81}(x) \cos^{19}(x) - \\
&6650134872937201800 \sin^{83}(x) \cos^{17}(x) + 253338471349988640 \sin^{85}(x) \cos^{15}(x) - \\
&7110542499799200 \sin^{87}(x) \cos^{13}(x) + 141629804643600 \sin^{89}(x) \cos^{11}(x) - 1902231808400 \sin^{91}(x) \cos^{9}(x) + \\
&16007560800 \sin^{93}(x) \cos^{7}(x) - 75287520 \sin^{95}(x) \cos^{5}(x) + 161700 \sin^{97}(x) \cos^{3}(x) - 100 \sin^{99}(x) \cos(x)
\end{aligned}
$$