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Convergence to the Barenblatt Solution for the Compressible Euler Equations with Damping and Vacuum

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Abstract

We study the asymptotic behavior of a compressible isentropic flow through a porous medium when the initial mass is finite. The model system is the compressible Euler equation with frictional damping. As $t \to \infty$, the density is conjectured to obey the well-known porous medium equation and the momentum is expected to be formulated by Darcy's law. In this paper, we give a definite answer to this conjecture without any assumption on smallness or regularity for the initial data. We prove that any L^∞ weak entropy solution to the Cauchy problem of damped Euler equations with finite initial mass converges, strongly in L^p with decay rates, to matching Barenblatt's profile of the porous medium equation. The density function tends to the Barenblatt's solution of the porous medium equation while the momentum is described by Darcy's law.

1. Introduction

We continue our study on the asymptotic behavior of compressible isentropic flow through a porous medium when vacuum occurs initially. The model system is the compressible Euler equation with frictional damping. As $t \to \infty$, the density is conjectured to obey the well-known porous medium equation and the momentum is expected to be formulated by Darcy's law. Although, many contributions have been made to the small smooth solutions or piecewise smooth Riemann solutions away from vacuum since the pioneer work of Nishida [29], some key problems in this topic remain open. Among them, the large-time asymptotic behavior for the solutions with vacuum has been a long-standing open problem. This work, together with the work of Huang & Pan [18–20], will give a complete answer to this problem. In fact, we showed that the L^∞ weak entropy solutions with vacuum selected by the physical entropy-flux pairs, converge strongly in L^p with decay rates, to the similarity solutions of the porous medium equation, determined by the end-states of the initial data and initial mass. New approaches are developed to deal

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with the nonlinear convection, nonlinear coupling and the singularity near vacuum based on the conservation of mass, the structure of the convection and the existence of mechanical energy function. This approach seems remarkable since we do not need smallness assumptions on the solutions.

We now formulate our results. Consider the compressible Euler equation with frictional damping,

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x &= -\alpha \rho u, \end{aligned} \tag{1.1}$$

with the initial data

$$\rho(x,0) = \rho_0(x) \geqq 0, \;\; m(x,0) = m_0(x). \tag{1.2}$$

Such a system occurs in the mathematical modeling of compressible flow though a porous medium. Here ρ, u and P denote respectively the density, velocity, and pressure; $m = \rho u$ is the momentum and the constant $\alpha > 0$ models friction. Assuming the flow is a polytropic perfect gas, then $P(\rho) = P_0\rho^\gamma,\ 1 < \gamma < 3$, with P_0 a positive constant, and γ the adiabatic gas exponent. Without loss of generality, α and P_0 are normalized to be 1 throughout this paper.

System (1.1) is hyperbolic with two characteristic speeds $\lambda_1 = u - \sqrt{P'(\rho)}$ and $\lambda_2 = u + \sqrt{P'(\rho)}$. Furthermore, (1.1) is strictly hyperbolic at the point away from vacuum where two characteristics coincide. Thus, this simple system involves three mechanisms: nonlinear convection, lower-order dissipation of damping, and the resonance due to vacuum. The interaction of these mechanisms leads to the big difference in qualitative behaviors of solutions from those of strictly hyperbolic conservation laws. For instance, the long-time behavior of the solutions to the Cauchy problem for strictly hyperbolic conservation laws was known to be that of the corresponding Riemann solutions, while the nonlinear diffusive phenomena should be expected in the large-time behavior of solutions to (1.1), (1.2).

In experiments, Darcy's law was observed in the same process. Thus, we have another model:

$$\begin{aligned} \rho_t &= P(\rho)_{xx}, \\ m &= -P(\rho)_x, \end{aligned} \tag{1.3}$$

where the second equation is the famous Darcy law and the first equation is the well-known porous medium equation. So, it is natural to expect some relationship between system (1.1) and system (1.3). Actually, we have the following conjecture; see [22].

Conjecture. *As $t \to \infty$, the system* (1.1) *is equivalent to the system* (1.3).

In the case away from vacuum, system (1.1) can be transferred to the p-system with damping by changing to the Lagrangian coordinates; see [35]. The conjecture has been justified by HSIAO & LIU [12, 13] for small smooth solutions away from vacuum, based on the energy estimates for derivatives. Since then, this problem has attracted considerable attention; see [11, 14, 15, 26, 28, 30–32, 36]. However, all these results are away from vacuum and/or require small smooth initial data. For more references on the p-system with damping, we refer to [6, 16, 17, 23, 27, 34, 38].

When a vacuum occurs in the solution, the difficulty of the problem is greatly increased. The main difficulties come from the interaction of nonlinear convection, lower-order dissipation of damping, and the resonance due to vacuum. It is known that the nonlinearity is the reason for shock formation in a hyperbolic system. For hyperbolic conservation laws, the self-similarity is an important feature in constructing fundamental Riemann solutions and in describing the large-time behaviors of solutions. The damping presents weak dissipation; it prevents the formation of singularity if the data is small and smooth. However, it breaks the self-similarity of the system. This is crucial for the large solutions. Another difficulty is due to the resonance near vacuum which develops a new singularity. In fact, LIU & YANG [24, 25] observed that the local smooth solutions of (1.1) blow up in finite time *before shock formation*. This implies the moving of the interface between the vacuum and the gas. Due to this new singularity, it is very difficult to obtain the solutions with any degree of regularity. This makes (1.1) difficult to understand analytically and makes the construction of effective numerical methods for computing solutions a highly non-trivial problem. Indeed, the only global weak solutions are constructed in L^∞ space by using the method of compensated compactness; see DING, CHEN & LUO [8] for $1 < \gamma \leqq \frac{5}{3}$ and HUANG & PAN [18] for $1 \leqq \gamma < 3$. Thus, to study the large-time behavior of the solution of (1.1), (1.2) with vacuum, it is suitable to consider the L^∞ weak solution.

Definition 1. We call $(\rho, m)(x,t) \in L^\infty$ an *entropy weak solution* of (1.1) and (1.2), if, for any non-negative test function $\phi \in \mathcal{D}(\mathbf{R}^2_+)$,

$$
\begin{aligned}
\iint_{t>0} (\rho\phi_t + m\phi_x)\, dxdt + \int_{\mathbf{R}} \rho_0(x)\phi(x,0)\, dx &= 0,\\
\iint_{t>0} [m\phi_t + (\frac{m^2}{\rho} + P(\rho))\phi_x - m\phi]\, dxdt + \int_{\mathbf{R}} m_0(x)\phi(x,0)\, dx &= 0,\\
\iint_{t>0} (\eta_e\phi_t + q_e\phi_x - \rho u^2\phi)\, dxdt + \int_{\mathbf{R}} \eta_e(x,0)\phi(x,0)\, dx &\geqq 0.
\end{aligned}
\tag{1.4}
$$

Here, the entropy-flux pair (η_e, q_e) is associated with mechanical energy:

$$
\begin{aligned}
\eta_e &= \frac{1}{2}\rho u^2 + \frac{1}{(\gamma-1)}\rho^\gamma,\\
q_e &= \frac{1}{2}\rho u^3 + \frac{\gamma}{\gamma-1}\rho^\gamma u.
\end{aligned}
\tag{1.5}
$$

As the L^∞ weak solution does not have any degree of regularity, the methods for the case away from vacuum are not applicable here. Recently, some essential progress was made by HUANG & PAN [18]; the authors followed the rescaling argument due to SERRE & HSIAO [34] and obtained the first justification for the conjecture for the vacuum case. They showed that the density in the L^∞ weak entropy solutions of (1.1), (1.2) converges to the similarity solution of the porous medium equation along the level curve of the diffusive similarity profiles provided that one of the initial end-states is nonzero. The long-time behavior of the momentum is not known however. This is far from satisfactory. In [19] and [20], HUANG & PAN developed the new technique based on the conservation of mass and entropy analysis to attack this conjecture. They showed that the L^∞ weak entropy solutions with vacuum converge, strongly in $L^p(R)$ ($p \geqq p_0$ for any $p_0 \geqq 2$) with decay rates, to the similarity solution of the porous medium equation determined uniquely by the end-states and the mass distribution of the initial data provided that one of the end-states is away from vacuum.

However, the case where the initial data are L^1 remains as an important open problem. This case has particular interest since the asymptotic behavior is expected to be the famous Barenblatt solution of the porous media equation. We give a definite answer to this expectation in the following theorem.

Theorem 1.1. *Suppose $\rho_0(x) \in L^1(\mathbf{R})$, and $M = \int_{-\infty}^{\infty} \rho_0(x)\,dx$. Let (ρ, m) be an L^∞ entropy weak solution of the Cauchy problem* (1.1), (1.2)*, satisfying the estimates*

$$0 \leqq \rho(x,t) \leqq C, \ |m(x,t)| \leqq C\rho(x,t), \tag{1.6}$$

and let $\bar\rho$ be the Barenblatt solution of (1.3) *with mass M and $\bar m = -P(\bar\rho)_x$. Then*

$$\begin{aligned} \|\bar\rho\|_{L^2}^2 &= O(1)(1+t)^{-\frac{1}{\gamma+1}}, \\ \|\bar\rho\|_{L^\gamma}^\gamma &= \int_{-\infty}^{\infty} P(\bar\rho)\,dx = O(1)(1+t)^{-\frac{\gamma-1}{\gamma+1}}. \end{aligned} \tag{1.7}$$

Define $y = -\int_{-\infty}^{x} (\rho-\bar\rho)(r,t)\,dr$. If $y(x,0) \in L^2(\mathbf{R})$, then there exist positive constants $k_1 = \min\{\frac{\gamma^2}{(\gamma+1)^2}, \frac{\gamma-1}{\gamma}\}$, $k_2 = \min\{\frac{\gamma^2}{(\gamma+1)^2}, \frac{1}{\gamma}\}$ and C such that for any $\varepsilon > 0$,

$$\begin{aligned} \|(\rho-\bar\rho)(x,t)\|_{L^2}^2 &\leqq C(1+t)^{-k_1+\varepsilon} \quad \text{if } 1 < \gamma \leqq 2, \\ \|(\rho-\bar\rho)(x,t)\|_{L^\gamma}^\gamma &\leqq C(1+t)^{-k_2+\varepsilon} \quad \text{if } \gamma > 2, \end{aligned} \tag{1.8}$$

where

$$\begin{aligned} k_1 &> \frac{1}{\gamma+1} \quad \text{for any } \gamma > \frac{1+\sqrt{5}}{2}, \\ k_2 &> \frac{\gamma-1}{\gamma+1} \quad \text{if } \gamma < 1+\sqrt{2}. \end{aligned} \tag{1.9}$$

Furthermore, for $1 < \gamma < 1+\sqrt{2}$, $P(\rho)$ decays as fast as $P(\bar\rho)$ in the sense

$$\int_{-\infty}^{\infty} P(\rho)\,dx = O(1)(1+t)^{-\frac{\gamma-1}{\gamma+1}}. \tag{1.10}$$

And for $1 < \gamma < 2$,

$$\|(\rho-\bar\rho)\|_{L^\gamma}^\gamma \leqq C(1+t)^{-\frac{\gamma-1}{\gamma+1}}. \tag{1.11}$$

Remark. (1) Condition (1.6) is fulfilled if the solutions are in the physical region initially. The invariant region theory verifies (1.6); see [4].

(2) The explicit form of the Barenblatt solution of (1.3) is given in Section 2.

(3) Theorem 1.1 states that any L^∞ entropy weak solutions of (1.1) and (1.2) satisfying the conditions in Theorem 1.1 must converge to the related Barenblatt solution of (1.3) with the same mass. Although there is not uniqueness for the solutions, our results indicate the unique asymptotic profile determined by the initial mass. Equation (1.10) indicates that $\|\rho\|_{L^\gamma}$ decays to zero as fast as the L^γ norm of Barenbaltt's solution $\bar{\rho}$.

(4) Since Barenblatt's solution $\bar{\rho}$ decays itself, it is interesting to compare the decay rate of $\bar{\rho}$ with that of $\rho - \bar{\rho}$. The inequality (1.9) shows that $\|\rho - \bar{\rho}\|_{L^2}$ decays faster than $\|\bar{\rho}\|_{L^2}$ when $\frac{1+\sqrt{5}}{2} < \gamma < 2$, and $\|\rho - \bar{\rho}\|_{L^\gamma}$ decays faster than $\|\bar{\rho}\|_{L^\gamma}$ if $2 \leqq \gamma < 1+\sqrt{2}$. The inequality (1.11) shows that $\|\rho - \bar{\rho}\|_{L^\gamma}$ decays at least as fast as $\|\bar{\rho}\|_{L^\gamma}$ if $1 < \gamma < 2$. This covers most interesting physical cases. However, the estimates in Section 4 strongly suggest that $\|\rho - \bar{\rho}\|_{L^\gamma}$ decays faster than the rates given in this Theorem and faster than the decay rates of the $\|\bar{\rho}\|_{L^\gamma}$ for any $\gamma > 1$. This will be carried out in a future paper.

Let us explain the basic ideas of this paper. Two main difficulties are the lack of regularity and the singularity near vacuum. Our ideas are based on the nature of the system: the conservation of mass, the structure of the pressure law, the dissipation of damping and the existence of a convex entropy (the mechanical energy). We want to explore these features to control the singularity and nonlinearity. Our first observation is that the mechanical energy will give a uniform estimate for the solutions (ρ, m). However, this estimate is not useful in the proof of the long-time behavior. We thus construct the proper functions by expanding the entropy around the Barenblatt profile $(\bar{\rho}, \bar{m})$; this might give the estimate for the difference between our solutions and Barenblatt's profiles $(\rho - \bar{\rho}, m - \bar{m})$. In order to obtain the large time convergence, higher-order estimates are necessary. There are several ways to get higher-order estimates if the solutions are smooth. However, our solutions are rather rough. Luckily, when we observe the conservation of the mass, we find the mass difference between our solutions and Barenblatt's profiles are zero. So, it is possible to introduce anti-derivative $y(x,t)$ for $(\rho - \bar{\rho})(x,t)$. Thus, our entropy estimate becomes the derivatives estimate for y. Furthermore, the equation of y is wave equation with source term. Thus, the normal energy method will give some kind of estimate on y and its derivatives. Coupling these two estimates in a clever way, the uniform estimates for both y and its derivatives are possible. However, life is not so easy. The singularity near vacuum makes our goal much harder to reach. In order to control the singularity near vacuum, we explore the structure of the convection and find some useful inequalities near vacuum; see Lemma 3.1 below. With the help of these inequalities, careful analysis on our two estimates gives the desired results. Then a weighted entropy estimate will give the decay rates. Our proof is somehow tricky and technical, this is due to the difficulties of the problem. Our argument becomes neat and simple when it is applied to the case away from vacuum.

Since (1.1) is hyperbolic, the entropy estimate is much more natural than the parabolic-type energy method used in [12]. Such an entropy analysis goes back to DAFERMOS [5] and DIPERNA [9], see the books by DAFERMOS [7] and SERRE [33] for more references. Our proof may be compared with the proof by LIU & HSIAO [12] for smooth small solutions away from vacuum. In [12], the estimates were obtained by a normal parabolic-type energy method for wave equations. To weaken and decouple the nonlinearity, smallness and the third-order estimates are necessary in order to close the argument. We can check that such a method is not applicable for our case. The nonlinear terms in convection cannot be controlled without higher-order derivative estimates. Here, we succeed in closing our argument on first-order estimates for large rough solutions. This is one of the remarkable advantages of our approach.

The arrangement of the present paper is as follows. In Section 2, some knowledge on the Barenblatt solutions are prepared carefully. The crucial uniform estimates are made in Section 3 and the decay estimates are done in Section 4.

2. The Barenblatt solution

We suspect that the large-time behavior of the solutions to (1.1) and (1.2) could be described by the fundamental solutions of porous media equations, i.e., the Barenblatt solution. By the results of [1], the solution of

$$\begin{aligned} \bar\rho_t &= (\bar\rho^{\gamma})_{xx}, \\ \bar\rho(-1,x) &= M\delta(x), \quad M>0, \end{aligned} \tag{2.1}$$

should take the form

$$\bar\rho(x,t) = (t+1)^{-\frac{1}{\gamma+1}}\{(A-B\xi^2)_+\}^{\frac{1}{\gamma-1}}, \tag{2.2}$$

with $\xi = x(t+1)^{-\frac{1}{\gamma+1}}$, $(f)_+ = \max\{0,f\}$, $B = \frac{\gamma-1}{2\gamma(\gamma+1)}$ and A determined by

$$2A^{\frac{\gamma+1}{2(\gamma-1)}}B^{-\frac{1}{2}}\int_0^{\frac{\pi}{2}}(\cos\theta)^{\frac{\gamma+1}{\gamma-1}}\,d\theta = M. \tag{2.3}$$

Here $\bar\rho$ is a weak solution to (2.1) such that

$$\int_{-\infty}^{\infty}\bar\rho\,dx = M, \tag{2.4}$$

and

$$\bar\rho = 0, \quad if\ \ |\xi| \geqq \sqrt{A/B}. \tag{2.5}$$

Hence, for any finite time $T>0$, $\bar\rho$ has compact support. This is the property of finite speed of propagation for the porous medium equation. Furthermore, the derivatives of $\bar\rho$ are not continuous across the interface between the gas and vacuum. This is because the porous medium equation is parabolic away from vacuum and is not at vacuum. For the definition of the weak solution to (2.1), we refer to [1, 2, 21].

KAMIN proved in [21] that (2.1) admits at most one solution. Here, we addressed the initial data at $t=-1$ to avoid the singularity at $t=0$. Thus, we have the following lemmas from (2.1)–(2.5).

Lemma 2.1. *If M is finite, then there is one and only one solution $\bar{\rho}(x,t)$ to* (2.1). *Furthermore,*

(1) *$\bar{\rho}(x,t)$ is continuous on R,*

(2) *there is a number $b=(\frac{A}{B})^{\frac{1}{2}}>0$, such that $\bar{\rho}(x,t)>0$ if $|x|<bt^{\frac{1}{\gamma+1}}$ and $\bar{\rho}(x,t)=0$ if $|x|\geqq bt^{\frac{1}{\gamma+1}}$,*

(3) *$\bar{\rho}(x,t)$ is smooth if $|x|<bt^{\frac{1}{\gamma+1}}$.*

In terms of the explicit form of $\bar{\rho}$, it is easy to check the following estimates.

Lemma 2.2. *For $\bar{\rho}$ defined in* (2.2) *and $t>0$,*

$$\begin{aligned} |\bar{\rho}| &\leqq C(1+t)^{-\frac{1}{\gamma+1}}, \\ |(\bar{\rho}^{\gamma-1})_x| &\leqq C(1+t)^{-\frac{\gamma}{\gamma+1}}, |(\bar{\rho}^{\gamma-1})_t| \leqq C(1+t)^{-\frac{2\gamma}{\gamma+1}}, \\ |(\bar{\rho}^{\gamma})_x| &\leqq C(1+t)^{-1}, |(\bar{\rho}^{\gamma})_t| \leqq C(1+t)^{-\frac{2\gamma+1}{\gamma+1}}, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} \bar{\rho}^2 dx &\leqq C(1+t)^{-\frac{1}{\gamma+1}}, \\ \int_{-\infty}^{\infty} \bar{\rho}^{\gamma} dx &\leqq C(1+t)^{-\frac{\gamma-1}{\gamma+1}}, \\ \int_{-\infty}^{\infty} (\bar{\rho}^{\gamma-1})_x^2\, dx &\leqq C(1+t)^{-\frac{2\gamma-1}{\gamma+1}}, \\ \int_{-\infty}^{\infty} (\bar{\rho}^{\gamma-1})_t^2\, dx &\leqq C(1+t)^{-\frac{4\gamma-1}{\gamma+1}}, \\ \int_{-\infty}^{\infty} (\bar{\rho}^{\gamma})_x^2\, dx &\leqq C(1+t)^{-\frac{2\gamma+1}{\gamma+1}}, \\ \int_{-\infty}^{\infty} (\bar{\rho}^{\gamma})_t^2\, dx &\leqq C(1+t)^{-\frac{4\gamma+1}{\gamma+1}}. \end{aligned} \tag{2.7}$$

3. Uniform Estimates

In this section, we are going to establish the basic estimates for the difference between solutions of (1.1), (1.2) and the related Barenblatt profile. Our approaches are based on the conservation of mass, the analysis of entropy inequality and the control of the singularity near vacuum states.

First of all, we give a generalized version of Lemma 4.1 of HUANG & PAN [18]. These simple inequalities play an important role in controlling the singularity near vacuum.

Lemma 3.1. *Let $0\leqq a,b\leqq\Lambda<\infty$. There are positive constants C_1 and C_2 such that*

(1) $|a-b|^{\gamma+1}\leqq(a-b)(P(a)-P(b))$,

(2) $C_1|a-b|^2 \leqq [P(a)-P(b)-P'(b)(a-b)] \leqq C_2|a-b|^\gamma$ *if* $1<\gamma\leqq 2$,
(3) $C_1|a-b|^\gamma \leqq [P(a)-P(b)-P'(b)(a-b)] \leqq C_2|a-b|^2$ *if* $\gamma>2$.

Proof. It suffices to prove

$$[P(a)-P(b)-P'(b)(a-b)] \leqq \begin{cases} C_2|a-b|^\gamma & \text{if } 1<\gamma\leqq 2, \\ C_2|a-b|^2 & \text{if } \gamma>2, \end{cases}$$

because all the other inequalities are given in [18].

Without loss of generality, we assume that $a \geqq b$ and thus $x=a-b\geqq 0$ and $x\leqq \Lambda$. For $\gamma>2$, $|P''(x)|\leqq M$, we define $F(x)=P(b+x)-P(b)-P'(b)x-Mx^2$ which satisfies $F(0)=0$, and

$$\begin{aligned} F'(x) &= P'(b+x)-P'(b)-2Mx \\ &\leqq (P''(b+\theta x)-2M)x \quad \text{for } \theta\in[0,1] \\ &\leqq -Mx \leqq 0. \end{aligned}$$

Hence, we have $F(x)\leqq 0$ for $x\in[0,\Lambda]$. And thus

$$[P(a)-P(b)-P'(b)(a-b)] \leqq M(a-b)^2.$$

For $1<\gamma\leqq 2$, we define $f(x)=P(b+x)-P(b)-P'(b)x-\gamma x^\gamma$ which satisfies $f(0)=0$. We observe that $f(x)=(1-\gamma)x^\gamma\leqq 0$ if $b=0$. Assume $b\geqq b_1>0$. We compute

$$\begin{aligned} f'(x) &= \gamma(b+x)^{\gamma-1}-\gamma b^{\gamma-1}-\gamma^2x^{\gamma-1} \\ &\leqq \gamma[(b+x)^{\gamma-1}-b^{\gamma-1}-\gamma x^{\gamma-1}]. \end{aligned}$$

Hence $f'(0)=0$. For $x\in(0,\Lambda]$, we compute

$$\begin{aligned} f''(x) &= \gamma(\gamma-1)[(b+x)^{\gamma-2}-\gamma x^{\gamma-2}] \\ &= \gamma(\gamma-1)x^{\gamma-2}[(\frac{x}{b+x})^{2-\gamma}-\gamma] \\ &\leqq -\gamma(\gamma-1)^2x^{\gamma-2}<0, \end{aligned}$$

and $\lim_{x\to 0^+} f''(x)<0$. Thus, we have $f'(x)\leqq 0$ for $x\geqq 0$. This implies $f(x)\leqq 0$ for $\Lambda\geqq x\geqq 0$ and $b\geqq 0$. Therefore,

$$[P(a)-P(b)-P'(b)(a-b)] \leqq \gamma|a-b|^\gamma. \quad \square$$

Suppose that (ρ,m) is a weak entropy solution of (1.1) and (1.2) satisfying the conditions in Theorem 1.1. Let $\bar\rho$ be the Barenblatt solution of (2.1) such that

$$M=\int_{-\infty}^{\infty}\bar\rho(x,t)\,dx=\int_{-\infty}^{\infty}\rho_0(x)\,dx.$$

Due to the conservation of mass, it is easy to see

$$M=\int_{-\infty}^{\infty}\rho(x,t)\,dx.$$

Let $\bar{m} = -P(\bar{\rho})_x$ and

$$\begin{aligned} w &= \rho - \bar{\rho}, \\ z &= m - \bar{m}, \end{aligned} \tag{3.2}$$

which satisfies

$$\begin{aligned} w_t + z_x &= 0 \\ z_t + \left(\frac{m^2}{\rho}\right)_x + (P(\rho) - P(\bar{\rho}))_x + z &= -\bar{m}_t, \end{aligned} \tag{3.3}$$

and

$$\int_{-\infty}^{\infty} w(x,t)\, dx = 0.$$

Define

$$y = -\int_{-\infty}^{x} w(r,t)dr. \tag{3.4}$$

We have

$$y_x = -w, \quad z = y_t. \tag{3.5}$$

Therefore the second equation of (3.3) turns into a wave equation with source term:

$$y_{tt} + \left(\frac{m^2}{\rho}\right)_x + (P(\rho) - P(\bar{\rho}))_x + y_t = -\bar{m}_t. \tag{3.6}$$

Multiplying y with (3.6) and integrating over $[0,t] \times (-\infty, \infty)$, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(y_t y + \frac{1}{2} y^2\right) dx - \int_0^t \int_{-\infty}^{\infty} y_t^2\, dx d\tau \\ &\quad + \int_0^t \int_{-\infty}^{\infty} (P(\rho) - P(\bar{\rho}))(\rho - \bar{\rho})\, dx d\tau \\ &\leqq C \|(y, (m - \bar{m}))(x,0)\|_{L^2}^2 + \int_0^t \int_{-\infty}^{\infty} \frac{m^2}{\rho} y_x\, dx d\tau \\ &\quad + \int_0^t \int_{-\infty}^{\infty} y_x (\bar{\rho}^\gamma)_t\, dx d\tau. \end{aligned} \tag{3.7}$$

Due to Lemma 3.1 and Lemma 2.2, we have the following inequalities:

$$(P(\rho) - P(\bar{\rho}))(\rho - \bar{\rho}) \geqq |\rho - \bar{\rho}|^{\gamma+1} = |y_x|^{\gamma+1}, \tag{3.8}$$

$$\begin{aligned} \int_0^t \int_{-\infty}^{\infty} y_x (\bar{\rho}^\gamma)_t\, dx d\tau &\leqq \int_0^t (1+\tau)^{-\frac{2\gamma+1}{\gamma+1}} \|y_x\|_{L^1}\, d\tau \\ &\leqq C. \end{aligned} \tag{3.9}$$

Thus, (3.7)–(3.9) imply the following lemma.

Lemma 3.2. *Let y be the function defined in* (3.4). *If* $y(x,0) \in L^2(\mathbf{R})$, *then*

$$\int_{-\infty}^{\infty} \left(y_t y + \frac{1}{2} y^2 \right) dx - \int_0^t \int_{-\infty}^{\infty} y_t^2 \, dxdt + \int_0^t \int_{-\infty}^{\infty} |y_x|^{\gamma+1} \, dxdt$$
$$\leqq C + \int_0^t \int_{-\infty}^{\infty} \frac{m^2}{\rho} y_x \, dxdt. \tag{3.10}$$

In order to deal with the nonlinearity and singularity near vacuum, we now use the entropy inequality, rather than the usual energy method. This, together with (3.10), will give one of our desired estimates in the following theorem.

Theorem 3.3. *Let y be the function defined in* (3.4) *such that* $y(x,0) \in L^2(\mathbf{R})$. *Then*

$$\int_{-\infty}^{\infty} (y^2 + y_t^2 + |y_x|^2) dx + \int_0^t \int_{-\infty}^{\infty} (|y_x|^{\gamma+1} + y_t^2) \, dxd\tau$$
$$\leqq C \quad if \ 1 < \gamma \leqq 2,$$
$$\int_{-\infty}^{\infty} (y^2 + y_t^2 + |y_x|^{\gamma}) dx + \int_0^t \int_{-\infty}^{\infty} (|y_x|^{\gamma+1} + y_t^2) \, dxd\tau$$
$$\leqq C \quad if \ \gamma > 2. \tag{3.11}$$

In order to prove Theorem 3.3, we choose

$$\eta_e = \frac{m^2}{2\rho} + \frac{1}{\gamma - 1} P(\rho)$$

to be the mechanical energy and q_e the related flux as in Definition 1. Then we define

$$\eta_* = \eta_e - \frac{1}{\gamma-1} P'(\bar{\rho})(\rho - \bar{\rho}) - \frac{1}{\gamma - 1} P(\bar{\rho}). \tag{3.12}$$

It is easy to check that

$$\frac{1}{\gamma-1} P(\bar{\rho})_t = (\cdots)_x - \frac{\bar{m}^2}{\bar{\rho}}. \tag{3.13}$$

Thus, by the definition of weak entropy solution, the following entropy inequality holds in the sense of distribution:

$$\eta_{*t} + \frac{1}{\gamma-1}[P'(\bar{\rho})(\rho - \bar{\rho})]_t + q_{ex} + (\cdots)_x + \left(\frac{m^2}{\rho} - \frac{\bar{m}^2}{\bar{\rho}} \right) \leqq 0. \tag{3.14}$$

By the theory of divergence-measure fields (see CHEN & FRID [3]), we have

$$\frac{d}{dt} \int_{-\infty}^{\infty} \eta_*(x,t) \, dx + \frac{d}{dt} \int_{-\infty}^{\infty} \frac{1}{\gamma-1} [P'(\bar{\rho})(\rho - \bar{\rho})] \, dx$$
$$+ \int_{-\infty}^{\infty} \left(\frac{m^2}{\rho} - \frac{\bar{m}^2}{\bar{\rho}} \right) dx \leqq 0. \tag{3.15}$$

Since

$$\int_{-\infty}^{\infty} P'(\bar\rho)(\rho-\bar\rho)\,dx \leqq CM, \tag{3.16}$$

we integrate (3.15) over $[0, t]$ and obtain

Lemma 3.4. *For any $t > 0$,*

$$\int_{-\infty}^{\infty} \eta_*(x,t)\,dx + \int_0^t \int_{-\infty}^{\infty} \left(\frac{m^2}{\rho} - \frac{\bar m^2}{\bar\rho}\right) dx \leqq C. \tag{3.17}$$

At this moment, no conclusion can be drawn from (3.17) since $\frac{m^2}{\rho} - \frac{\bar m^2}{\bar\rho}$ varies in sign. One role in the proof of Theorem 3.3 is the study of the term $\frac{m^2}{\rho} - \frac{\bar m^2}{\bar\rho}$. By the Taylor expansion of $\frac{m^2}{\rho}$ around $(\bar m, \bar\rho)$, we have

$$\frac{m^2}{\rho} = \frac{\bar m^2}{\bar\rho} + \frac{2\bar m}{\bar\rho} z - \frac{\bar m^2}{\bar\rho^2}(\rho-\bar\rho) + Q, \tag{3.18}$$

where

$$Q = \frac{m^2}{\rho} - \frac{2\bar m}{\bar\rho} m + \frac{\bar m^2}{\bar\rho^2}\rho = \left(\frac{m}{\sqrt{\rho}} - \frac{\bar m}{\bar\rho}\rho\right)^2 \geqq 0, \tag{3.19}$$

due to the convexity of $\frac{m^2}{\rho}$. Then we have

Lemma 3.5. *For any $t > 0$,*

$$\int_{-\infty}^{\infty} \eta_*(x,t)\,dx + \int_0^t \int_{-\infty}^{\infty} Q(x,\tau)\,dx \leqq C. \tag{3.20}$$

Proof. By Lemma 3.4 and (3.18), it is clear that

$$\begin{aligned} &\int_{-\infty}^{\infty} \eta_*(x,t)\,dx + \int_0^t \int_{-\infty}^{\infty} Q(x,\tau)\,dx \\ &\leqq C + \left| \int_0^t \int_{-\infty}^{\infty} \frac{2\bar m}{\bar\rho} z\,dxd\tau \right| + \left| \int_0^t \int_{-\infty}^{\infty} \frac{\bar m^2}{\bar\rho^2}(\rho-\bar\rho)\,dxd\tau \right|. \end{aligned} \tag{3.21}$$

However,

$$\begin{aligned} &\left| \int_0^t \int_{-\infty}^{\infty} \frac{\bar m^2}{\bar\rho^2}(\rho-\bar\rho)\,dxd\tau \right| \\ &\leqq C \int_0^t \left| \frac{\bar m^2}{\bar\rho^2} \right|_{L^\infty} \|(\rho-\bar\rho)\|_{L^1}\,d\tau \\ &\leqq CM \int_0^t (1+\tau)^{-\frac{2\gamma}{\gamma+1}}\,d\tau \\ &\leqq C, \end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
&\left|\int_0^t \int_{-\infty}^{\infty} \frac{2\bar{m}}{\bar{\rho}} z \, dx d\tau\right| \\
&\quad = \left|\int_0^t \int_{-\infty}^{\infty} \frac{2\bar{m}}{\bar{\rho}} y_t \, dx d\tau\right| \\
&\quad \leqq C \left|\int_0^t \int_{-\infty}^{\infty} (\bar{\rho}^{\gamma-1} y_x)_t \, dx d\tau\right| + C \left|\int_0^t \int_{-\infty}^{\infty} (\bar{\rho}^{\gamma-1})_t y_x \, dx d\tau\right| \\
&\quad \leqq C + C \int_0^t (1+\tau)^{-\frac{2\gamma}{\gamma+1}} \, d\tau \\
&\quad \leqq C,
\end{aligned} \tag{3.23}$$

due to $\|y_x\|_{L^1} \leqq C$. Therefore, (3.20) follows from (3.21)–(3.23). □

Since $Q \geqq 0$, it is obvious that

Corollary 3.6. *For any $t > 0$,*

$$\int_{-\infty}^{\infty} \eta_*(x,t) \, dx \leqq C,$$

$$\int_0^t \int_{-\infty}^{\infty} Q(x,\tau) \, dx d\tau \leqq C,$$

$$\int_0^t \int_{-\infty}^{\infty} \left(\frac{m^2}{\rho} - \frac{\bar{m}^2}{\bar{\rho}}\right) dx d\tau \leqq C.$$

Now we are able to improve the estimate in Lemma 3.2 as follows.

Lemma 3.7. *For any $t > 0$,*

$$\int_{-\infty}^{\infty} \left(y_t y + \frac{1}{2} y^2\right) dx - \frac{3}{2} \int_0^t \int_{-\infty}^{\infty} y_t^2 \, dx dt + \int_0^t \int_{-\infty}^{\infty} |y_x|^{\gamma+1} \, dx dt \leqq C.$$

Proof. We are going to bound the last term in (3.10). From (3.18) we have

$$\frac{m^2}{\rho} y_x = \left(Q + \frac{2\bar{m}}{\bar{\rho}} z + \frac{\bar{m}^2}{\bar{\rho}^2} y_x\right) y_x + \frac{\bar{m}^2}{\bar{\rho}} y_x,$$

thus by the estimates in (3.22) and (3.23), we have

$$\begin{aligned}
&\int_0^t \int_{-\infty}^{\infty} \frac{m^2}{\rho} y_x \, dx d\tau \\
&\quad = \int_0^t \int_{-\infty}^{\infty} \left[\left(Q + \frac{2\bar{m}}{\bar{\rho}} z + \frac{\bar{m}^2}{\bar{\rho}^2} y_x\right) y_x + \frac{\bar{m}^2}{\bar{\rho}} y_x\right] dx d\tau \\
&\quad \leqq C + C \left|\int_0^t \int_{-\infty}^{\infty} \left(Q|y_x| + \frac{2\bar{m}}{\bar{\rho}} z y_x + \frac{\bar{m}^2}{\bar{\rho}^2} y_x^2\right) dx d\tau\right| \\
&\quad \leqq C + \int_0^t \int_{-\infty}^{\infty} \left(\frac{1}{2} z^2 + C \frac{\bar{m}^2}{\bar{\rho}^2} y_x^2\right) dx d\tau \\
&\quad \leqq C + \frac{1}{2} \int_0^t \int_{-\infty}^{\infty} y_t^2 \, dx d\tau.
\end{aligned} \tag{3.24}$$

Then, (3.24) and (3.10) imply Lemma 3.7. □

We now proceed with further analysis on $(\frac{m^2}{\rho} - \frac{\bar{m}^2}{\bar{\rho}})$. Let

$$\Omega_1 = \{(x,t) : \rho(x,t) < \bar{\rho}\}, \quad \Omega_2 = \{(x,t) : \rho(x,t) \geqq \bar{\rho}\},$$
$$\Omega_{1t} = \{x : \rho(x,t) < \bar{\rho}\}, \qquad \Omega_{2t} = \{x : \rho(x,t) \geqq \bar{\rho}\},$$

and

$$F(\Omega_1) = \iint_{\Omega_1} \left(\frac{m^2}{\rho} - \frac{\bar{m}^2}{\bar{\rho}}\right) dx d\tau, \quad F(\Omega_2) = \iint_{\Omega_2} \left(\frac{m^2}{\rho} - \frac{\bar{m}^2}{\bar{\rho}}\right) dx d\tau.$$

In $\Omega_1(t)$, we have $|y_x| \leqq \bar{\rho}$ and

$$\begin{aligned}
&\frac{m^2}{\rho} - \frac{\bar{m}^2}{\bar{\rho}} \\
&\quad = \frac{m^2}{\rho\bar{\rho}} y_x + \frac{1}{\bar{\rho}}(m^2 - \bar{m}^2) \\
&\quad = \frac{z^2}{\bar{\rho}} + \frac{2z\bar{m}}{\bar{\rho}} + \frac{y_x}{\bar{\rho}}\left(Q + \frac{2\bar{m}}{\bar{\rho}} z + \frac{\bar{m}^2}{\bar{\rho}^2} y_x + \frac{\bar{m}^2}{\bar{\rho}}\right).
\end{aligned} \tag{3.25}$$

Hence, we have

$$\begin{aligned}
&\iint_{\Omega_1} \frac{z^2}{\bar{\rho}}\, dx d\tau \\
&\quad \leqq F(\Omega_1) + \left|\iint_{\Omega_1} \left[\frac{2z\bar{m}}{\bar{\rho}} + \frac{y_x}{\bar{\rho}}\left(Q + \frac{2\bar{m}}{\bar{\rho}} z + \frac{\bar{m}^2}{\bar{\rho}^2} y_x + \frac{\bar{m}^2}{\bar{\rho}}\right)\right] dx d\tau\right| \\
&\quad \leqq C + F(\Omega_1) + \left|\iint_{\Omega_1} \frac{y_x}{\bar{\rho}} \frac{2\bar{m}}{\bar{\rho}} z\, dx d\tau\right| \\
&\quad \leqq C + F(\Omega_1) + \frac{1}{2}\iint_{\Omega_1} \frac{z^2}{\bar{\rho}}\, dx d\tau + C\iint_{\Omega_1} \frac{\bar{m}^2}{\bar{\rho}^2} \frac{|y_x|}{\bar{\rho}} |y_x|\, dx d\tau \\
&\quad \leqq C + F(\Omega_1) + \frac{1}{2}\iint_{\Omega_1} \frac{z^2}{\bar{\rho}}\, dx d\tau,
\end{aligned} \tag{3.26}$$

which implies that

$$\iint_{\Omega_1} \frac{z^2}{\bar{\rho}}\, dx d\tau \leqq C + 2F(\Omega_1). \tag{3.27}$$

Here, the integral in (3.26) could be bounded in the same manner as in the case for the domain $[0,t] \times (-\infty, \infty)$.

On the other hand, in $\Omega_2(t)$, we have $|y_x| \leqq \rho$ and

$$\begin{aligned}
\frac{m^2}{\rho} - \frac{\bar{m}^2}{\bar{\rho}} &= \frac{z^2}{\rho} + \frac{2\bar{m}}{\rho} z + \frac{\bar{m}^2}{\rho\bar{\rho}} y_x \\
&= \frac{z^2}{\rho} + \frac{2\bar{m}}{\bar{\rho}} z + \frac{2\bar{m}}{\rho\bar{\rho}} z y_x + \frac{\bar{m}^2}{\rho\bar{\rho}} y_x.
\end{aligned} \tag{3.28}$$

Thus, we have

$$\begin{aligned}
&\iint_{\Omega_2} \frac{z^2}{\rho}\, dx d\tau \\
&\quad \leqq C + F(\Omega_2) + \left| \iint_{\Omega_2} \left(\frac{2\bar{m}}{\bar{\rho}} z + \frac{2\bar{m}}{\rho\bar{\rho}} z y_x + \frac{\bar{m}^2}{\rho\bar{\rho}} y_x \right) dx d\tau \right| \\
&\quad \leqq C + F(\Omega_2) + C \left| \iint_{\Omega_2} \frac{2\bar{m}}{\rho\bar{\rho}} z y_x \, dx d\tau \right| + C \left| \iint_{\Omega_2} \frac{\bar{m}^2}{\bar{\rho}^2} \bar{\rho}\, dx d\tau \right| \\
&\quad \leqq C + F(\Omega_2) + \frac{1}{2} \iint_{\Omega_2} \frac{z^2}{\rho}\, dx d\tau + C \iint_{\Omega_2} \frac{\bar{m}^2}{\bar{\rho}^2} \frac{|y_x|}{\rho} |y_x| \, dx d\tau \\
&\quad \leqq C + F(\Omega_2) + \frac{1}{2} \iint_{\Omega_2} \frac{z^2}{\rho}\, dx d\tau,
\end{aligned}$$

which implies that

$$\iint_{\Omega_2} \frac{z^2}{\rho}\, dx d\tau \leqq C + 2F(\Omega_2). \tag{3.29}$$

We thus proved the following result.

Lemma 3.8. *Let y be defined as in* (3.4). *Then*,

$$\int_0^t \int_{-\infty}^{\infty} y_t^2 \, dx d\tau \leqq C. \tag{3.30}$$

We now turn to explore the estimates on η_*. In Corollary 3.6, we have

$$\int_{-\infty}^{\infty} \eta_* \, dx \leqq C.$$

By Lemma 3.1 and $|y_x| \leqq C$, we thus have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left(\frac{m^2}{\rho} + |y_x|^2 \right) dx \leqq C \quad \text{if } 1 < \gamma \leqq 2, \\
&\int_{-\infty}^{\infty} \left(\frac{m^2}{\rho} + |y_x|^\gamma \right) dx \leqq C \quad \text{if } \gamma > 2,
\end{aligned} \tag{3.31}$$

and hence

$$\int_{-\infty}^{\infty} \frac{m^2}{\rho} \, dx \leqq C. \tag{3.32}$$

In Ω_{1t}, we have $|y_x| \leqq \bar{\rho}$ and (3.25). Therefore,

$$\begin{aligned}
\int_{\Omega_{1t}} \frac{z^2}{\bar{\rho}} \, dx &\leqq \left| \int_{\Omega_{1t}} \left(\frac{m^2}{\rho} - \frac{\bar{m}^2}{\bar{\rho}} - \frac{m^2}{\rho\bar{\rho}} y_x - \frac{2\bar{m}}{\bar{\rho}} z \right) dx \right| \\
&\leqq C + \left| \int_{\Omega_{1t}} \frac{2\bar{m}}{\bar{\rho}} z \, dx \right| \\
&\leqq C + C \int_{\Omega_{1t}} \left| \frac{\bar{m}}{\bar{\rho}} \right| dx \\
&\leqq C,
\end{aligned}$$

which implies

$$\int_{\Omega_{1t}} \frac{z^2}{\bar{\rho}}\, dx \leqq C$$

and

$$\int_{\Omega_{1t}} y_t^2\, dx \leqq C. \tag{3.33}$$

On the other hand, in Ω_{2t}, we have $|y_x| \leqq \rho$ and (3.28). Therefore,

$$\begin{aligned}\int_{\Omega_{2t}} \frac{z^2}{\rho}\, dx &\leqq \left|\int_{\Omega_{2t}} \left(\frac{m^2}{\rho} - \frac{\bar{m}^2}{\bar{\rho}} - \frac{2\bar{m}}{\bar{\rho}} z - \frac{2\bar{m}}{\bar{\rho}} z \frac{y_x}{\rho} - \frac{\bar{m}^2}{\bar{\rho}} \frac{y_x}{\rho}\right) dx\right| \\ &\leqq C,\end{aligned}$$

which implies

$$\int_{\Omega_{2t}} \frac{z^2}{\rho}\, dx \leqq C$$

and

$$\int_{\Omega_{2t}} y_t^2\, dx \leqq C. \tag{3.34}$$

Thus, (3.31)–(3.34) gives

Lemma 3.9. *For any $t \geqq 0$,*

$$\begin{aligned}&\int_{-\infty}^{\infty} (|y_x|^2 + y_t^2)\, dx \leqq C \quad \text{if } 1 < \gamma \leqq 2, \\ &\int_{-\infty}^{\infty} (|y_x|^\gamma + y_t^2)\, dx \leqq C \quad \text{if } \gamma \geqq 2.\end{aligned}$$

Lemmas 3.7, 3.8 and 3.9 imply Theorem 3.3.

4. Decay rates

We now prove the decay rate for the difference between the solutions of damped Euler equations and the Barenblatt solutions for porous media equations. Our main technique is the weighted entropy estimates. In fact, we have

Theorem 4.1. *For any $t \geqq 0$, there are constants $k_1 = \min\{\frac{\gamma^2}{(\gamma+1)^2}, \frac{\gamma-1}{\gamma}\}$, $k_2 = \min\{\frac{\gamma^2}{(\gamma+1)^2}, \frac{1}{\gamma}\}$ and $C > 0$ such that for any $\varepsilon > 0$,*

$$\begin{aligned}&(1+t)^{k_1-\varepsilon} \int_{-\infty}^{\infty} (\eta_* + y_t^2)(x,t)\, dx \leqq C \quad \textit{if } 1 < \gamma \leqq 2, \\ &(1+t)^{k_2-\varepsilon} \int_{-\infty}^{\infty} (\eta_* + y_t^2)(x,t)\, dx \leqq C \quad \textit{if } \gamma \geqq 2.\end{aligned} \tag{4.1}$$

Namely,

$$\int_{-\infty}^{\infty}(|\rho-\bar\rho|^2+(m-\bar m)^2)(x,t)\,dx \leqq C(1+t)^{-k_1+\varepsilon} \quad if\ 1<\gamma\leqq 2,$$
$$\int_{-\infty}^{\infty}(|\rho-\bar\rho|^\gamma+(m-\bar m)^2)(x,t)\,dx \leqq C(1+t)^{-k_2+\varepsilon} \quad if\ \gamma\geqq 2. \tag{4.2}$$

Proof. We multiply the equation (3.15) with $(1+t)^k$ to obtain

$$(1+t)^k\frac{d}{dt}\int_{-\infty}^{\infty}(\eta_*(x,t)+\frac{1}{\gamma-1}[P'(\bar\rho)(\rho-\bar\rho)])\,dx$$
$$+(1+t)^k\int_{-\infty}^{\infty}\left(\frac{m^2}{\rho}-\frac{\bar m^2}{\bar\rho}\right)dx \leqq 0. \tag{4.3}$$

Thus, we have

$$\frac{d}{dt}\int_{-\infty}^{\infty}(1+t)^k\eta_*(x,t)dx+\frac{d}{dt}\int_{-\infty}^{\infty}(1+t)^k\frac{1}{\gamma-1}[P'(\bar\rho)(\rho-\bar\rho)]\,dx$$
$$+(1+t)^k\int_{-\infty}^{\infty}\left(\frac{m^2}{\rho}-\frac{\bar m^2}{\bar\rho}\right)dx$$
$$\leqq k(1+t)^{k-1}\int_{-\infty}^{\infty}\eta_*(x,t)+\frac{1}{\gamma-1}[P'(\bar\rho)(\rho-\bar\rho)]\,dx.$$

Integrating the above over $[0,t]$, we have

$$(1+t)^k\int_{-\infty}^{\infty}(\eta_*(x,t)+\frac{1}{\gamma-1}[P'(\bar\rho)(\rho-\bar\rho)])\,dx$$
$$+\int_0^t\int_{-\infty}^{\infty}(1+\tau)^k\left(\frac{m^2}{\rho}-\frac{\bar m^2}{\bar\rho}\right)dxd\tau$$
$$\leqq C+\int_0^t\int_{-\infty}^{\infty}k(1+\tau)^{k-1}\eta_*(x,\tau)dxd\tau$$
$$+k\int_0^t\int_{-\infty}^{\infty}(1+\tau)^{k-1}\frac{1}{\gamma-1}[P'(\bar\rho)(\rho-\bar\rho)]\,dxd\tau. \tag{4.4}$$

First of all, we bound the terms on the right-hand side of (4.4). We observe for some positive δ that

$$k\int_0^t\int_{-\infty}^{\infty}(1+\tau)^{k-1}\frac{1}{\gamma-1}[P'(\bar\rho)(\rho-\bar\rho)]\,dxd\tau$$
$$\leqq C\int_0^t\int_{-\infty}^{\infty}(1+\tau)^{(2k-2)+(1+\delta)}(\bar\rho^{\gamma-1})_x^2\,dxd\tau$$
$$+C\int_0^t(1+\tau)^{-1-\delta}\|y(\cdot,\tau)\|_{L^2}^2\,d\tau$$
$$\leqq C(\delta)+C\int_0^t(1+\tau)^{2k-1+\delta-\frac{2\gamma-1}{\gamma+1}}\,d\tau$$
$$\leqq C \tag{4.5}$$

if

$$2k < \frac{2\gamma - 1}{\gamma + 1}$$

and

$$\delta \leqq \frac{1}{4}\left(\frac{2\gamma - 1}{\gamma + 1} - 2k\right).$$

Due to Lemma 3.1, we have

$$[P(\rho) - P(\bar{\rho}) - P'(\bar{\rho})(\rho - \bar{\rho})] \leqq \begin{cases} C|\rho - \bar{\rho}|^{\gamma} & \text{if } 1 < \gamma < 2, \\ C|\rho - \bar{\rho}|^{2} & \text{if } \gamma \geqq 2. \end{cases} \tag{4.6}$$

Hence, for $\gamma \geqq 2$, we have

$$\begin{aligned} &\int_0^t \int_{-\infty}^{\infty} k(1+\tau)^{k-1}\eta_*(x,\tau)dxd\tau \\ &\leqq k\int_0^t \int_{-\infty}^{\infty} (1+\tau)^{k-1}\frac{m^2}{2\rho}\,dxd\tau \\ &\quad + C\int_0^t \int_{-\infty}^{\infty} (1+\tau)^{k-1} y_x^2\,dxd\tau \\ &\leqq \frac{k}{2}\int_0^t \int_{-\infty}^{\infty} (1+\tau)^{k-1}\left(\frac{m^2}{\rho} - \frac{\bar{m}^2}{\bar{\rho}}\right)dxd\tau \\ &\quad + \frac{k}{2}\int_0^t \int_{-\infty}^{\infty} (1+\tau)^{k-1}\frac{\bar{m}^2}{\bar{\rho}}\,dxd\tau \\ &\quad + C\int_0^t \int_{-\infty}^{\infty} (1+\tau)^{k-1} y_x^2\,dxd\tau, \end{aligned} \tag{4.7}$$

where

$$\int_0^t \int_{-\infty}^{\infty} (1+\tau)^{k-1}\frac{\bar{m}^2}{\bar{\rho}}\,dxd\tau \leqq C\int_0^t (1+\tau)^{k-1-\frac{2\gamma}{\gamma+1}}\,d\tau \leqq C, \tag{4.8}$$

and

$$\begin{aligned} &\int_0^t \int_{-\infty}^{\infty} (1+\tau)^{k-1} y_x^2\,dxd\tau \\ &\leqq C_1\int_0^t \int_{-\infty}^{\infty} ((1+\tau)^{k-1}|y_x|^{\frac{\gamma-1}{\gamma}})^{\frac{\gamma}{\gamma-1}}\,dxd\tau \\ &\quad + C_2\int_0^t \int_{-\infty}^{\infty} (|y_x|^{\frac{1+\gamma}{\gamma}})^{\gamma}\,dxd\tau \\ &\leqq C\int_0^t \int_{-\infty}^{\infty} (1+\tau)^{(k-1)(\frac{\gamma}{\gamma-1})}|y_x|\,dxd\tau \\ &\quad + C\int_0^t \int_{-\infty}^{\infty} |y_x|^{\gamma+1}dxd\tau \\ &\leqq C \end{aligned} \tag{4.9}$$

if $(k-1)(\frac{\gamma}{\gamma-1}) < -1$, i.e., $k < \frac{1}{\gamma}$.

For $1 \leqq \gamma < 2$, we have

$$\begin{aligned}
&\int_0^t \int_{-\infty}^{\infty} k(1+\tau)^{k-1}\eta_*(x,\tau)dxd\tau \\
&\quad \leqq \frac{k}{2}\int_0^t \int_{-\infty}^{\infty}(1+\tau)^{k-1}\left(\frac{m^2}{\rho}-\frac{\bar{m}^2}{\bar{\rho}}\right) dxd\tau \\
&\qquad +\frac{k}{2}\int_0^t \int_{-\infty}^{\infty}(1+\tau)^{k-1}\frac{\bar{m}^2}{\bar{\rho}}\, dxd\tau \\
&\qquad +C\int_0^t \int_{-\infty}^{\infty}(1+\tau)^{k-1}|y_x|^{\gamma}\, dxd\tau \\
&\quad \leqq C+\frac{k}{2}\int_0^t \int_{-\infty}^{\infty}(1+\tau)^{k-1}\left(\frac{m^2}{\rho}-\frac{\bar{m}^2}{\bar{\rho}}\right) dxd\tau \\
&\qquad +C\int_0^t \int_{-\infty}^{\infty}(1+\tau)^{k-1}|y_x|^{\gamma}\, dxd\tau.
\end{aligned} \tag{4.10}$$

However,

$$\begin{aligned}
&\int_0^t \int_{-\infty}^{\infty}(1+\tau)^{k-1}|y_x|^{\gamma}\, dxd\tau \\
&\quad \leqq C_3\int_0^t \int_{-\infty}^{\infty}((1+\tau)^{k-1}|y_x|^{\frac{1}{\gamma}})^{\gamma}\, dxd\tau \\
&\qquad +C_4\int_0^t \int_{-\infty}^{\infty}(|y_x|^{(\gamma-\frac{1}{\gamma})})^{\frac{\gamma}{\gamma-1}}\, dxd\tau \\
&\quad \leqq C\int_0^t \int_{-\infty}^{\infty}(1+\tau)^{\gamma(k-1)}|y_x|\, dxd\tau \\
&\qquad +C\int_0^t \int_{-\infty}^{\infty}|y_x|^{\gamma+1}\, dxd\tau \\
&\quad \leqq C,
\end{aligned} \tag{4.11}$$

if $\gamma(k-1) < -1$, i.e., $k < 1-\frac{1}{\gamma}$.

It is observed that

$$\begin{cases} \frac{2\gamma-1}{2(\gamma+1)} \geqq 1-\frac{1}{\gamma} & \text{if } 1<\gamma\leqq 2,\\ \frac{2\gamma-1}{2(\gamma+1)} \geqq \frac{1}{\gamma} & \text{if } \gamma \geqq 2. \end{cases}$$

Thus, we conclude from (4.4)–(4.11) that for any $\varepsilon > 0$,

$$\begin{aligned}
&(1+t)^{\frac{\gamma-1}{\gamma}-\varepsilon}\int_{-\infty}^{\infty}(\eta_*(x,t)+\frac{1}{\gamma-1}[P'(\bar{\rho})(\rho-\bar{\rho}))\, dx \\
&\quad +\int_0^t \int_{-\infty}^{\infty}(1+\tau)^{\frac{\gamma-1}{\gamma}-\varepsilon}\left(\frac{m^2}{\rho}-\frac{\bar{m}^2}{\bar{\rho}}\right) dxd\tau \leqq C \text{ if } 1<\gamma\leqq 2,\\
&(1+t)^{\frac{1}{\gamma}-\varepsilon}\int_{-\infty}^{\infty}(\eta_*(x,t)+\frac{1}{\gamma-1}[P'(\bar{\rho})(\rho-\bar{\rho}))\, dx \\
&\quad +\int_0^t \int_{-\infty}^{\infty}(1+\tau)^{\frac{1}{\gamma}-\varepsilon}\left(\frac{m^2}{\rho}-\frac{\bar{m}^2}{\bar{\rho}}\right) dxd\tau \leqq C \text{ if } \gamma\leqq 2.
\end{aligned}$$

Since

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} (1+t)^k \frac{1}{\gamma-1} [P'(\bar\rho)(\rho-\bar\rho)]\, dx \right| \\
&\quad \leqq C \left| \int_{-\infty}^{\infty} (1+t)^k (\bar\rho^{\gamma-1})_x y\, dx \right| \\
&\quad \leqq \int_{-\infty}^{\infty} y^2\, dx + C \int_{-\infty}^{\infty} (1+t)^{2k} (\bar\rho^{\gamma-1})_x^2\, dx \\
&\quad \leqq C + C(1+t)^{2k-\frac{2\gamma-1}{\gamma+1}} \leqq C
\end{aligned} \tag{4.12}$$

if $k < \frac{2\gamma-1}{2(\gamma+1)}$, we arrive at

$$\begin{aligned}
(1+t)^k \int_{-\infty}^{\infty} \eta_*(x,t)dx + \int_0^t \int_{-\infty}^{\infty} \left(1 - \frac{k}{2(1+\tau)}\right) (1+\tau)^k \\
\times \left(\frac{m^2}{\rho} - \frac{\bar m^2}{\bar\rho} \right) dx d\tau \leqq C,
\end{aligned} \tag{4.13}$$

where

$$k < q(\gamma) = \begin{cases} \frac{\gamma-1}{\gamma} & \text{if } 1 < \gamma \leqq 2, \\ \frac{1}{\gamma} & \text{if } \gamma \geqq 2. \end{cases}$$

From now on, we choose $k < q(\gamma)$. By (3.18), we deduce that

$$\begin{aligned}
&(1+t)^k \int_{-\infty}^{\infty} \eta_*(x,t)dx \\
&\quad + \int_0^t \int_{-\infty}^{\infty} \left(1 - \frac{k}{2(1+\tau)}\right) (1+\tau)^k Q(x,\tau)\, dx d\tau \\
&\leqq C + \left| \int_0^t \int_{-\infty}^{\infty} \left(1 - \frac{k}{2(1+\tau)}\right) (1+\tau)^k \left[\frac{2\bar m z}{\bar\rho} + \frac{\bar m^2}{\bar\rho^2} y_x \right] dx d\tau \right|.
\end{aligned} \tag{4.14}$$

However,

$$\begin{aligned}
&\left| \int_0^t \int_{-\infty}^{\infty} \left(1 - \frac{k}{2(1+\tau)}\right) (1+\tau)^k \frac{\bar m^2}{\bar\rho^2} y_x\, dx d\tau \right| \\
&\quad \leqq C \int_0^t \int_{-\infty}^{\infty} \left[(1+\tau)^k \frac{\bar m^2}{\bar\rho^2} \right]^{\frac{\gamma+1}{\gamma}} dx d\tau + C \int_0^t \int_{-\infty}^{\infty} |y_x|^{\gamma+1} dx d\tau \\
&\quad \leqq C + C \int_0^t (1+\tau)^{k\frac{\gamma+1}{\gamma} - 2 + \frac{1}{\gamma+1}} d\tau \leqq C
\end{aligned} \tag{4.15}$$

if $k < \frac{\gamma^2}{(\gamma+1)^2}$. Here, we used the following estimate:

$$\int_{-\infty}^{\infty} \left(\frac{\bar m^2}{\bar\rho^2} \right)^{\frac{\gamma+1}{\gamma}} dx \leqq C(1+t)^{-2+\frac{1}{\gamma+1}},$$

which is from Lemma 2.2.

We now choose $k < \min\{q(\gamma), \frac{\gamma^2}{(\gamma+1)^2}\}$. By (4.5), we have

$$\begin{aligned}
&\left|\int_0^t \int_{-\infty}^{\infty} (1+\tau)^k \frac{\bar{m}}{\bar{\rho}} z \, dx d\tau\right| \\
&\quad \leqq \left|\int_0^t \int_{-\infty}^{\infty} \left[(1+\tau)^k \frac{\bar{m}}{\bar{\rho}} y\right]_t \, dx d\tau\right| + \left|\int_0^t \int_{-\infty}^{\infty} k(1+\tau)^{k-1} \frac{\bar{m}}{\bar{\rho}} y \, dx d\tau\right| \\
&\qquad + \left|\int_0^t \int_{-\infty}^{\infty} (1+\tau)^k y_x (\bar{\rho}^{\gamma-1})_t \, dx d\tau\right| \\
&\quad \leqq C + C\left|\int_0^t \int_{-\infty}^{\infty} (1+\tau)^k y_x (\bar{\rho}^{\gamma-1})_t \, dx d\tau\right| \\
&\quad \leqq C + C\int_0^t \int_{-\infty}^{\infty} |y_x|^{\gamma+1} dx d\tau + C\int_0^t \int_{-\infty}^{\infty} [(1+\tau)^k (\bar{\rho}^{\gamma-1})_t]^{\frac{\gamma+1}{\gamma}} dx d\tau \\
&\quad \leqq C,
\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}
&\left|\int_0^t \int_{-\infty}^{\infty} (1+\tau)^{k-1} \frac{\bar{m}}{\bar{\rho}} z \, dx d\tau\right| \\
&\quad \leqq C\int_0^t \int_{-\infty}^{\infty} (1+\tau)^{2(k-1)} (\bar{\rho}_x)^2 \, dx d\tau + \int_0^t \int_{-\infty}^{\infty} y_t^2 \, dx d\tau \\
&\quad \leqq C.
\end{aligned} \tag{4.17}$$

Thus, we conclude from (4.14)–(4.17) that

$$(1+t)^k \int_{-\infty}^{\infty} \eta_*(x,t) dx + \int_0^t \int_{-\infty}^{\infty} (1+\tau)^k Q(x,\tau) \, dx d\tau \leqq C \tag{4.18}$$

and

$$(1+t)^k \int_{-\infty}^{\infty} \eta_*(x,t) dx \leqq C \tag{4.19}$$

for positive k such that $k < \min\left\{q(\gamma), \frac{\gamma^2}{(\gamma+1)^2}\right\}$.

As the consequences of (4.19) and Lemma 3.1, we have, for any $\varepsilon > 0$,

$$\begin{aligned}
&(1+t)^{k_1-\varepsilon} \int_{-\infty}^{\infty} \left(|y_x|^2 + \frac{m^2}{\rho}\right) dx \leqq C \text{ if } 1 < \gamma \leqq 2, \\
&(1+t)^{k_2-\varepsilon} \int_{-\infty}^{\infty} \left(|y_x|^\gamma + \frac{m^2}{\rho}\right) dx \leqq C \text{ if } \gamma \geqq 2,
\end{aligned} \tag{4.20}$$

where $k_1 = \min\left\{\frac{\gamma^2}{(\gamma+1)^2}, \frac{\gamma-1}{\gamma}\right\}$, $k_2 = \min\left\{\frac{\gamma^2}{(\gamma+1)^2}, \frac{1}{\gamma}\right\}$.

We now establish the decay estimates for y_t based on (4.20).

In Ω_{1t}, we have $|y_x| \leqq \bar{\rho}$ and (3.25). Therefore,

$$\begin{aligned}
&(1+t)^k \int_{\Omega_{1t}} \frac{z^2}{\bar{\rho}}\, dx \\
&\leqq C + \left| \int_{\Omega_{1t}} (1+t)^k \left[\frac{m^2}{\rho} - \frac{\bar{m}^2}{\bar{\rho}} - \frac{m^2}{\rho\bar{\rho}} y_x - \frac{2\bar{m}}{\bar{\rho}} z \right] dx \right| \\
&\leqq C + \left| \int_{\Omega_{1t}} (1+t)^k \frac{2\bar{m}}{\bar{\rho}} z\, dx \right| \\
&\leqq C,
\end{aligned}$$

which implies

$$(1+t)^k \int_{\Omega_{1t}} y_t^2\, dx \leqq C. \tag{4.21}$$

On the other hand, in Ω_{2t}, we have $|y_x| \leqq \rho$ and (3.28). Therefore,

$$\begin{aligned}
&(1+t)^k \int_{\Omega_{2t}} \frac{z^2}{\rho}\, dx \\
&\leqq C + \left| \int_{\Omega_{2t}} (1+t)^k \left[\frac{m^2}{\rho} - \frac{\bar{m}^2}{\bar{\rho}} - \frac{2\bar{m}}{\bar{\rho}} z - \frac{2\bar{m}}{\bar{\rho}} z \frac{y_x}{\rho} - \frac{\bar{m}^2}{\bar{\rho}} \frac{y_x}{\rho} \right] dx \right| \\
&\leqq C + \left| \int_{\Omega_{2t}} (1+t)^k \frac{2\bar{m}}{\bar{\rho}} z\, dx \right| \\
&\leqq C.
\end{aligned} \tag{4.22}$$

The inequality (4.22) implies that

$$(1+t)^k \int_{\Omega_{2t}} y_t^2\, dx \leqq C. \tag{4.23}$$

We thus conclude from (4.21) and (4.23) that

$$(1+t)^k \int_{-\infty}^{\infty} y_t^2\, dx \leqq C. \tag{4.24}$$

The inequalities (4.19) and (4.24) give the results in Theorem 4.1. $\square$

With the help of Theorem 4.1, a more careful analysis of η_* leads to the proof of Theorem 1.1.

Proof of Theorem 1.1. First of all, it is easy to check that

$$\frac{\gamma^2}{(\gamma+1)^2} > \frac{\gamma-1}{\gamma+1}, \ \frac{\gamma-1}{\gamma} > \frac{\gamma-1}{\gamma+1}, \tag{4.25}$$

while

$$\frac{1}{\gamma} > \frac{\gamma-1}{\gamma+1} \text{ if } 1 < \gamma < 1+\sqrt{2}, \tag{4.26}$$

and

$$k_1 > \frac{1}{\gamma+1} \text{ if } \frac{1+\sqrt{5}}{2} < \gamma. \tag{4.27}$$

Hence, (1.8) and (1.9) follow from Theorem 4.1.

For (1.10), we assume $1 < \gamma < 1+\sqrt{2}$ and choose $\frac{\gamma-1}{\gamma+1} < k < \min\left\{q(\gamma), \frac{\gamma^2}{(\gamma+1)^2}\right\}$, which is possible by (4.25)–(4.27). We also have (4.19):

$$(1+t)^k \int_{-\infty}^{\infty} \eta_*(x,t)dx \leqq C.$$

This implies the following estimates:

$$(1+t)^k \int_{-\infty}^{\infty} F(\rho,\bar\rho)(x,t)\,dx \leqq C, \tag{4.28}$$

where

$$F(\rho,\bar\rho) = P(\rho) - P(\bar\rho) - P'(\bar\rho)(\rho-\bar\rho).$$

We observe that $F(\rho,\bar\rho) \geqq 0$ for any $\rho \geqq 0$ and $\bar\rho \geqq 0$. Hence,

$$\int_{-\infty}^{\infty} P(\rho)\,dx = \int_{-\infty}^{\infty} F(\rho,\bar\rho)\,dx + \int_{-\infty}^{\infty} P(\bar\rho)\,dx + \int_{-\infty}^{\infty} P'(\bar\rho)(\rho-\bar\rho)\,dx. \tag{4.29}$$

By (4.12) and (4.19), we have

$$\int_{-\infty}^{\infty} F(\rho,\bar\rho)\,dx \leqq C(1+t)^{-k}, \quad \left|\int_{-\infty}^{\infty} P'(\bar\rho)(\rho-\bar\rho)\,dx\right| \leqq C(1+t)^{-k}.$$

Since $\int_{-\infty}^{\infty} P(\bar\rho)\,dx$ decays at a rate $(1+t)^{-\frac{\gamma-1}{\gamma+1}}$, we conclude from (4.26) that $P(\rho)$ decays at the same rate as $P(\bar\rho)$, and

$$\int_{-\infty}^{\infty} P(\rho)\,dx = \int_{-\infty}^{\infty} \rho^\gamma\,dx = O(1)(1+t)^{-\frac{\gamma-1}{\gamma+1}}. \tag{4.30}$$

This is for (1.10).

We now prove (1.11). Assuming $1 < \gamma < 2$, we observe that

$$|P(\rho) - P(\bar\rho)| \leqq |F(\rho,\bar\rho)| + |P'(\bar\rho)(\rho-\bar\rho)|,$$

and

$$|\rho-\bar\rho|^\gamma \leqq |P(\rho) - P(\bar\rho)|,$$

which is obtained by dividing the first inequality of Lemma 3.1 with $|a-b|$. Thus, we have

$$\begin{aligned}
&(1+t)^{\frac{\gamma-1}{\gamma+1}} \int_{-\infty}^{\infty} |\rho-\bar\rho|^\gamma\,dx \\
&\quad \leqq (1+t)^{\frac{\gamma-1}{\gamma+1}} \int_{-\infty}^{\infty} F(\rho,\bar\rho)\,dx + (1+t)^{\frac{\gamma-1}{\gamma+1}} \int_{-\infty}^{\infty} P'(\bar\rho)|\rho-\bar\rho|\,dx
\end{aligned}$$

$$\leqq C + C(1+t)^{\frac{\gamma-1}{\gamma+1}} \|P'(\bar{\rho})\|_{L^\infty}$$
$$\leqq C. \tag{4.31}$$

This ends the proof of Theorem 1.1. □

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