Initial Boundary Value Problem For Compressible Euler Equations With Damping

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Abstract. We construct global L^∞ entropy weak solutions to the initial boundary value problem for the damped compressible Euler equations on bounded domain with physical boundaries. Time asymptotically, the density is conjectured to satisfy the porous medium equation and the momentum obeys to the classical Darcy's law. Based on entropy principle, we showed that the physical weak solutions converges to steady states exponentially fast in time. We also proved that the same is true for the related initial boundary value problems of porous medium equation and thus justified the validity of Darcy's law in large time.

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1 Introduction

We consider the compressible Euler equation with frictional damping:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = -\alpha \rho u. \end{cases} \tag{1.1}$$

Such a system occurs in the mathematical modelling of compressible flow through a porous medium. Here ρ, u, and P denote the density, velocity and pressure; the constant $\alpha > 0$ models friction. Assuming the flow is a polytropic perfect gas, then $P(\rho) = P_0\rho^\gamma$, $\gamma > 1$, with P_0 a positive constant, and γ the adiabatic gas exponent. Without loss of generality, we take $P_0 = \frac{1}{\gamma}$, $\alpha = 1$ throughout this paper.

After introducing the momentum $m = \rho u$, we can rewrite (1.1) as follows:

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + \Big(\dfrac{m^2}{\rho} + P(\rho)\Big)_x = -m. \end{cases} \tag{1.2}$$

The system (1.2) is supplemented by the following initial value and boundary conditions:

$$\begin{cases} \rho(x,0) = \rho_0(x), \ m(x,0) = m_0(x), \quad 0 < x < 1, \\ m(0,t) = 0, \quad m(1,t) = 0, \quad t \geq 0, \\ \displaystyle\int_0^1 \rho_0(x)\, dx = \rho_* > 0. \end{cases} \tag{1.3}$$

Where, the last condition is imposed to avoid the trivial case, $\rho \equiv 0$.

For large time, it is conjectured that Darcy's law is valid and (1.2) is well approximated by the decoupled system

$$\begin{cases} \tilde{\rho}_t = P(\tilde{\rho})_{xx}, \\ \tilde{m} = -P(\tilde{\rho})_x. \end{cases} \tag{1.4}$$

Where, the first equation is well-known porous medium equation while the second equation states Darcy's law. The initial boundary conditions turn into

$$\begin{cases} \tilde{\rho}(x,0) = \tilde{\rho}_0(x), \quad 0 < x < 1, \\ P_x|_{x=0} = 0, \quad P_x|_{x=1} = 0, \quad t \geq 0. \end{cases} \tag{1.5}$$

When the initial data is small smooth and is away from vacuum, the global existence and large time behavior of the solutions to (1.2)–(1.3) were established by [16], [17]. However, when initial data is large or rough, shock will develop in finite time [46], and one has to consider weak entropy solutions. One of the main difficulties is that the weak solution may contain the vacuum state, where the system (1.2) experiences resonance since two family of characteristics coincide, [24], [25] and [26]. In this paper, we will first construct L^∞ weak entropy solution to (1.2)–(1.3) for physical initial data, and then prove that any L^∞ entropy weak solution of (1.2)–(1.3) converges exponentially to equilibrium state. We then prove that the solutions of the related diffusion problem (1.4)–(1.5) tend to the same equilibrium state exponentially fast in time provided that

$$\int_0^1 \tilde{\rho}_0(x)dx = \int_0^1 \rho_0(x)dx. \tag{1.6}$$

We thus justified the validity of Darcy's law in large time.

Due to strong physical background and significant mathematical challenge, system (1.2) and its time-asymptotic behavior have received considerable attentions. Intensive literatures are available for Cauchy problem. In this direction, the readers are referred to [31], [13], [14] and [12] for existence of small smooth solutions; to [27], [4], and [6] for solutions in BV; to [7] and [20] for L^∞ solutions. For large time behavior of solutions, we refer [13], [14], [33], [34] and [45] fore small smooth solutions; and we refer [19], [21], [39] and [47] for weak solutions. For initial boundary value problems, see [16], [28] and [35] for small smooth solutions. There are also some results on non-isentropic flows, see [15], [17], [18], [29], [36] and [37].

In this paper, we continue the study of [16] and [17] on bounded domain with typical physical boundary condition (1.3). We will study the global existence and large time behavior of weak solutions. The existence of entropy weak solutions will be achieved by means of Godunov scheme [10] and the compensation compactness frameworks established by [7], [9], [22], [23], [30] and [42]. The proof is in the spirit of [44] and [38]. For the large time behavior, we adopt the new framework introduced by [19] and [20] based on entropy dissipation. The exponential decay rates are obtained in this case on bounded domain.

The plan of the rest of this paper is as follows. In section 2, we give some elementary notions and basic facts that will be used in this paper. The main results will be stated. In section 3, we construct the approximate solutions and prove the uniform L^∞ bound for the approximate solutions. In section 4, the compensated compactness theory will be applied to the approximate solutions to show the convergence, up to subsequence, to an entropy weak solution. The boundary conditions are verified in the sense of trace. Finally, we will prove the large time behavior of any entropy L^∞ weak solution and decay rates in section 5.

2 Preliminaries and Main results

In this section, we first introduce some basic facts about system (1.2) and the homogeneous compressible Euler equations. For more details, see [5], and [40]. It is convenient to use vector form of the the systems. Set

$$v = (\rho, m)^T,\ f(v) = \Big(m, \frac{m^2}{\rho} + \frac{\rho^\gamma}{\gamma}\Big)^T,\ g(v) = (0, -m)^T, \tag{2.1}$$

we rewrite (1.2)–(1.3) as

$$\begin{cases} v_t + f(v)_x = g(v), \\ v(x,0) = v_0(x),\ x \in (0,1), \\ m(0,t) = m(1,t) = 0. \end{cases} \tag{2.2}$$

Clearly, the Jacobian matrix of flux f is

$$\nabla f = \begin{pmatrix} 0 & 1 \\ -\frac{m^2}{\rho^2} + \rho^{\gamma-1} & \frac{2m}{\rho} \end{pmatrix}, \tag{2.3}$$

which has eigenvalues

$$\lambda_1 = \frac{m}{\rho} - \rho^\theta,\quad \lambda_2 = \frac{m}{\rho} + \rho^\theta, \tag{2.4}$$

and the so-called Riemann invariants are

$$w = \frac{m}{\rho} + \frac{\rho^\theta}{\theta},\quad z = \frac{m}{\rho} - \frac{\rho^\theta}{\theta}, \tag{2.5}$$

where $\theta = \frac{\gamma-1}{2}$.

We now give the definition of the weak solutions of (1.2)–(1.3), or equivalently (2.2).

Definition 2.1. *For every $T > 0$, we define a weak solution of* (1.2)-(1.3) *to be a pair of bounded measurable functions $v(x,t) = \big(\rho(x,t), m(x,t)\big)$ satisfying the following pair of integral identities*:

$$\int_0^T \int_0^1 (\rho\psi_t + m\psi_x)\, dx\, dt + \int_{t=0} \rho_0 \psi\, dx = 0, \tag{2.6}$$

$$\int_0^T \int_0^1 \Big(m\psi_t + \Big(\frac{m^2}{\rho} + P(\rho)\Big)\psi_x\Big)\, dx\, dt - \int_0^T \int_0^1 m\psi\, dx\, dt + \int_{t=0} m_0 \psi\, dx = 0, \tag{2.7}$$

for all $\psi \in C_0^\infty(I_T)$ satisfying $\psi(x,T) = 0$ for $0 \le x \le 1$ and $\psi(0,t) = \psi(1,t) = 0$ for $t \ge 0$, where $I_T = (0,1) \times (0,T)$, and $\frac{m}{\rho}$ vanishes when $\rho = 0$. *Moreover, m satisfy the initial boundary condition* (1.3) *in the sense of trace, defined in (4.8) below.*

An interesting feature of nonlinear hyperbolic balance laws is that when weak solution is concerned, the uniqueness is lost. In order to select the physical relevant solutions, one often imposes entropy admissible conditions. We now define the entropy and entropy flux pairs.

Definition 2.2. A pair of mappings $\eta: \ \mathbb{R}^2 \to \mathbb{R}$ and $q: \ \mathbb{R}^2 \to \mathbb{R}$ is called an *entropy-entropy flux pair* if it satisfies the following equation

$$\nabla q = \nabla \eta \nabla f.$$

Let $\tilde{\eta}(\rho, m/\rho) = \eta(\rho, m)$. If $\tilde{\eta}(0, u) = 0$, then η is called a weak entropy.

Among all entropies, the most natural entropy is the mechanical energy

$$\eta_e(\rho, m) = \frac{m^2}{2\rho} + \frac{\rho^\gamma}{\gamma(\gamma - 1)}, \tag{2.8}$$

which plays a very important role in estimates for entropy dissipation measures. It is easy to check that η_e is a weak and convex entropy.

Definition 2.3. The weak solution $v(x,t) = \big(\rho(x,t), m(x,t)\big)$ defined in Definition 2.1 is said to be *entropy admissible* if for any convex entropy η and associated entropy flux q, the following *entropy inequality* holds

$$\eta_t + q_x + \eta_m m \leq 0, \tag{2.9}$$

in the sense of distribution.

Typically, in order to construct approximate solutions to non-homogeneous hyperbolic systems, fractional step scheme (operator splitting) is applied. In each time step, one first solves the associated homogeneous system, then apply the ODE correction ignoring fluxes. In this paper, we will use many results of the homogeneous compressible Euler equations:

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + \Big(\dfrac{m^2}{\rho} + P(\rho)\Big)_x = 0. \end{cases}$$

or equivalently,

$$v_t + f(v)_x = 0. \tag{2.10}$$

One of the building blocks is the Riemann problem

$$\begin{cases} (2.10), \quad t > 0, \quad x \in \mathbb{R}, \\ (\rho, m)|_{t=0} = \begin{cases} (\rho_l, m_l), \ x < 0, \\ (\rho_r, m_r), \ x > 0, \end{cases} \end{cases} \tag{2.11}$$

where $\rho_l,\ \rho_r,\ m_l$, and m_r are constants satisfying $0 \leq \rho_l,\ \rho_r,\ |m_l/\rho_l|,\ |m_r/\rho_r| < \infty$. There are two distinct types of rarefaction waves and shock waves, called *elementary waves* , which are labelled 1-rarefaction or 2-rarefaction waves and 1-shock or 2-shock waves, respectively.

Lemma 2.1. *There exists a global weak entropy solution of* (2.11) *which is piecewise smooth function satisfying*

$$
\begin{aligned}
&w(x,t) = w(\frac{x}{t}) \leq \max\{w(\rho_l, m_l), w(\rho_r, m_r)\},\\
&z(x,t) = z(\frac{x}{t}) \geq \min\{z(\rho_l, m_l), z(\rho_r, m_r)\},\\
&w(x,t) - z(x,t) \geq 0.
\end{aligned}
$$

It follows that the region $\Lambda = \{(\rho, m):\ \ w \leq w_0,\ z \geq z_0,\ w - z \geq 0\}$ is an invariant region for the Riemann problem (2.11). More precisely, if the Riemann data lies in Λ, then the solution of (2.11) lies in Λ, too.

Lemma 2.2. *If* $\{(\rho, m):\ a \leq x \leq b\} \subset \Lambda$, *then*

$$
\left(\frac{1}{b-a}\int_a^b \rho\ dx,\ \frac{1}{b-a}\int_a^b m\ dx\right) \in \Lambda. \tag{2.12}
$$

Account to boundary in our problem, the boundary Riemann solver is applied.

Lemma 2.3. *For the mixed problem*

$$
\begin{cases}
(2.10), & t > 0,\ \ x > 0,\\
(\rho, m)|_{t=0} = (\rho_0, m_0), & x > 0,\\
m|_{x=0} = 0, & t \geq 0,
\end{cases} \tag{2.13}
$$

where (ρ_0, m_0) *are constants, there exists a weak entropy solution in the region* $\big\{(x,t):\ x \geq 0,\ t \geq 0\big\}$ *satisfying the following estimates*

$$
\begin{aligned}
&w(x,t) \leq \max\{w(\rho_0, m_0), -z(\rho_0, m_0)\},\\
&z(x,t) \geq z(\rho_0, m_0), \quad \text{and} \quad w(x,t) - z(x,t) \geq 0.
\end{aligned}
$$

The term $-z(\rho_0, m_0)$ is new to the mixed problem because of the shock waves reflecting off or coming out at the boundary $x = 0$. Similar to (2.13), we can solve the following mixed problem in the region $\big\{(x,t):\ x \leq 1,\ t \geq 0\big\}$:

$$
\begin{cases}
(2.10), & t > 0,\ \ x < 1,\\
(\rho, m)|_{t=0} = (\rho_0, m_0), & x < 1,\\
m|_{x=1} = 0, & t \geq 0,
\end{cases} \tag{2.14}
$$

The weak entropy solution of (2.14) satisfies the following estimates:

$$\begin{aligned}&z(x,t) \geq \min\{z(\rho_0,m_0), -w(\rho_0,m_0)\},\\ &w(x,t) \leq w(\rho_0,m_0), \quad \text{and} \quad w(x,t)-z(x,t) \geq 0.\end{aligned}$$

Lemma 2.4. *Suppose that* $\big(\rho(x,t), m(x,t)\big)$ *is a solution of* (2.11) *or* (2.13) *and or* (2.14). *Then, the jump strength of* $m(x,t)$ *across an elementary wave can be dominated by that of* $(\rho(x,t))$ *across the same elementary wave,* i.e.,

$$\begin{aligned}&\text{across a shock wave :} \qquad |m_r - m_l| \leq C|\rho_r - \rho_l|,\\ &\text{across a rarefaction wave :} \quad |m - m_l| \leq C|\rho - \rho_l| \leq C|\rho_r - \rho_l|,\end{aligned}$$

where C *depends only on the bounds of* ρ *and* $|m|$.

Lemma 2.5. *For any* $\varepsilon > 0$, *there exist constants* $h > 0$ *and* $k > 0$ *such that the solution of* (2.11) *in the region* $\big\{(x,t): \ |x| < h, \ 0 \leq t < k\big\}$ *satisfies*

$$\int_{-h}^{h} |\rho(x,t) - \rho(x,0)| \, dx \ \leq \ Ch\varepsilon, \quad 0 \leq t \leq k, \tag{2.15}$$

where C *depends only on the bounds of* ρ *and* $|m|$, *and the mesh lengths* h *and* k *satisfy* $\max_{i=1,2} \ \sup|\lambda_i(\rho,m)| < \frac{h}{2k}$.

The following two theorems are the main results of this paper.

Theorem 1. *Suppose that the initial data* (ρ_0, m_0) *satisfy the conditions*

$$0 \leq \rho_0(x) \leq M_1, \quad \rho_0 \not\equiv 0, \quad |m_0(x)| \leq M_2\rho_0(x),$$

for some positive constants $M_i (i = 1,2)$. *Then, for* $\gamma > 1$, *the initial-boundary value problem* (1.2)-(1.3) *has a global weak solution* $(\rho(x,t), m(x,t))$, *as defined in Definition 2.1, satisfying the following estimates and entropy condition*:

$$\begin{aligned}&0 \leq \rho \leq C, \quad |m| \leq C\rho \ \ \text{a.e. for a constant } C>0, \text{ and}\\ &\int_0^T\int_0^1 \big(\eta(\rho,m)\tilde{\psi}_t + q(\rho,m)\tilde{\psi}_x\big)\,dx\,dt - \int_0^T\int_0^1 \eta_m(\rho,m)m\tilde{\psi}\,dx\,dt \geq 0,\end{aligned} \tag{2.16}$$

for all weak and convex entropy pairs (η, q) *for* (1.2)-(1.3) *and for all nonnegative smooth functions* $\tilde{\psi} \in C_0^1(I_T)$.

Theorem 2. *Suppose* $\displaystyle\int_0^1 \rho_0(x)dx = \rho_*$. *Let* (ρ, m) *be any* L^∞ *entropy weak solution of the initial boundary problem* (1.2)-(1.3) *defined in Definition 2.1, satisfying the estimates*

$$0 \leq \rho(x,t) \leq \Lambda < \infty, \quad |m(x,t)| \leq M_1\rho(x,t),$$

where M_1, Λ *are positive constants and let* $(\tilde{\rho}, \tilde{m})$ *be the weak solution of* (1.4)–(1.5) *with mass* ρ_* *and* $\tilde{m} = -P(\tilde{\rho})_x$*. Then, there exist constants* $C,\ \delta > 0$ *depending on* $\gamma,\ \rho_*, \Lambda,$ *and initial data such that*

$$\big\| \big((\rho - \tilde{\rho}),\ (m - \tilde{m}) \big)(\cdot, t) \big\|^2_{L^2([0,1])} \le C e^{-\delta t}. \tag{2.17}$$

The proof of Theorem 1 is in the spirit of [44] and [38]. In Section 3, we construct the approximate solutions v_h derived by the Godunov scheme [10]. The L^∞ norm of approximate solutions is established. The compensated compactness framework is then applied to the sequence of approximate solutions to obtain a global weak entropy solution in section 4. The boundary conditions are verified in the sense of trace.

In section 5, we prove Theorem 2, the exponential decay rate of the L^2-norm of the difference between solutions of (1.2)-(1.3) and (1.4)–(1.5). We will see that an easy lemma plays an important role. It should be pointed out that the key approach in [13—21] is to compare the solution of (1.2)-(1.3) with the similarity solution of (1.4) via energy estimates. Unfortunately, the exponential decay rate cannot be achieved by this approach, due to the boundary effects. Instead of comparing two solutions directly, we first show that the large time asymptotic state for both solutions is a constant state $(\rho_*, 0)$ and both solutions tend to the constant state exponentially fast. Hence by the triangular inequality we can see that the solution of (1.2)-(1.3) tends to that of (1.4)–(1.5) exponentially fast as time goes to infinity.

3 Approximate solutions

The approximate solutions will be constructed by Godunov scheme [10] with operator splitting. We choose the space mesh length $h = \frac{1}{N}$, where N is a positive integer. The time mesh length $k = k(h)$ will be chosen later so that the Courant-Friedrich-Levy condition

$$\max_{i=1,2} \left(\sup |\lambda_i(v)| \right) < \frac{h}{2k} \tag{3.1}$$

holds for a given $T > 0$. We partition the interval $[0, 1]$ into cells, with the j^{th} cell centered at $x_j = jh,\ j = 1, \cdots, N-1$. Set $x_0 = 0$ and $x_N = 1$. We now use the Godunov scheme to construct a sequence of approximate solutions of (2.2). Namely, we solve the Riemann problems (2.11) in the region $R^1_j \equiv \big\{(x,t):\ x_{j-\frac{1}{2}} \le x < x_{j+\frac{1}{2}},\ 0 \le t < k\big\}$:

$$\begin{cases} \dfrac{\partial}{\partial t}\, \underline{v}_h \ + \ \dfrac{\partial}{\partial x}\, f(\underline{v}_h) \ = \ 0, \\ \underline{v}_h|_{t=0} = \begin{cases} (\rho^0_j, m^0_j), & x < x_j, \\ (\rho^0_{j+1}, m^0_{j+1}), & x > x_j, \quad j = 1, \cdots,\ N-1, \end{cases} \end{cases}$$

where

$$\rho_j^0 = \frac{1}{h}\int_{x_{j-1}}^{x_j} \rho_0(x)\,dx, \quad m_j^0 = \frac{1}{h}\int_{x_{j-1}}^{x_j} m_0(x)\,dx, \quad \text{for } j = 1, \cdots, N.$$

We also solve the mixed problems (2.13) and (2.14) with (ρ_1^0, m_1^0) and (ρ_N^0, m_N^0), in regions $\big\{(x,t):\ 0 \le x < x_{\frac{1}{2}},\ 0 \le t < k\big\}$ and $\big\{(x,t):\ x_{N-\frac{1}{2}} \le x < 1,\ 0 \le t < k\big\}$, respectively. Then we set

$$v_h(x,t) = \underline{v}_h(x,t) \ + \ V(\underline{v}_h(x,t))t, \quad 0 \le x \le 1, \quad 0 \le t < k, \tag{3.2}$$

where $V(v) = \big(V_1(v), V_2(v)\big) \equiv (0, -m)$, and

$$v_j^1 = \frac{1}{h}\int_{x_{j-1}}^{x_j} v_h(x, t_1 - 0)\,dx, \quad j = 1, \cdots, N. \tag{3.3}$$

Suppose that we have defined approximate solutions $v_h(x,t)$ for $0 \le t < t_i$. We then define

$$v_h(x,t) = \underline{v}_h(x,t) + V(\underline{v}_h(x,t))(t - t_i), \quad t_i \le t < t_{i+1}, \tag{3.4}$$

where $\underline{v}_h(x,t)$ are piecewise smooth functions defined as solutions of the Riemann problems in the region $R_j^i\big\{(x,t):\ x_{j-\frac{1}{2}} \le x < x_{j+\frac{1}{2}},\ t_i \le t < t_{i+1}\big\}$

$$\begin{cases} (2.10), \\ \underline{v}_h(x,t)|_{t=t_i} = \begin{cases} v_j^i, & x < x_j, \\ v_{j+1}^i, & x > x_j, \quad j = 1, \cdots, N-1, \end{cases} \end{cases} \tag{3.5}$$

and as solutions of mixed problems in the two regions R_0^i and R_N^i:

$$\begin{aligned} & R_0^i = \{(x,t):\ 0 \le x < x_{\frac{1}{2}},\ t_i \le t < t_{i+1}\}, \\ & \qquad \begin{cases} (2.10), & x > 0,\ t > t_i, \\ \underline{v}_h(x,t)|_{t=t_i} = v_1^i, \quad x > 0, \\ \underline{m}_h|_{x=0} = 0. \end{cases} \\ & R_N^i = \{(x,t):\ x_{N-\frac{1}{2}} \le x < 1,\ t_i \le t < t_{i+1}\}, \\ & \qquad \begin{cases} (2.10), & x < 1,\ t > t_i, \\ \underline{v}_h(x,t)|_{t=t_i} = v_1^i, \quad x < 1, \\ \underline{m}_h|_{x=1} = 0. \end{cases} \end{aligned} \tag{3.5$'$}$$

Next, we set

$$v_j^{i+1} = \frac{1}{h}\int_{x_{j-1}}^{x_j} v_h(x, t_{i+1} - 0)dx, \quad 1 \le j \le N. \tag{3.6}$$

Therefore, inductively, the approximate solutions $v_h = (\rho_h, m_h) \equiv (\underline{\rho}_h, m_h)$ are well-defined, since $\underline{\rho}_h \geq 0$. We summarize the above process as follows:

$$v^{i+1} = A_h \circ R \circ E_k(\cdot, v^i), \tag{3.7}$$

where A_h is the cell-averaging operator (3.6), $E_k(x, v^i)$ is the Riemann solver (3.5) (or boundary Riemann solver (3.5')), and R is the reconstruction step (3.4).

For $t_i \leq t < t_{i+1}$, we set

$$w_h(x,t) = \underline{w}_h(x,t) - \frac{\underline{w}_h(x,t) + \underline{z}_h(x,t)}{2}(t - t_i), \tag{3.8}$$

$$z_h(x,t) = \underline{z}_h(x,t) - \frac{\underline{w}_h(x,t) + \underline{z}_h(x,t)}{2}(t - t_i), \tag{3.9}$$

where $\underline{w}_h$ and $\underline{z}_h$ are Riemann invariants corresponding to the Riemann solutions $\underline{v}_h$.

With the help of $w_h(x,t)$ and $z_h(x,t)$ defined by (3.8) and (3.9), we prove the following uniform bound for the approximate solutions.

Theorem 3.1. *Suppose that the initial data* (ρ_0, m_0) *satisfy the following conditions*:

$$0 \leq \rho_0(x) \leq M_1, \quad \rho_0(x) \not\equiv 0, \quad |m_0(x)| \leq M_2 \rho_0(x). \tag{3.10}$$

Then, the approximate solutions (ρ_h, m_h) *derived by the Godunov scheme are uniformly bounded in the region* $\bar{I}_T \equiv \{(x,t):\ 0 \leq x \leq 1,\ 0 \leq t \leq T\}$ *for any* $T > 0$; *that is, there is a constant* $C > 0$ *independent of* t *such that*

$$0 \leq \rho_h(x,t) \leq C, \quad |m_h(x,t)| \leq C \rho_h(x,t). \tag{3.12}$$

Proof. Assume that $0 < k < 1$. For $t_i \leq t < t_{i+1}$ ($i \geq 0$ integers), the Riemann invariant properties imply that

$$\begin{aligned}
w_h(x,t) &= \underline{w}_h(x,t)\Big(1 - \frac{t-t_i}{2}\Big) - \underline{z}_h(x,t)\frac{t-t_i}{2} \\
&\leq \sup_x \underline{w}_h(x, t_i + 0)\Big(1 - \frac{t-t_i}{2}\Big) - \inf_x \underline{z}_h(x, t_i + 0)\frac{t-t_i}{2}, \\
z_h(x,t) &= \underline{z}_h(x,t)\Big(1 - \frac{t-t_i}{2}\Big) - \frac{\underline{w}_h(x,t)}{2}(t - t_i) \\
&\geq \inf_x \underline{z}_h(x, t_i + 0)\Big(1 - \frac{t-t_i}{2}\Big) - \sup_x \underline{w}_h(x, t_i + 0)\frac{t-t_i}{2}.
\end{aligned}$$

In particular, we obtain

$$\begin{aligned}
\sup_x w_h(x, t_{i+1} - 0) &\leq \sup_x \underline{w}_h(x, t_i + 0)\Big(1 - \frac{k}{2}\Big) - \inf_x \underline{z}_h(x, t_i + 0)\frac{k}{2}, \\
\inf_x z_h(x, t_{i+1} - 0) &\geq \inf_x \underline{z}_h(x, t_i + 0)\Big(1 - \frac{k}{2}\Big) - \sup_x \underline{w}_h(x, t_i + 0)\frac{k}{2}.
\end{aligned}$$

Let $\alpha_i = \max\Big\{\sup_x \underline{w}_h(x, t_i+0),\ -\inf_x \underline{z}_h(x, t_i+0)\Big\}$. Then

$$\max\Big\{\sup_x w_h(x, t_{i+1}-0),\ -\inf_x z_h(x, t_{i+1}-0)\Big\} \leq \alpha_i. \tag{3.13}$$

By (3.6) we know that

$$\sup_x w_h(x, t_{i+1}+0) \leq \sup_x w_h(x, t_{i+1}-0), \quad \inf_x z_h(x, t_{i+1}+0) \leq \inf_x \underline{z}_h(x, t_{i+1}-0). \tag{3.14}$$

Therefore

$$\alpha_{i+1} \leq \alpha_i, \quad \text{and} \quad \alpha_i \leq \alpha_0, \ \ 0 \leq i \leq n, \tag{3.15}$$

where $\alpha_0 = \max\Big\{\sup_x w_0(x),\ -\inf_x z_0(x)\Big\}$. Then,

$$\begin{aligned} & w_h(x,t) \leq \alpha_0, \quad z_h(x,t) \geq -\alpha_0, \quad \text{and} \\ & w_h(x,t) - z_h(x,t) \geq 0. \end{aligned}$$

Then there is a constant $C > 0$ independent of h, k and t such that

$$0 \leq \rho_h(x,t) \leq C, \quad |m_h(x,t)| \leq C\rho_h(x,t).$$

This completes the proof the Theorem 3.1.

Now, we can choose the time mesh length $k = k(h)$. Let

$$\lambda = \max_{i=1,2}\Big\{\sup_{0\leq\rho\leq C,\ |m|\leq C\rho} |\lambda_i(\rho, m)|\Big\},$$

then we take

$$k = \frac{T}{n}, \quad \text{where} \ \ n = \max\Big\{\Big[\frac{4\lambda T}{h}\Big] + 1,\ \Big[\frac{T}{2}\Big] + 1\Big\}. \tag{3.16}$$

For this k, both the CFL condition and $0 < k < 1$ hold.

4 Global existence of weak solutions

In this section, we will show that the approximate solutions, constructed in last section, admit a convergent subsequence whose limit is a weak entropy solution of problem (1.2)–(1.3). The convergence is achieved by the compensated compactness, the boundary conditions are verified in the sense of trace.

With the uniform L^∞ estimates given in Theorem 3.1, and the specific structure of system (1.2), now it is standard to apply the compensated compactness framework ([7], [9], [22], [23]) to the approximate solution $\{v_h\}$, to conclude that there exists a convergent subsequence $\{v_{h_j}\}_{j=1}^{\infty}$ such that $h_j \to 0$ as $j \to \infty$ and

$$\big(\rho_{h_j}(x,t), m_{h_j}(x,t)\big) \to \big(\rho(x,t), m(x,t)\big) \ \text{ a.e.} \tag{4.1}$$

Furhtermore, such a limit $(\rho, m)(x,t)$ satisfies (2.6) and (2.7) for any test function $\psi(x,t) \in C_0^\infty(I_T)$ for any $T > 0$. Also, the entropy inequality holds in the sense of distribution. Clearly, there is a constant $C > 0$ such that

$$0 \le \rho(x,t) \le C, \quad |m(x,t)| \le C\rho(x,t) \ \text{ a.e.} \tag{4.2}$$

Now we turn to the initial and boundary conditions of weak solutions. First, we need to determine the traces of weak solutions whose exact meaning will be stated below. Let $v = (\rho, m)$ be a weak solution of (1.2) obtained in (4.1). We introduce the generalized function $\mathcal{A}: \ C_0^1(\mathbb{R}^2) \longrightarrow \mathbb{R}^2$ as follows: for $\psi \in C_0^1(\mathbb{R}^2)$,

$$\mathcal{A}(\psi) = -\int_0^T \int_0^1 \big(v\psi_t + f(v)\psi_x + g(v)\psi\big) dxdt. \tag{4.3}$$

We take smooth $\zeta_0(t), \zeta_T(t), \xi_0(x), \xi_1(x)$ with

$$\begin{aligned} &\zeta_0(0) = 1, \quad \zeta_0(T) = 0; \quad \zeta_T(0) = 0, \quad \zeta_T(T) = 1; \\ &\xi_0(0) = 1, \quad \xi_0(1) = 0; \quad \xi_1(0) = 0, \quad \xi_1(1) = 1. \end{aligned} \tag{4.4}$$

For any $\chi(x)$, we define the generalized functions:

$$\begin{aligned} &v^\star(\cdot, 0)(\chi) = \mathcal{A}(\chi \cdot \zeta_0) - \chi(0)\mathcal{A}(\xi_0 \cdot \zeta_0) - \chi(1)\mathcal{A}(\xi_1 \cdot \zeta_0), \\ &v^\star(\cdot, T)(\chi) = -\mathcal{A}(\chi \cdot \zeta_T) + \chi(0)\mathcal{A}(\xi_0 \cdot \zeta_T) + \chi(1)\mathcal{A}(\xi_1 \cdot \zeta_T), \\ &f^\star(v)(0, \cdot)(\chi) = \mathcal{A}(\xi_0 \cdot \chi), \\ &f^\star(v)(1, \cdot)(\chi) = -\mathcal{A}(\xi_1 \cdot \chi), \end{aligned} \tag{4.5}$$

where $(\chi \cdot \zeta_0)(x,t) = \chi(x)\zeta_0(t)$ and so on mean the tensor product.

Then we can define the trace of v along the segments $(0,1) \times \{0\}$ and $(0,1) \times \{T\}$, and the trace of $f(v)$ along the segments $\{0\} \times (0,T)$ and $\{1\} \times (0,T)$ respectively as $v^\star(\cdot, 0)$, $v^\star(\cdot, T)$, $f^\star(v)(0, \cdot)$ and $f^\star(v)(1, \cdot)$. Similarly, for any $t \in (0,T)$, we can also define $v^\star(\cdot, t)$ as the trace of v along the segment $(0,1) \times \{t\}$. For any $x \in (0,1)$, define $f^\star(v)(x, \cdot)$ as the trace of $f(v)$ along the segment $\{x\} \times (0,T)$.

Similar to [11], we have

Lemma 4.1. *Let v satisfy* (1.2) *in distributional sense, then,*

$$\begin{aligned} v^\star(\cdot, 0)|_{(0,1)}, \quad v^\star(\cdot, T)|_{(0,1)} \ &\in \ L_{loc}^\infty(0,1); \\ f^\star(v)(0, \cdot)|_{(0,T)}, \quad f^\star(v)(1, \cdot)|_{(0,T)} \ &\in \ L_{loc}^\infty(0,T), \end{aligned} \tag{4.6}$$

and for any $\psi \in C_0^1(\mathbb{R}^2)$,

$$\begin{aligned}
&\int_0^T \int_0^1 \big(v\psi_t + f(v)\psi_x + g(v)\psi\big)dxdt \\
&= \int_0^1 v^\star(x,T)\psi(x,T)dx - \int_0^1 v^\star(x,0)\psi(x,0)dx \\
&\quad + \int_0^T f^\star(v)(1,t)\psi(1,t)dt - \int_0^T f^\star(v)(0,t)\psi(0,t)dt.
\end{aligned} \tag{4.7}$$

Lemma 4.2. *Let* $v_{h_j} = (\rho_{h_j}, m_{h_j})$ *be the convergent sequence of approximate solutions of* (1.2)-(1.3) *constructed in section* 3 *and* $v = (\rho, m)$ *is the limit function obtained in* (4.1). *Then* $v(x,t)$ *satisfies the initial-boundary conditions*:

$$m^\star(0,t) = m^\star(1,t) = 0, \quad t \in (0,T); \tag{4.8}$$

$$v^\star(x,0) = v_0(x), \qquad x \in (0,1). \tag{4.9}$$

Proof. From (2.6)–(2.7), it is easy to see, for any $\psi \in C_0^1(\mathbb{R}^2)$, that

$$\lim_{j\to+\infty} \Big[\int_0^T \int_0^1 (v_{h_j}\psi_t + f(v_{h_j})\psi_x + g(v_{h_j})\psi)dxdt + \int_{t=0} v_{h_j}\psi dx - \int_{t=T} v_{h_j}\psi dx \Big] = 0, \tag{4.10}$$

which implies

$$\int_0^T \int_0^1 (v\psi_t + f(v)\psi_x + g(v)\psi)dxdt + \lim_{j\to+\infty} [\int_{t=0} v_{h_j}\psi dx - \int_{t=T} v_{h_j}\psi dx] = 0. \tag{4.11}$$

Therefore, (4.7) and (4.11) give

$$\begin{aligned}
&\lim_{j\to+\infty} (\int_{t=T} v_{h_j}\psi dx - \int_{t=0} v_{h_j}\psi dx) \\
&= \int_0^1 v^\star(x,T)\psi(x,T)dx - \int_0^1 v^\star(x,0)\psi(x,0)dx \\
&\quad + \int_0^T f^\star(v)(1,t)\psi(1,t)dt - \int_0^T f^\star(v)(0,t)\psi(0,t)dt.
\end{aligned} \tag{4.12}$$

The first component of (4.12) reads

$$\begin{aligned}
&\int_0^1 \rho^\star(x,T)\psi(x,T)dx - \int_0^1 \rho^\star(x,0)\psi(x,0)dx + \int_0^T m^\star(1,t)\psi(1,t)dt \\
&\quad - \int_0^T m^\star(0,t)\psi(0,t)dt - (\int_{t=T} \rho\psi dx - \int_{t=0} \rho\psi \ dx) = 0.
\end{aligned} \tag{4.13}$$

Taking $\psi(x,t)=\zeta(x)\chi(t)\in C_0^1(\mathbb{R}^2)$ with $\zeta,\chi\in C_0^1(\mathbb{R})$, and $\chi(0)=1$, $\chi(T)=0$, $\zeta(1)=\zeta(0)=0$ in (4.13), we get

$$\int_0^1 \rho^\star(x,0)\zeta(x)dx=\int_0^1 \rho_0(x)\zeta(x)dx,$$

which implies $\rho^\star(x,0)=\rho_0(x)$ on $(0,1)$.

Taking $\psi(x,t)=\zeta(x)\chi(t)\in C_0^1(\mathbb{R}^2)$ with $\zeta,\chi\in C_0^1(\mathbb{R})$, and $\chi(0)=\chi(T)=0$, $\zeta(1)=0$, $\zeta(0)=1$ in (4.13), we get

$$\int_0^T m^\star(0,t)\chi(t)dx=0.$$

Thus $m^\star(0,t)=0$ on $(0,T)$. It is similar to show that $m^\star(1,t)=0$ on $(0,T)$. Using the second component of (4.12), it is easy to show $m^\star(x,0)=m_0(x)$ on $(0,1)$. This completes the proof of Lemma 4.2.

Collecting all results obtained above, we thus conclude the proof of Theorem 1. However, we remark that Lemma 4.2 might not apply to all weak solutions which satisfies (2.6)-(2.7). However, in the same spirit, one could show that the weak solutions obtained as vanishing viscosity limit with the same boundary condition (1.3) verifies (4.8) and (4.9). This explains the last line in Definition 2.1.

5 Large Time Behavior of Weak Solution

In this section, we investigate the large time asymptotic behavior of any entropy weak solution for the initial boundary value problem (1.2)-(1.3), including the one obtained in section 4. In deed, we have

Theorem 5.1. *Let* (ρ,m) *be any* L^∞ *entropy weak solution of the initial boundary problem* (1.2)-(1.3), *defined in Definition 2.1, satisfying* $\int_0^1 \rho_0(x)dx=\rho_*$ and

$$0\le\rho(x,t)\le\Lambda<\infty,\quad |m(x,t)|\le M_1\rho(x,t), \tag{5.1}$$

where M_1,Λ *are positive constants. Then, there exist constants* $C,\ \delta>0$ *depending on* $\gamma,\ \rho_*,\Lambda,$ *and initial data such that*

$$\big\|(\rho-\rho_*,\ m)(\cdot,t)\big\|^2_{L^2([0,1])}\le Ce^{-\delta t}. \tag{5.2}$$

Due to dissipation of momentum equation and the boundary condition, the kinetic energy is expected to vanish as time tends to infinity while the potential energy will converge to a constant. Furthermore, it is easy to see

$$\int_0^1 \rho(x,t)dx = \int_0^1 \rho_0(x)dx = \rho_*, \tag{5.3}$$

due to conservation law of total mass. This suggests that the asymptotic state of $(\rho, m)(x,t)$ should be $(\rho_*, 0)$.

We now turn to prove Theorem 5.1. First of all, we give a lemma which will play an important role in controlling the singularity near vacuum.

Lemma 5.2. *Let $0 \le \rho \le \Lambda < \infty$. There is a positive constant C_1 such that*

$$\big[P(\rho) - P(\rho_*) - P'(\rho_*)(\rho - \rho_*)\big] \le C_1 \big[P(\rho) - P(\rho_*)\big](\rho - \rho_*). \tag{5.4}$$

Proof. Consider

$$\Gamma(\rho) = \frac{\gamma}{\rho_*}(P(\rho) - P(\rho_*))(\rho - \rho_*) - \big[P(\rho) - P(\rho_*) - P'(\rho_*)(\rho - \rho_*)\big]. \tag{5.5}$$

Clearly, $\Gamma(\rho)$ is continuous for $\rho \ge 0$. Since

$$\Gamma(0) = P(\rho_*) > 0,$$

there exists $d \in (0, \rho_*)$ such that

$$\Gamma(\rho) > \frac{1}{2}P(\rho_*) > 0, \quad for\ \rho \in [0, d]. \tag{5.6}$$

For $\rho > d > 0$, we can see that

$$P'(d)(\rho - \rho_*)^2 \le \big[P(\rho) - P(\rho_*)\big](\rho - \rho_*), \tag{5.7}$$

and

$$P(\rho) - P(\rho_*) - P'(\rho_*)(\rho - \rho_*) \le \begin{cases} \dfrac{P''(d)}{2}(\rho - \rho_*)^2, & 1 < \gamma \le 2, \\ \dfrac{P''(\Lambda)}{2}(\rho - \rho_*)^2, & \gamma > 2. \end{cases} \tag{5.8}$$

Choosing

$$C_1 = \max\{\frac{\gamma}{\rho_*},\ \frac{P''(d)}{2P'(d)},\ \frac{P(\Lambda)}{2P'(d)}\}, \tag{5.9}$$

we thus have

$$P(\rho) - P(\rho_*) - P'(\rho_*)(\rho - \rho_*) \le C_1\big[P(\rho) - P(\rho_*)\big](\rho - \rho_*). \tag{5.10}$$

This completes the proof of Lemma 5.2.

We then set

$$w = \rho - \rho_*, \quad z = m, \tag{5.11}$$

which satisfy

$$\begin{cases} w_t + z_x = 0 \\ z_t + \Big(\dfrac{m^2}{\rho}\Big)_x + \big[P(\rho) - P(\rho_*)\big]_x + z = 0, \end{cases} \tag{5.12}$$

and

$$\int_0^1 w(x,t)dx = 0. \tag{5.13}$$

Define

$$y = -\int_0^x w(\sigma,t)d\sigma, \tag{5.14}$$

which implies that

$$y_x = -w = \rho_* - \rho, \quad y_t = z. \tag{5.15}$$

Since

$$\int_0^1 \rho(x,t)dx = \int_0^1 \rho_0(x)dx = \rho_*,$$

we have

$$y(0) = y(1) = 0. \tag{5.16}$$

Therefore the second equation of (5.12) turns into

$$y_{tt} + \Big(\frac{m^2}{\rho}\Big)_x + \big[P(\rho) - P(\rho_*)\big]_x + y_t = 0. \tag{5.17}$$

Multiplying y with (5.17) and integrating over $[0,1]$, we have

$$\frac{d}{dt}\int_0^1 \Big(y_t y + \frac{1}{2}y^2\Big)dx - \int_0^1 y_t^2 dx + \int_0^1 \big[P(\rho) - P(\rho_*)\big](\rho - \rho_*)dx = \int_0^1 \frac{m^2}{\rho} y_x dx. \tag{5.18}$$

Since $\rho, u = m/\rho, m = y_t \in L^\infty[0,1]$, we get

$$\frac{d}{dt}\int_0^1 \Big(y_t y + \frac{1}{2}y^2\Big)dx - \int_0^1 y_t^2 dx + \int_0^1 \big[P(\rho) - P(\rho_*)\big](\rho - \rho_*)dx = \int_0^1 \frac{\rho_*}{\rho} y_t^2 dx - \int_0^1 y_t^2 dx, \tag{5.19}$$

i.e.

$$\frac{d}{dt}\int_0^1 \Big(y_t y + \frac{1}{2}y^2\Big)dx + \int_0^1 \big[P(\rho) - P(\rho_*)\big](\rho - \rho_*)dx = \int_0^1 y_t^2 \frac{\rho_*}{\rho} dx. \tag{5.20}$$

In order to deal with the nonlinearity, we now use the entropy inequality, rather than the usual energy method. Let

$$\eta_e = \frac{m^2}{2\rho} + \frac{P(\rho)}{\gamma - 1}, \qquad q_e = \frac{m^3}{2\rho^2} + \frac{\rho^{\gamma-1}m}{\gamma - 1}$$

be the mechanical energy and related flux. We define

$$\eta_* = \eta_e - \frac{1}{\gamma - 1}P'(\rho_*)(\rho - \rho_*) - \frac{1}{\gamma - 1}P(\rho_*). \tag{5.21}$$

Thus, by the definition of weak entropy solution, the following entropy inequality holds in the sense of distribution:

$$\eta_{*t} + \frac{1}{\gamma - 1}[P'(\rho_*)(\rho - \rho_*)]_t + q_{ex} + \frac{m^2}{\rho} \le 0. \tag{5.22}$$

Since ρ_* is a constant, we get

$$\eta_{*t} + \frac{P'(\rho_*)}{\gamma - 1}(\rho - \rho_*)_t + q_{ex} + \frac{m^2}{\rho} \le 0. \tag{5.23}$$

By the conservation of mass and theory of divergence-measure fields [3] , we have

$$\frac{d}{dt}\int_0^1 \eta_* dx + \int_0^1 \frac{m^2}{\rho} dx \le 0,$$

i.e.,

$$\frac{d}{dt}\int_0^1 \eta_* dx + \int_0^1 \frac{y_t^2}{\rho} dx \le 0. \tag{5.24}$$

Choosing $K = \max\{2, \ \ 2\Lambda + \rho_*\}$, we add (5.20) to (5.24) $\times K$,

$$\frac{d}{dt}\int_0^1 \Big(K\eta_* + yy_t + \frac{1}{2}y^2\Big)dx + \int_0^1 \big[P(\rho) - P(\rho_*)\big](\rho - \rho_*)dx + \int_0^1 \frac{K - \rho_*}{\rho} y_t^2 dx \le 0, \tag{5.25}$$

Using the expression of η_* we get

$$\begin{aligned}&\frac{d}{dt}\int_0^1 \Big(\frac{K}{2\rho}y_t^2 + yy_t + \frac{1}{2}y^2 + \frac{K}{\gamma - 1}\big[P(\rho) - P(\rho_*) - P'(\rho_*)(\rho - \rho_*)\big]\Big)dx \\ &+ \int_0^1 \big[P(\rho) - P(\rho_*)\big](\rho - \rho_*)dx + \int_0^1 \frac{K - \rho_*}{\rho} y_t^2 dx \le 0.\end{aligned} \tag{5.26}$$

Clearly, Lemma 5.2 implies

$$\int_0^1 \frac{K}{\gamma-1}\big[P(\rho)-P(\rho_*)-P'(\rho_*)(\rho-\rho_*)\big]dx \le \frac{C_1K}{\gamma-1}\int_0^1 \big[P(\rho)-P(\rho_*)\big](\rho-\rho_*)dx. \quad (5.27)$$

On the other hand, since P is a convex function, the Lemma 4.1 of [20] and Poincaré's inequality imply that there are positive constants C_2 and C_3 such that

$$\begin{aligned}\int_0^1 \Big(\frac{K}{2\rho}y_t^2 + yy_t + \frac{1}{2}y^2\Big)dx &\le \int_0^1 \Big(\frac{K}{2\rho}y_t^2 + \frac{1}{2}y_t^2 + y^2\Big)dx \\ &\le C_2\int_0^1 \frac{K-\rho_*}{\rho}y_t^2dx + \int_0^1 y^2dx \\ &\le C_2\int_0^1 \frac{K-\rho_*}{\rho}y_t^2dx + \int_0^1 y_x^2dx \\ &\le C_2\int_0^1 \frac{K-\rho_*}{\rho}y_t^2dx + C_3\int_0^1 \big[P(\rho)-P(\rho_*)\big](\rho-\rho_*)dx.\end{aligned} \quad (5.28)$$

Therefore, for $C_4 = \max\{C_2, C_3\}$, it holds

$$\int_0^1 \Big(K\eta_* + yy_t + \frac{1}{2}y^2\Big)dx \le C_4\Big(\int_0^1 \big[P(\rho)-P(\rho_*)\big](\rho-\rho_*)dx + \int_0^1 \frac{K-\rho_*}{\rho}y_t^2dx\Big). \quad (5.29)$$

Therefore, from (5.26)–(5.29), we conclude that there is a positive constant C_5 such that

$$\frac{d}{dt}\int_0^1 \Big(K\eta_* + yy_t + \frac{1}{2}y^2\Big)dx + C_5\int_0^1 \Big(K\eta_* + yy_t + \frac{1}{2}y^2\Big)dx \le 0. \quad (5.30)$$

Furthermore, since $K > 2\Lambda \ge 2\rho$, we know that

$$\begin{aligned}&K\eta_* + yy_t + \frac{1}{2}y^2 \\ &\ge 2y_t^2 + yy_t + \frac{1}{2}y^2 + \frac{K}{\gamma-1}\big[P(\rho)-P(\rho_*)-P'(\rho_*)(\rho-\rho_*)\big] \\ &\ge y_t^2 + C_6(\rho-\rho_*)^2,\end{aligned} \quad (5.31)$$

where C_6 is a positive constant. Hence, (5.30) implies that

$$\int_0^1 \Big(K\eta_* + yy_t + \frac{1}{2}y^2\Big)dx \le C_7\exp\{-C_5t\}, \quad (5.32)$$

and

$$\int_0^1 y_t^2 + (\rho-\rho_*)^2dx \le C_8\exp\{-C_5t\}. \quad (5.33)$$

This completes the proof of Theorem 5.1.

As indicated in introduction, we also expect that (1.2)–(1.3) is captured by (1.4)–(1.5) time asymptotically if

$$\int_0^1 \tilde{\rho}_0(x)\ dx = \rho_*. \tag{5.34}$$

In view of Theorem 5.1, we will show that the large time asymptotic state of (1.4)–(1.5) is also the constant state $(\rho_*,\ 0)$. Then by applying the triangular inequality we can prove Theorem 2.

Consider

$$\begin{cases} \tilde{\rho}_t - \tilde{P}_{xx} = 0, \\ \tilde{\rho}(x,0) = \tilde{\rho}_0(x), \qquad 0 \le x \le 1, \\ \tilde{P}_x(0,t) = \tilde{P}_x(1,t) = 0, \quad t \ge 0, \end{cases} \tag{5.35}$$

where $\tilde{P} = P(\tilde{\rho})$, and $\tilde{P}_0'(0) = \tilde{P}_0'(1) = 0$, for $\tilde{P}_0(x) = P(\tilde{\rho}_0(x))$. The initial data $\tilde{\rho}_0$ satisfies

$$0 \le \tilde{\rho}_0(x) \le \Lambda, \quad \text{and} \quad \int_0^1 \tilde{\rho}_0(x)dx = \int_0^1 \rho_0(x)dx = \rho_*. \tag{5.36}$$

The global existence and large time behavior of weak solutions of (5.35) has been established in [1], see also [43]. Here, we give a proof in different version including the decay of momentum.

Theorem 5.3. *Let $\tilde{\rho}_0(x)$ satisfy* (5.36). *Then for the global weak solution $\tilde{\rho}(x,t)$ of* (5.35) *and $\tilde{m} = -\tilde{P}_x$, there exist positive constants c_1 and $\delta_1 > 0$ such that*

$$\int_0^1 \big((\tilde{\rho} - \rho_*)^2 + \tilde{m}^2\big)dx \le c_1 \exp\{-\delta_1 t\}, \qquad \text{as} \ \ t \to +\infty. \tag{5.37}$$

Proof. First, we note that $0 \le \tilde{\rho}(x,t) \le \Lambda$ due to the comparison principle [43]. Second, there is a $T > 0$ such that $\rho(x,t) > 0$ is a classical solution for $t > T$, see [1]. Then, for $t > T$, we consider the equation

$$(\tilde{\rho} - \rho_*)_t = (\tilde{P} - P_*)_{xx}, \tag{5.38}$$

which is equivalent to $(5.35)_1$, where $\tilde{P} = P(\tilde{\rho}), P_* = P(\rho_*)$. Let

$$\psi(x,t) = \tilde{\rho}(x,t) - \rho_*, \tag{5.39}$$

and

$$\phi = \int_0^x \psi(r,t)\ dr, \tag{5.40}$$

then

$$\phi_x = \psi = \tilde{\rho} - \rho_*. \tag{5.41}$$

Due to the conservation of mass we have

$$\phi(0) = \phi(1) = 0. \tag{5.42}$$

Integrating (5.38) over $[0, x]$ and use the boundary condition we get

$$\phi_t = (\tilde{P} - P_*)_x. \tag{5.43}$$

Multiplying (5.43) by ϕ and integrating over $[0, 1]$ we get

$$\frac{d}{dt}\int_0^1 \frac{1}{2}\phi^2 dx + \int_0^1 (\tilde{P} - P_*)(\tilde{\rho} - \rho_*)dx = 0. \tag{5.44}$$

Multiplying (5.38) by $\tilde{\rho} - \rho_*$ and integrating over $[0, 1]$ we get

$$\frac{d}{dt}\int_0^1 \frac{1}{2}(\tilde{\rho} - \rho_*)^2 dx + \int_0^1 (\tilde{P} - P_*)_x(\tilde{\rho} - \rho_*)_x dx = 0. \tag{5.45}$$

Since $(\tilde{P} - P_*)_x = \tilde{P}_x = P'(\tilde{\rho})\tilde{\rho}_x = P'(\tilde{\rho})(\tilde{\rho} - \rho_*)_x$, one has

$$\frac{d}{dt}\int_0^1 \frac{1}{2}(\tilde{\rho} - \rho_*)^2 dx + \int_0^1 P'(\tilde{\rho})(\tilde{\rho} - \rho_*)_x(\tilde{\rho} - \rho_*)_x dx = 0,$$

i.e.

$$\frac{d}{dt}\int_0^1 \frac{1}{2}(\tilde{\rho} - \rho_*)^2 dx + \int_0^1 P'(\tilde{\rho})[(\tilde{\rho} - \rho_*)_x]^2 dx = 0. \tag{5.46}$$

Multiplying (5.38) by $(\tilde{P} - P_*)$ and integrating over $[0, 1]$ we get

$$\int_0^1 [P(\tilde{\rho}) - P(\rho_*)](\tilde{\rho} - \rho_*)_t dx + \int_0^1 [(\tilde{P} - P_*)_x]^2 dx = 0. \tag{5.47}$$

Now, we define

$$F(\tilde{\rho} - \rho_*) = \int_0^{\tilde{\rho} - \rho_*} [P(\rho_* + \xi) - P(\rho_*)]d\xi, \tag{5.48}$$

then we have

$$F_t = [P(\tilde{\rho}) - P(\rho_*)](\tilde{\rho} - \rho_*)_t.$$

So (5.47) turns out to be

$$\frac{d}{dt}\int_0^1 F dx + \int_0^1 [(\tilde{P} - P_*)_x]^2 dx = 0. \tag{5.49}$$

From the definition of F, we know

$$F = \int_0^{\tilde{\rho} - \rho_*} P'(\zeta)\xi d\xi,$$

where ζ is between $\tilde\rho$ and ρ_*. Since $0 \le \tilde\rho, \rho_* \le \Lambda$ we know that

$$0 \le F \le \frac{P'(\Lambda)}{2}(\tilde\rho - \rho_*)^2. \tag{5.50}$$

Since $P(\tilde\rho) = \tilde\rho^\gamma/\gamma$, then $\tilde\rho = (\gamma\tilde P)^{\frac{1}{\gamma}}$, and so $\tilde\rho_t = (\gamma\tilde P)^{\frac{1}{\gamma}-1}\tilde P_t$. Then we consider the equation of $\tilde P$

$$\tilde P_t = (\gamma\tilde P)^{1-\frac{1}{\gamma}}\tilde P_{xx},$$

i.e.

$$(\tilde P - P_*)_t = (\gamma\tilde P)^{1-\frac{1}{\gamma}}(\tilde P - P_*)_{xx}. \tag{5.51}$$

Multiplying (5.51) by $(\tilde P - P_*)_{xx}$ and integrating over $[0,1]$ we get

$$\frac{d}{dt}\int_0^1 \frac{1}{2}[(\tilde P - P_*)_x]^2 dx + \int_0^1 (\gamma\tilde P)^{1-\frac{1}{\gamma}}[(\tilde P - P_*)_{xx}]^2 dx = 0. \tag{5.52}$$

Doubling (5.45), (5.46), and (5.52), adding the results to (5.49), and notice that $P'(\tilde\rho) \ge 0$ and $(\gamma\tilde P)^{1-\frac{1}{\gamma}} \ge 0$, we arrive at

$$\frac{d}{dt}\int_0^1 \Big\{\phi^2 + (\tilde\rho - \rho_*)^2 + F + [(\tilde P - P_*)_x]^2\Big\}dx + \int_0^1 \Big\{(\tilde P - P_*)(\tilde\rho - \rho_*) + [(\tilde P - P_*)_x]^2\Big\}dx \le 0, \tag{5.53}$$

where we have thrown some non-negative terms (in the second part of the left hand side) away. Since $\phi_x = \tilde\rho - \rho_*$, by Poincaré's inequality and (5.50) we obtain

$$\int_0^1 \Big\{\phi^2 + (\tilde\rho - \rho_*)^2 + F + [(\tilde P - P_*)_x]^2\Big\}dx \le \int_0^1 \Big\{\Big(2 + \frac{P'(\Lambda)}{2}\Big)(\tilde\rho - \rho_*)^2 + [(\tilde P - P_*)_x]^2\Big\}dx. \tag{5.54}$$

Now, from Lemma 4.1 in [20], we know that

$$C_9(\tilde\rho - \rho_*)^2 \le (\tilde P - P_*)(\tilde\rho - \rho_*), \tag{5.55}$$

where C_9 is a constant. Combining (5.54) and (5.55) we obtain

$$\int_0^1 \Big\{\phi^2 + (\tilde\rho - \rho_*)^2 + F + [(\tilde P - P_*)_x]^2\Big\}dx \le \int_0^1 \Big\{C_{10}(\tilde P - P_*)(\tilde\rho - \rho_*) + [(\tilde P - P_*)_x]^2\Big\}dx, \tag{5.56}$$

where $C_{10} = \Big(2 + \frac{P'(\Lambda)}{2}\Big)/C_9$. Therefore, (5.56) implies that

$$\int_0^1 \Big\{\phi^2 + (\tilde\rho - \rho_*)^2 + F + [(\tilde P - P_*)_x]^2\Big\}dx \le C_{11}\int_0^1 \Big\{(\tilde P - P_*)(\tilde\rho - \rho_*) + [(\tilde P - P_*)_x]^2\Big\}dx, \tag{5.57}$$

where $C_{11} = \max\{C_{10}, 1\}$. Combining (5.53) and (5.57) we get

$$\frac{d}{dt}\int_0^1 \Big\{\phi^2 + (\tilde\rho - \rho_*)^2 + F + [(\tilde P - P_*)_x]^2\Big\}dx + \frac{1}{C_{11}}\int_0^1 \Big\{\phi^2 + (\tilde\rho - \rho_*)^2 + F + [(\tilde P - P_*)_x]^2\Big\}dx \le 0,$$

which implies that

$$\int_0^1 \Big\{\phi^2 + (\tilde{\rho} - \rho_*)^2 + F + [(\tilde{P} - P_*)_x]^2\Big\}dx \le C_{12}\exp\Big\{-\frac{t}{C_{11}}\Big\}, \tag{5.58}$$

where C_{12} is a constant depending on the initial data.

Since $F \ge 0$, we obtain

$$\int_0^1 \Big\{\phi^2 + (\tilde{\rho} - \rho_*)^2 + [(\tilde{P} - P_*)_x]^2\Big\}dx \le C_{12}\exp\Big\{-\frac{t}{C_{11}}\Big\}, \tag{5.59}$$

i.e.

$$\int_0^1 \Big\{\phi^2 + (\tilde{\rho} - \rho_*)^2 + \tilde{m}^2\Big\}dx \le C_{12}\exp\Big\{-\frac{t}{C_{11}}\Big\}. \tag{5.60}$$

This completes the proof of Theorem 5.3.

Theorem 2 is a immediate consequence of Theorem 5.1 and Theorem 5.3.

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