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Darcy's law as long-time limit of adiabatic porous media flow

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Abstract

It is conjectured that Darcy's law governs the motion of compressible porous media flow in large time. This has been justified for one-dimensional *isentropic* flows. In this work, we show the conjecture is true for one-dimensional *adiabatic* flows with generic small smooth initial data.

Keywords: Darcy's Law; Damping mechanism; L^1-estimates; Weighted energy estimates; Self-similarity

1. Introduction

In this paper, we study the large-time asymptotic behavior of smooth solutions to the following Cauchy problem:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v,s)_x = -\alpha u, \ \ \alpha > 0, \\ s_t = 0, \\ (v,u,s)(x,0) = (v_0(x), u_0(x), s_0(x)), \quad x \in R, \\ (v_0,u_0,s_0)(x) \to (v_\pm, u_\pm, s_\pm), \ \ as \ \ x \to \pm\infty, \end{cases} \tag{1.1}$$

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with $v_\pm > 0$. This system models one-dimensional gas flows through porous media when the process admits smooth solutions. If singularities occur in the process, the third equation in the above system will be replaced by the energy equation, and one has the classical compressible Euler equations with damping:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v,s)_x = -\alpha u, \quad \alpha > 0, \\ (e(v,s) + \frac{1}{2}u^2)_t + (pu)_x = -\alpha u^2. \end{cases} \tag{1.2}$$

Here, (x, t) are space and time variables in Lagrangian coordinates, v, u and s denote the specific volume, particle velocity and specific entropy, respectively. p is the gas pressure with $p_v(v, s) < 0$, for $v > 0$, and $e(v, s)$ is the specific internal energy, for which one has $e_s \neq 0$ and $e_v + p = 0$ (due to the second law of thermodynamics). For smooth solutions, the third equation of (1.2) is equivalent to $s_t = 0$.

In practice, Darcy's law was observed experimentally in such a process, and one has an alternative model:

$$\begin{cases} v_t - u_x = 0, \\ p(v,s)_x = -\alpha u, \quad \alpha > 0, \\ s_t = 0, \end{cases}$$

which is equivalent to the following (decoupled) system:

$$\begin{cases} v_t + \frac{1}{\alpha} p(v,s)_{xx} = 0, \\ u = -\frac{1}{\alpha} p(v,s)_x, \\ s_t = 0. \end{cases} \tag{1.3}$$

Hence, it is not surprise that (1.1) and (1.3) are conjectured to be equivalent as time goes to infinity. That is, one expects that Darcy's law is valid for compressible porous media flows time-asymptotically. More precisely, the conjecture reads that the solutions of (1.1) converge to the corresponding solutions of (1.3) as $t \to +\infty$.

For isentropic flows, where $s = const.$, the conjecture was verified first by Hsiao and Liu [5] for small smooth solutions away from vacuum. Since then, this problem has attracted considerable attentions of mathematicians. There are several improvements of [5]; see for instance [6,14,16,17,19,21,23] and the book [4] and the references therein. However, the more important issue is to generalize [5] to more general physical settings. Apparently, there are two lines for such kind of generalizations. The first one is to verify the conjecture for isentropic flows with large data. This was achieved recently by Huang et al. [11–13] where they proved this conjecture for general L^∞ weak entropy solutions with or without vacuum. The second one is to extend the result of [5] to the full system of adiabatic flows where $s \neq const.$. In this direction, the first attempt was made by Hsiao and Serre [10] for a small smooth perturbation problem about constant state

$(\bar{v}, 0, \bar{s})$. For a better decay rate in this case, we refer to [18]. Later, Hsiao and Luo [7] set the adiabatic flow as the perturbation near the isentropic flows. Their proof relies on comparison principles and a technical condition. More recently, by combining L^1 technique and weighted energy estimates, Marcati and Pan [15] generalized [10,7] to the cases of perturbation around a steady solution or isentropic flows. However, the case for generic small smooth solutions remains open.

The main focus of current paper is to fill this blank. For this purpose, we study the following Cauchy problem:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v, s)_x = -u, \\ s_t = 0, \\ (v, u, s)(x, 0) = (v_0(x), u_0(x), s_0(x)), \quad x \in R, \\ (v_0, u_0, s_0)(x) \to (v_\pm, u_\pm, s_\pm), \ as \ x \to \pm\infty, \end{cases} \tag{1.4}$$

with $v_\pm > 0$. For sake of simplicity, we have taken $\alpha = 1$. Furthermore, from now on, we choose $p(v, s) = (\gamma - 1)v^{-\gamma}e^s$, with $\gamma > 1$, which is the case for the polytropic gas dynamics.

The global existence with small smooth initial data of smooth solutions for the Cauchy problem (1.4) has been studied first in [9,10,22]. This is based on the fact that the dissipation of damping prevents the formation of singularity from small smooth initial data. However, such dissipation is not strong enough, the shock will develop in finite time if initial data is large or rough, see [22]. On the other hand, no theory is known on the global existence of weak solutions for (1.4), although [1] provides a global BV solution to isentropic flow. Thus we will focus on the small smooth solutions. According to the conjecture mentioned before, these smooth solutions are expected to converge to certain solutions of

$$\begin{cases} \tilde{v}_t = -p(\tilde{v}, s)_{xx}, \\ \tilde{u} = -p(\tilde{v}, s)_x, \\ s_t = 0, \\ \tilde{v}(x, 0) = \tilde{v}_0(x), \ \ s(x, 0) = s_0(x), \\ \tilde{v}_0(\pm\infty) = v_\pm, \ \ s_0(\pm\infty) = s_\pm. \end{cases} \tag{1.5}$$

Partial evidences of such expectation are given in [10] for the case $v_- = v_+ = \bar{v}$ and $s_- = s_+ = \bar{s}$; in [7,15] for the case $s_- = s_+ = \bar{s}$; and in [15] for the case $p(v_-, s_-) = p(v_+, s_+)$. Thus, the case where $p(v_-, s_-) \neq p(v_+, s_+)$ and $s_- \neq s_+$ remains open while some initial boundary value problems have been solved completely; see [8,20]. In this article, we shall give a definite answer to the conjecture for adiabatic flows with generic small smooth initial data away from vacuum. We are going to prove that the small smooth solutions of (1.4) converge to those of (1.5) provided that the initial data satisfies certain mass law. This justified the Darcy's law in the time-asymptotic point of view. Furthermore, the large-time asymptotic profiles are constructed explicitly. The striking point is that such a diffusive profile has singularity which is different from the isentropic case [5] and the cases in [7,10,15,18].

Comparing with isentropic case, the first main difficulty in our case is the existence and large-time behavior of the solutions to (1.5) is not known in literatures. This difficulty prevented us to adopt the approach for isentropic flows, where the corresponding large-time behavior for the diffusive problems is known as the self-similar solutions [2,3]. This is because that the equations in (1.5) reduce to porous media equation when entropy is a constant, which is invariant under diffusive rescalings. However, such invariance breaks down when entropy varies. Thus, one cannot work on (1.4) directly like in isentropic case. The further study of problem (1.5) is necessary. In Sections 2–3, we will establish the global existence and large-time behavior of solutions to (1.5), the large-time profile will be constructed explicitly by means of self-similar solution to the *asymptotic problem* of (1.5) (not (1.5) itself) for a carefully chosen variable. Such self-similar solutions are singular in general (striking!). This differs from all previous results. The proof is based on the framework developed in [15]. However, the singularity in asymptotic profiles requires the modification of several key steps in the argument. One more feature distinguishing from isentropic case is that the asymptotic profile for specific volume is not self-similar! In our case, the asymptotic profiles for pressure and velocity are well-defined from asymptotic problem, while that for specific volume is not a solution of the asymptotic problem.

The second main difficulty follows from the singularity in the asymptotic profiles of solutions to (1.5). In previous work, the Darcy's law was verified by comparing the solutions of (1.4) with those of (1.5) using energy estimates up to second-order derivatives of solutions to (1.4). Such approach is not suitable for our case due to the singularity mentioned above. In Section 4, we thus adopt a modified version of the framework introduced in [12] by means of weighted entropy estimates to prove that the difference between solutions of (1.4) and (1.5) approaches to zero as time goes to infinity with a uniform estimate in lower order. Then, we can adopt the argument of [15] with regularity of the solutions of (1.5) obtained by standard theory of parabolic equations.

To end this introduction, we make some remarks. (a) Our purpose here is the verification of Darcy's law in large-time behavior of compressible adiabatic flows, the decay rates in some of our theorems are not optimal. It is possible to improve them by means of Green's function approach as in [18]. It is of course an interesting problem, but not the main concern of current paper. (b) It is remarkable that Theorem 4.2 below provided a framework for possible large weak solutions with uniform L^∞ bound. This leads to the chance of a proof of the conjecture for large weak solutions once they are obtained in the future.

2. Asymptotic profiles and main results

As explained in last section, one of the key difficulty in our problem is the mystery of the asymptotic profile of the solutions of (1.5). This will be discovered by a guess-and-check method. In this section, we first construct the asymptotic diffusive profiles serving as long-time ansatz for (1.4) and (1.5) by means of asymptotic self-similarity. To this end, we introduce the change of variables:

$$
\begin{aligned}
&a(x) = (\gamma-1)^{-\frac{1}{\gamma}} e^{-\frac{1}{\gamma} s(x)},\\
&w \equiv a(x)\tilde{v} = p(\tilde{v}, s)^{-\frac{1}{\gamma}},
\end{aligned}
\tag{2.1}
$$

then problem (1.4) is equivalent to the following one:

$$
\begin{cases}
w_t + a(x)(w^{-\gamma})_{xx} = 0,\\
\tilde{u} = -(w^{-\gamma})_x,\\
s(x,t) = s_0(x),\\
w(x,0) = w_0(x) = a(x)\tilde{v}_0(x),\\
w(\pm\infty) = w_\pm > 0, s(\pm\infty) = s_\pm.
\end{cases}
\tag{2.2}
$$

We recall that the asymptotic diffusive profiles were similarity solutions found by diffusive rescaling invariance for isentropic flow. For adiabatic flows, the first equation in (2.2) is not invariant under diffusive rescaling. However, one still can hope the asymptotic profiles are similarity solutions invariant under diffusive rescaling for $w(x,t)$. One evidence was given by Marcati and Pan [15] for the case when $s_- = s_+$. Roughly speaking, due to parabolicity of the first equation in (2.2), one expects that the information of solutions will be dominated by the end states in large time since the local information will diffuse. Thus, we expect the large-time asymptotic behavior of (2.2) will be described by

$$
\begin{cases}
\bar{w}_t + a_1(\bar{w}^{-\gamma})_{xx} = 0,\\
\bar{w}(\pm\infty) = w_\pm,
\end{cases}
\tag{2.3}
$$

where, $a_1 = (\gamma-1)^{-\frac{1}{\gamma}} e^{-\frac{1}{\gamma} s_-}$, if $x < 0$; while $a_1 = (\gamma-1)^{-\frac{1}{\gamma}} e^{-\frac{1}{\gamma} s_+}$, if $x > 0$. This equation is invariant under diffusive rescaling, thus it has self-similar solutions. We denote by $\bar{w}(\eta)$ (with $\eta = \frac{x}{\sqrt{t+1}}$) the similarity solution of (2.3).

We now construct this profile. Inserting $\bar{w}(\eta)$ into (2.3), one has a two-point boundary value problem for a second-order O.D.E.:

$$
\begin{cases}
-\frac{1}{2}\eta\bar{w}' + a_1(\eta)(\bar{w}^{-\gamma})'' = 0,\\
\bar{w}(\pm\infty) = w_\pm.
\end{cases}
\tag{2.4}
$$

Define $W = \bar{w}^{-\gamma}$, the first equation in (2.4) becomes

$$
W'' = -\frac{\eta}{2\gamma a_1(\eta)} W^{-1-\frac{1}{\gamma}} W',
\tag{2.5}
$$

which can be integrated as

$$W'(\eta) = W'(\eta_0) \exp\left\{-\int_{\eta_0}^{\eta} W^{-1-\frac{1}{\gamma}}(r) \frac{r\,dr}{2\gamma a_1(r)}\right\}. \tag{2.6}$$

Thus, W was obtained by integrating (2.6) once more. Hence, the similarity profile $\bar{w}$ was constructed. It is easy to see from above argument that $\bar{w}$ has continuous first-order derivatives and bounded second-order derivatives. However, $\bar{w}''(\eta)$ has a jump at $\eta = 0$ in general if a_1 has a jump. However, $a_1(\bar{w}^{-\gamma})''$ is continuous by (2.4). Moreover, it is clear that $\bar{w}$ has the properties listed in the following lemma.

Lemma 2.1. *Let $\bar{w}(\eta)$ be the similarity solution to (2.3) with $\bar{w}(\pm\infty) = w_\pm$ and $\eta = \frac{x}{\sqrt{1+t}}$. There are positive constants C, C_1 and C_2 such that*

$$|\bar{w}'(\eta)| + |\bar{w}''(\eta)| \leqslant C_1 |w_+ - w_-| \exp\{-C_2\eta^2\},$$
$$|\bar{w}(\eta) - w_-|_{\eta<0} + |\bar{w}(\eta) - w_+|_{\eta>0} \leqslant C_1 |w_+ - w_-| \exp\{-C_2\eta^2\},$$

$$\bar{w}_x = (1+t)^{-\frac{1}{2}} \bar{w}'(\eta), \quad \bar{w}_t = -\frac{1}{2}(1+t)^{-1}\eta\bar{w}'(\eta), \quad (\bar{w}^{-\gamma})_{xx} = -\frac{\bar{w}_t}{a_1},$$

$$\|\bar{w}_t(\cdot,t)\|_{L^2}^2 + \|\bar{w}_{xx}(\cdot,t)\|_{L^2}^2 \leqslant C|w_+ - w_-|^2 (1+t)^{-\frac{3}{2}},$$

$$\|D_x^i D_x^j \bar{w}(\cdot,t)\|_{L^\infty} \leqslant C_1 |w_+ - w_-| (1+t)^{-(i+\frac{1}{2}j)}$$

for $i = 0, 1, 2$, $j = 0, 1, 2$ and $i + j \geqslant 1$.

We conjecture that $\bar{w}$ constructed above is the asymptotic profile of the solution to (2.2). This will be stated in Theorem 2.1 below and be proved in the next section.

With the asymptotic profiles constructed above, we can state our main results in mathematical rigors. Since $s_t = 0$ in both (1.4) and (1.5), then $s(x,t) = s(x) = s_0(x)$. This is the advantage to use entropy instead of energy as variable. Let $\bar{w}$ be the similarity profiles defined above, $\bar{v} = a^{-1}\bar{w}$ and $\bar{u} = -(\bar{w}^{-\gamma})_x$. The global existence and large time behavior of solutions to (1.5) can be established by comparing w with $\bar{w}$ through a combination of L^1 argument and weighted energy estimates. Thus, our first main result is the following theorem.

Theorem 2.1. *Assume that $w_0(x)$ and $s_0(x)$ are C^2 functions such that $w_0(x) - \bar{w}(x,0) \in H^2(R) \cap L^1(R)$ and*

$$x(s_0(x) - s_-) \in L^1(R_-), \quad x(s_0(x) - s_+) \in L^1(R_+). \tag{2.7}$$

There exists $\delta_0 > 0$ such that, if $0 < \delta < \delta_0$ and

$$|w_+ - w_-| + \|w_0(x) - \bar{w}(x,0)\|_{H^2} \leqslant \delta,$$

then (1.5) *has a unique global smooth solution* $(\tilde{v}, \tilde{u}, s)(x, t)$, *satisfying*

$$w(x,t) - \bar{w} \in C^0([0,t]; H^2), \quad for\ all\ t > 0.$$

Moreover, there exist positive constants $C > 0$, $\beta_1 > \frac{1}{3}$ *and* $\beta_2 > \frac{1}{2}$, *such that*

$$\begin{aligned} &\|(\tilde{v} - \bar{v})(\cdot, t)\|_{L^\infty} \leqslant C(1+t)^{-\frac{1}{2}}(1 + \log(1+t))^{\beta_1}, \\ &\|(\tilde{u} - \bar{u})(\cdot, t)\|_{L^\infty} \leqslant C(1+t)^{-1}(1 + \log(1+t))^{\beta_2}. \end{aligned} \tag{2.8}$$

Remark 1. (a) Our results in Theorem 2.1 generalize those ones in [2], to the adiabatic case and extend to a larger class of initial data. The decay rate here is better than in [2] and it is almost optimal.

(b) Condition (2.7) can be weaken by replacing x with $|x|^\beta$ for $\beta > 0$. This is clear from our proof below.

(c) $(\bar{v}, \bar{u})$ is not a solution of the asymptotic problem (2.3), but they are the right asymptotic profile for the solutions of (1.5) although they do not satisfy the equation of mass conservation law.

We now compare the solutions of (1.4) and (1.5). Following [5], we define

$$\begin{aligned} &m(x,t) \equiv -(u_+ - u_-)m_0(x)e^{-t}, \\ &u_m(x,t) \equiv u_- e^{-t} + \int_{-\infty}^{x} m_t(\xi, t)\, d\xi, \end{aligned} \tag{2.9}$$

where $m_0(x)$ is a smooth function with compact support such that

$$\int_{-\infty}^{+\infty} m_0(x)\, dx = 1.$$

Denote by $(\tilde{v}, \tilde{u}, s)$ the solution to (1.5) obtained in Theorem 2.1. In addition, we assume

$$\int_{-\infty}^{+\infty} (v_0(x) - \tilde{v}_0(x))\, dx = -(u_+ - u_-). \tag{2.10}$$

A special choice of $\tilde{v}_0$ is given in Remark 2 below. Let us denote by $y(x,t) = \int_{-\infty}^{x} (v - \tilde{v} - m)(\xi, t)\, d\xi$, then y satisfies

$$\begin{cases} y_{tt} + [p(y_x + \tilde{v} + m, s) - p(\tilde{v}, s)]_x + y_t = p(\tilde{v}, s)_{xt}, \\ y(x,0) = y_0(x) = \int_{-\infty}^{x} (v_0(\xi) - \tilde{v}_0(\xi) - m(\xi, 0))\, d\xi, \\ y_t(x,0) = y_1(x) = u_0(x) - \tilde{u}(x,0) - u_m(x,0). \end{cases} \tag{2.11}$$

The following theorem shows that the solutions of (1.4) converge to the corresponding solutions of (1.5) sharing the same asymptotic profile.

Theorem 2.2. *Under the conditions of Theorem* 2.1 *and* (2.10), *if there exists* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$ *and* $\|y_0\|_{H^3} + \|y_1\|_{H^2} \leqslant \varepsilon$, *system* (2.11) *admits a unique global smooth solution* y, *such that*

$$y \in C^0([0,t]; H^3), \quad y_t \in C^0([0,t]; H^2)$$

for all $t > 0$. *Moreover, there exists* $C > 0$, *such that*

$$\|y_x(\cdot,t)\|_{L^\infty} \leqslant C(1+t)^{-\frac{3}{4}}, \quad \|y_t(\cdot,t)\|_{L^\infty} \leqslant C(1+t)^{-\frac{5}{4}}. \tag{2.12}$$

Hence, $v(x,t) = \tilde{v} + m + y_x$ *and* $u(x,t) = \tilde{u} + u_m + y_t$, *is the* (*unique*) *global smooth solution* (v,u,s) *to* (1.4), *such that*

$$\|(v-\tilde{v})(\cdot,t)\|_{L^\infty} \leqslant C(1+t)^{-\frac{3}{4}}, \quad \|(u-\tilde{u})(\cdot,t)\|_{L^\infty} \leqslant C(1+t)^{-\frac{5}{4}}. \tag{2.13}$$

Furthermore, *in view of Theorem* 2.1, *it holds that*

$$\begin{aligned} &\|(v-\bar{v})(\cdot,t)\|_{L^\infty} \leqslant C(1+t)^{-\frac{1}{2}}(1+\log(1+t))^{\beta_1}, \\ &\|(u-\bar{u})(\cdot,t)\|_{L^\infty} \leqslant C(1+t)^{-1}(1+\log(1+t))^{\beta_2}, \end{aligned} \tag{2.14}$$

where β_1 *and* β_2 *are the same as before*.

Remark 2. (a) The decay rates in Theorem 2.2 are not optimal. Since we focus on the verification of Darcy's law, we did not attempt to obtain the optimal decay rates in current paper.

(b) Condition (2.10) is the restriction on the initial data which comes from conservation law of mass in (1.4) and (1.5). This condition enables us to choose potential y through mass conservation law. In general, one cannot expect the stability of background if the perturbation carries unbounded excessive mass. There is a large class of functions $\tilde{v}_0(x)$ which can be chosen (for any given $v_0(x)$ in (1.4)). A special choice is $\tilde{v}_0(x) = a^{-1}\bar{w}(x+x_0,0)$, where x_0 is uniquely determined by

$$\int_{-\infty}^{+\infty} (v_0(x) - a^{-1}\bar{w}(x+x_0,0))\,dx = -(u_+ - u_-).$$

Apparently, such a $\tilde{v}_0(x)$ satisfies the condition in Theorem 2.1.

(c) In Section 4, we will only prove Theorem 2.2 for the case $u_- = u_+ = 0$ where $m(x,t) = 0$ and $u_m = 0$. The general case can be treated in the similar way since $m(x,t)$ and u_m decay to zero exponentially fast.

3. Nonlinear diffusive problems

In this section, we are going to compare the solutions of (2.2) with $\bar{w}(\eta)$ constructed in last section. We will prove that (2.2) has a global smooth solution with $\bar{w}(\eta)$ as the large-time asymptotic profile.

Let $\phi = w - \bar{w}$. From (2.2) and (2.3), we have the following equation:

$$\begin{cases} \phi_t + a(x)(\psi(\bar{w})\phi)_{xx} + (a - a_1)(\bar{w}^{-\gamma})_{xx} + a(x)(g(\phi, \tilde{w})\phi^2)_{xx} = 0, \\ \phi(x,0) = \phi_0(x) = w_0(x) - \bar{w}(x,0). \end{cases} \tag{3.1}$$

Here

$$\psi(\bar{w}) = -\gamma \bar{w}^{-(\gamma+1)},$$
$$g(\phi, \bar{w})\phi^2 = (\phi + \bar{w})^{-\gamma} - \bar{w}^{-\gamma} - \psi(\bar{w})\phi.$$

Now let $F = -\psi(\bar{w})\phi$, the corresponding problem on F is given by

$$\begin{cases} F_t + a(x)\psi(\bar{w})F_{xx} - \psi(\bar{w})(a - a_1)(\bar{w}^{-\gamma})_{xx} \\ \quad -\psi_1(\bar{w})F\bar{w}_t - a\psi(\bar{w})(fF^2)_{xx} = 0, \\ F(x,0) = F_0(x) = -\psi(\bar{w}(x,0))\phi_0(x), \end{cases} \tag{3.2}$$

where

$$-\psi_1(\bar{w})F = \psi'(\bar{w})\phi, \quad fF^2 = g\phi^2.$$

We will establish the global existence and large-time behavior, for the solution F to (3.2), in the Banach space $X(0,T)$, defined, for all $T > 0$, by

$$X(0,t) = \{F \in C^0([0,t]; H^2),\ 0 \leqslant t \leqslant T\}$$

and equipped with the norm

$$N^2(t) = \sup_{0 \leqslant \tau \leqslant t} \|F(\tau)\|_{H^2}^2.$$

To simplify notations, we will use $\|f\|$ to denote L^2 norm of function f in x throughout this paper. The main result of this subsection is the following theorem.

Theorem 3.1. *Assume that $F_0(x)$ and $s(x) = s_0(x)$ are C^2* functions, *such that $F_0 \in H^2(R) \cap L^1(R)$, and*

$$x(a(x) - a_1) \in L^1(R). \tag{3.3}$$

Then there exist constants $\varepsilon_0 > 0$ *and* $\delta > 0$, *such that if* $|w_+ - w_-| \leqslant \delta$ *and* $\|F_0\|_{H^2} \leqslant \varepsilon_0$, *then*, (3.2) *has a unique global smooth solution* F *satisfying*

$$\sum_{j=0}^{2} w_{j+1}(t)\|\partial_x^j F(\cdot,t)\|^2 + \int_0^t \sum_{j=1}^{3} w_j(\tau)\|\partial_x^j F(\cdot,\tau)\|^2\, d\tau \leqslant C,$$

where the weight functions $w_j(t)$ *are given by*

$$w_1(t) = (1+t)^{\frac{1}{2}}(1+\log(1+t))^{-k}, \quad w_j(t) = (1+t)^{j-1}w_1(t)$$

for $j, k > 1$.

Remark 3. (a) Condition (3.3) plays an important role in our proof of Theorem 2.2 (see Lemmas 3.2–3.10 below). This condition enables us to bound the L^1-norm of F (or ϕ) for all time. It requires decay properties of $s(x)$ at infinity. This condition can be replaced by the weaker one such as

$$|x|^{\beta}(a(x) - a_1) \in L^1(R) \tag{3.3'}$$

for some $\beta > 0$. This is clear following our proof. Eq. (3.3) is equivalent to (2.7) in Theorem 2.1.

(b) In general, we cannot bound the L^1-norm of F for all time without the conditions on the decay properties of $a(x) - a_1(x)$ as $x \to \pm\infty$ such as (3.3′). One even cannot bound the total mass of F uniformly in time under the condition $a(x) - a_1 \in L^1$. From this point of view, (3.3′) is optimal.

The local existence and uniqueness of the solution to (3.2) in $X(0,T)$ is standard, then to get the global existence, we will prove uniform estimates on the solution of (3.2). Hence, from now on, we assume the local existence in $X(0,T)$, for some $T > 0$.

The following L^1-estimate follows from the standard contraction property of the porous media type equation and will play a fundamental role in the analysis.

Lemma 3.2. *Under the conditions of Theorem* 3.1, *as long as the solution exists in* $X(0,T)$, *there exist positive constants* C_1 *and* C_2, *such that*

$$\|\phi(\cdot,t)\|_{L^1} \leqslant C_1\|F(\cdot,t)\|_{L^1} \leqslant C_2(\|\phi_0\|_{L^1} + \delta). \tag{3.4}$$

Proof. We present here a formal argument which can be easily made rigorous by using any sequence approximating the sign function and passing into the limit by means of the *Lebesgue Dominated Convergence Theorem*. Observe that $h = sign(\phi) = sign(F)$. Let us multiply the equation in (3.1) by $a^{-1}h$, then by integrating over $[0,t] \times (-\infty, +\infty)$, it follows

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} a^{-1}|\phi|(x,t)\,dx + \int_0^t \int_{-\infty}^{+\infty} sign'(F) F_x^2\,dx\,d\tau \\
&\leqslant C \int_{-\infty}^{+\infty} a^{-1}|\phi_0|(x)\,dx + C\left|\int_0^t \int_{-\infty}^{+\infty} (a-a_1)\bar{w}_t\, sign(F)\,dx\,d\tau\right| \\
&\quad + \left|\int_0^t \int_{-\infty}^{+\infty} (fF^2)_x F_x\, sign'(F)\,dx\,d\tau\right| \\
&\leqslant C(\|\phi_0\|_{L^1} + \delta).
\end{aligned} \tag{3.5}
$$

Here, we have used the following facts:

$$
\begin{aligned}
&\left|\int_0^t \int_{-\infty}^{+\infty} (a-a_1)\bar{w}_t\, sign(F)\,dx\,d\tau\right| \\
&\leqslant C \int_0^t \int_{-\infty}^{+\infty} |a-a_1\|\bar{w}_t|\,dx\,d\tau \\
&\leqslant C \int_0^t \int_{-\infty}^{+\infty} (1+t)^{-\frac{3}{2}} |x(a-a_1)\|\bar{w}'(\eta)|\,dx\,d\tau \\
&\leqslant C\delta,
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
&\int_0^t \int_{-\infty}^{+\infty} (fF^2)_x F_x\, sign'(F)\,dx\,d\tau \\
&= \int_0^t \int_{-\infty}^{+\infty} F_x(2fF_x + f_F F F_x + f_{\bar{w}} F \bar{w}_x) F \delta_{\{F=0\}}\,dx\,d\tau \\
&= 0.
\end{aligned} \tag{3.7}
$$

This gives the proof of (3.4). □

With the help of Lemma 3.2, we can make the energy estimates on F.

Lemma 3.3. *Under the hypotheses of Theorem* 3.1, *there exists* $\varepsilon_* > 0$ *such that if* $0 < \varepsilon < \varepsilon_*$ *and* $N(T) \leqslant \varepsilon$, *then it holds*, for $0 \leqslant t \leqslant T$ *that*

$$
\|F(\cdot,T)\|^2 + \int_0^t \|F_x(\cdot,\tau)\|^2\,d\tau \leqslant C(\|F_0\|^2 + \delta). \tag{3.8}
$$

Proof. Let us multiply Eq. (3.1) by $a^{-1}F$ and integrate the result over $[0,t] \times (-\infty,+\infty)$, then we get

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{1}{2} a^{-1} F\phi(x,t)\,dx + \int_0^t \int_{-\infty}^{+\infty} F_x^2\,dx\,d\tau \\
&\leqslant \int_{-\infty}^{+\infty} \frac{1}{2} a^{-1} F_0\phi_0\,dx + \left|\int_0^t \int_{-\infty}^{+\infty} a^{-1}(a-a_1)(\bar{w}^{-\gamma})_{xx} F\,dx\,d\tau\right| \\
&\quad + \left|\int_0^t \int_{-\infty}^{+\infty} \frac{1}{2} a^{-1}\psi_2(\bar{w}) F^2 \bar{w}_t\,dx\,d\tau\right| + \left|\int_0^t \int_{-\infty}^{+\infty} (fF^2)_x F_x\,dx\,d\tau\right| \\
&\equiv \int_{-\infty}^{+\infty} \frac{1}{2} a^{-1} F_0\phi_0\,dx + I_1 + I_2 + I_3,
\end{aligned} \tag{3.9}$$

with $\psi_2(\bar{w})F^2 = \phi^2\psi'(\bar{w})$.

We estimate I_1–I_3 step by step as follows:

$$\begin{aligned}
I_1 &= \left|\int_0^t \int_{-\infty}^{+\infty} a^{-1}(a-a_1)(\bar{w}^{-\gamma})_{xx} F\,dx\,d\tau\right| \\
&\leqslant C\delta\varepsilon \int_0^t (1+\tau)^{-\frac{3}{2}} \|x(a-a_1)\|_{L^1}\,d\tau \\
&\leqslant C\delta\varepsilon,
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
I_2 &= \left|\int_0^t \int_{-\infty}^{+\infty} \frac{1}{2} a^{-1}\psi_2(\bar{w}) F^2 \bar{w}_t\,dx\,d\tau\right| \\
&\leqslant C \int_0^t \|F\|_{L^\infty} \|\bar{w}_t\|_{L^\infty} \|F\|_{L^1}\,dx\,d\tau \\
&\leqslant C\delta \int_0^t \|F\|^{\frac{1}{2}} \|F_x\|^{\frac{1}{2}} (1+\tau)^{-1}\,d\tau \\
&\leqslant C\delta \left(\int_0^t \|F\|^2 \|F_x\|^2\,d\tau + \int_0^t (1+\tau)^{-\frac{4}{3}}\,d\tau\right) \\
&\leqslant C\delta \left(1 + \varepsilon^2 \int_0^t \|F_x\|^2\,d\tau\right),
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
I_3 &= \left|\int_0^t \int_{-\infty}^{+\infty} (fF^2)_x F_x\,dx\,d\tau\right| \\
&\leqslant \left(\frac{1}{2} + C\varepsilon\right) \int_0^t \|F_x\|^2\,d\tau + C\delta^2 \int_0^t \|F\|_{L^\infty}^4\,d\tau
\end{aligned}$$

$$\leqslant \left(\frac{1}{2}+C\varepsilon\right)\int_0^t \|F_x\|^2\,d\tau + C\delta^2\int_0^t \|F\|^2\|F_x\|^2\,d\tau$$

$$\leqslant \left(\frac{1}{2}+C\varepsilon\right)\int_0^t \|F_x\|^2\,d\tau. \tag{3.12}$$

Owing to the smallness of δ and ε, we conclude from (3.9)–(3.12) that

$$\|F(\cdot,t)\|^2+\int_0^t \|F_x(\cdot,\tau)\|^2\,d\tau \leqslant C(\|F_0\|^2+\delta), \tag{3.13}$$

which completes the proof of Lemma 3.3. □

For higher-order estimates, we use problem (3.2) to obtain the following results.

Lemma 3.4. *Under the same conditions of Lemma* 3.3, *F satisfies*

$$\|(F_x,F_t,F_{xx})(\cdot,t)\|^2+\int_0^t \|(F_{xx},F_{tx})(\cdot,\tau)\|^2\,d\tau \leqslant C(\|F_0\|_{H^2}^2+\delta). \tag{3.14}$$

Proof. Multiplying the equation in (3.2) by F_{xx}, and integrating over $[0,t]\times(-\infty,+\infty)$, we have

$$\begin{aligned}\int_{-\infty}^{+\infty} F_x^2(x,t)\,dx &+ \int_0^t\int_{-\infty}^{+\infty} F_{xx}^2(x,\tau)\,dx\,d\tau\\ &\leqslant C\left(\|F_{0x}\|^2+\left|\int_0^t\int_{-\infty}^{+\infty}\bar{w}_t F_{xx}\,dx\,d\tau\right|\right.\\ &\quad\left.+\left|\int_0^t\int_{-\infty}^{+\infty}(fF^2)_{xx}F_{xx}\,dx\,d\tau\right|\right),\end{aligned} \tag{3.15}$$

which implies, with the help of Cauchy–Schwartz inequality and Lemma 2.1, that

$$\begin{aligned}\int_{-\infty}^{+\infty} F_x^2(x,t)\,dx &+ \int_0^t\int_{-\infty}^{+\infty} F_{xx}^2(x,\tau)\,dx\,d\tau\\ &\leqslant C(\|F_{0x}\|^2+\delta^2) + C\int_0^t\int_{-\infty}^{+\infty}(fF^2)_{xx}^2\,dx\,d\tau.\end{aligned} \tag{3.16}$$

The last term in (3.16) is estimated as follows

$$\int_0^t\int_{-\infty}^{+\infty}(fF^2)_{xx}^2\,dx\,d\tau$$

$$\leqslant C\int_0^t\int_{-\infty}^{+\infty}[(|F|+|F_x|+|w_x|)^2F_x^2+F^2F_{xx}^2+F^4(\bar{w}_{xx}^2+\bar{w}_x^4)]\,dx\,d\tau$$

$$\leqslant C\varepsilon^2\delta^2+C\varepsilon\int_0^t\int_{-\infty}^{+\infty}F_{xx}^2(\tau,x)\,dx\,d\tau.$$

This, together with (3.16), gives

$$\|F_x(\cdot,t)\|^2+\int_0^t\|F_{xx}(\cdot,\tau)\|^2\,d\tau\leqslant C(\|F_{0x}\|^2+\delta^2). \tag{3.17}$$

Now, differentiating the first equation in (3.2) in t, one has

$$\begin{aligned}&F_{tt}+a\psi(\bar{w})F_{txx}+a\psi'(\bar{w})\bar{w}_tF_{xx}-[\psi(\bar{w})(a-a_1)(\bar{w}^{-\gamma})_{xx}]_t\\&-(\psi_1(\bar{w})F\bar{w}_t)_t-[a\psi(\bar{w})(fF^2)_{xx}]_t=0.\end{aligned} \tag{3.18}$$

Multiplying (3.18) by F_t and then integrating over $[0,t]\times(-\infty,+\infty)$, we have

$$\|F_t(\cdot,t)\|^2+\int_0^t\|F_{tx}(\cdot,\tau)\|^2\,d\tau\leqslant C(\|F_0\|_{H^2}^2+\delta).$$

The estimate for $\|F_{xx}(\cdot,t)\|^2$ can be easily proved by direct computation using Eq. (3.2). □

Lemmas 3.3 and 3.4 provide uniform bounds of $F(x,t)$ in $X(0,T)$ for any $T>0$. This, with local results, gives the global existence and uniqueness of the solution to (3.2). Now we will use weighted energy method to prove the following decay estimates.

Lemma 3.5. *The solution F of (3.2) obtained in Theorem 3.1 satisfies*

$$\begin{aligned}&w_1(t)\|F(\cdot,t)\|^2+w_2(t)\|F_x(\cdot,t)\|^2\\&\quad+\int_0^t(w_1(\tau)\|F_x(\cdot,\tau)\|^2+w_2(\tau)\|F_{xx}(\cdot,\tau)\|^2)\,d\tau\leqslant C.\end{aligned}$$

Proof. Let us multiply (3.1) by $a^{-1}w_1(t)F$, rearrange terms, we have

$$\begin{aligned}&\left(\frac{1}{2}F\phi a^{-1}w_1(t)\right)_t+w_1(t)F_x^2-\frac{1}{2}w_1'(t)a^{-1}\psi_1(\bar{w})F^2\\&\quad=\frac{1}{2}a^{-1}w_1(t)F^2\bar{w}_t-a^{-1}w_1(t)(a-a_1)F(\bar{w}^{-\gamma})_{xx}\\&\qquad+w_1(t)F_x(fF^2)_x+\{\cdots\}_x.\end{aligned} \tag{3.19}$$

Here $\{\cdots\}_x$ denotes the term which does not need to be computed explicitly, since it will vanish after integration in x. When integrated on $[0,t]\times(-\infty,+\infty)$, (3.19) yields

$$
\begin{aligned}
& w_1(t)\|F(\cdot,t)\|^2 + \int_0^t w_1(t)\|F_x(\tau)\|^2\,d\tau \\
&\leqslant C_1 \left(\|F_0\|^2 + \left| \int_0^t \int_{-\infty}^{+\infty} w_1'(\tau) F^2\,dx\,d\tau \right| \right. \\
&\quad + \left| \int_0^t \int_{-\infty}^{+\infty} w_1(\tau) F^2 \bar{w}_t\,dx\,d\tau \right| + \left| \int_0^t \int_{-\infty}^{+\infty} w_1(\tau) \bar{w}_t F(a-a_1)\,dx\,d\tau \right| \\
&\quad \left. + \left| \int_0^t \int_{-\infty}^{+\infty} w_1(\tau)(fF^2)_x^2\,dx\,d\tau \right| \right). \qquad (3.20)
\end{aligned}
$$

Observe that the following inequality on F,

$$\|F\|_{L^\infty} \leqslant C\|F_x\|^{\frac{2}{3}}, \qquad (3.21)$$

Since

$$
\begin{aligned}
\|F\|_{L^\infty} &\leqslant C\|F\|^{\frac{1}{2}}\|F_x\|^{\frac{1}{2}} \\
&\leqslant C\|F\|_{L^\infty}^{\frac{1}{4}}\|F_x\|^{\frac{1}{2}}\|F\|_{L^1}^{\frac{1}{4}}.
\end{aligned}
$$

We have the following estimates:

$$
\begin{aligned}
& \int_0^t \int_{-\infty}^{+\infty} (|w_1'(\tau)F^2| + |w_1(\tau)\bar{w}_t(a-a_1)F| + |w_1(\tau)\bar{w}_t F^2|)\,dx\,d\tau \\
&\leqslant C \int_0^t (1+\tau)^{-1} w_1(\tau)\|F\|_{L^\infty}\,d\tau \\
&\leqslant C \int_0^t (1+\tau)^{-1} w_1(\tau)\|F_x(\cdot,\tau)\|^{\frac{2}{3}}\,d\tau \\
&\leqslant C + \frac{1}{2}\int_0^t w_1(\tau)\|F_x(\cdot,\tau)\|^2\,d\tau, \qquad (3.22)
\end{aligned}
$$

$$
\begin{aligned}
& \int_0^t \int_{-\infty}^{+\infty} w_1(\tau)(fF^2)_x^2\,dx\,d\tau \\
&\leqslant C\varepsilon \int_0^t w_1(\tau)\|F_x(\cdot,\tau)\|^2\,d\tau + C\varepsilon \int_0^t (1+\tau)^{-1} w_1(\tau)\|F\|_{L^\infty}\,d\tau \\
&\leqslant C\varepsilon + C\varepsilon \int_0^t w_1(\tau)\|F_x(\cdot,\tau)\|^2\,d\tau, \qquad (3.23)
\end{aligned}
$$

where we have used

$$|w_i'(t)| \leqslant C(1+t)^{-1} w_i(t), \quad \text{for } i = 1, 2, \ldots .$$

Hence, by the smallness of ε, we conclude from (3.20)–(3.23) that

$$w_1(t)\|F(\cdot,t)\|^2 + \int_0^t w_1(t)\|F_x(\cdot,\tau)\|^2 \, d\tau \leqslant C. \tag{3.24}$$

For second-order estimate, we multiply (3.2) by $w_2(t)F_{xx}$ and hence

$$\left(\frac{1}{2} w_2(t) F_x^2\right)_t - a\psi(\bar{w}) w_2(t) F_{xx}^2 - \frac{1}{2} w_2'(t) F_x^2 - \psi_1(\bar{w}) F F_{xx} \bar{w}_t w_2(t)$$
$$= -w_2(t)\psi(\bar{w})(a-a_1)(\bar{w}^{-\gamma})_{xx} F_{xx} - a\psi(\bar{w})(fF^2)_{xx} F_{xx} w_2(t) + \{\cdots\}_x.$$

Then one has

$$\begin{aligned}
&w_2(t)\|F_x(\cdot,t)\|^2 + \int_0^t w_2(\tau)\|F_{xx}(\cdot,\tau)\|^2 \, d\tau \\
&\quad \leqslant C + C\left(\left|\int_0^t \int_{-\infty}^{+\infty} \bar{w}_t^2 w_2(\tau)(a-a_1)^2 \, dx \, d\tau\right|\right. \\
&\qquad \left. + \left|\int_0^t \int_{-\infty}^{+\infty} F^2 \bar{w}_t^2 w_2(\tau) \, dx \, d\tau\right| + \int_0^t \int_{-\infty}^{+\infty} w_2(\tau)(fF^2)_{xx}^2 \, dx \, d\tau\right).
\end{aligned} \tag{3.25}$$

Since

$$\begin{aligned}
&\left|\int_0^t \int_{-\infty}^{+\infty} \bar{w}_t^2 w_2(\tau)(a-a_1)^2 \, dx \, d\tau\right| + \left|\int_0^t \int_{-\infty}^{+\infty} F^2 \bar{w}_t^2 w_2(\tau) \, dx \, d\tau\right| \\
&\quad \leqslant C\delta^2 \int_0^t (1+\tau)^{-3} w_2(\tau) \, d\tau + C\delta^2 \int_0^t (1+\tau)^{-1} w_1(\tau)\|F\|_{L^\infty} \, d\tau \\
&\quad \leqslant C\delta^2
\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
(fF^2)_{xx} &= (2FF_x f + f_F F^2 F_x + f_{\bar{w}} \bar{w}_x F^2)_x \\
&= (2fF + f_F F^2) F_{xx} + (2f + 4f_F F + f_{FF} F^2) F_x^2 \\
&\quad + (4f_{\bar{w}} F + 2f_{F\bar{w}} F^2) F_x \bar{w}_x + (f_{\bar{w}} \bar{w}_{xx} + f_{\bar{w}\bar{w}} \bar{w}_x^2) F^2,
\end{aligned}$$

it follows that

$$\left|\int_0^t \int_{-\infty}^{+\infty} w_2(\tau)(fF^2)_{xx}^2\,dx\,d\tau\right|$$

$$\leqslant C + C\varepsilon \int_0^t w_2(\tau)\|F_{xx}(\cdot,\tau)\|^2\,d\tau + C\int_0^t \int_{-\infty}^{+\infty} F_x^4 w_2(\tau)\,dx\,d\tau \tag{3.27}$$

and

$$\int_0^t \int_{-\infty}^{+\infty} F_x^4 w_2(\tau)\,dx\,d\tau$$

$$\leqslant C\varepsilon^2 \int_0^t w_2(\tau)\|F_{xx}(\cdot,\tau)\|^2\,d\tau$$

$$+C\int_0^t w_2(\tau)\|F_x(\cdot,\tau)\|^2\|F_x(\cdot,\tau)\|^2\,d\tau. \tag{3.28}$$

Owing to the smallness of ε, we deduce from (3.25)–(3.28) that

$$w_2(t)\|F_x(\cdot,t)\|^2 + \int_0^t w_2(\tau)\|F_{xx}(\cdot,\tau)\|^2\,d\tau$$

$$\leqslant C\left(1+\int_0^t w_2(\tau)\|F_x(\cdot,\tau)\|^2\|F_x(\cdot,\tau)\|^2\,d\tau\right). \tag{3.29}$$

Therefore Gronwall's inequality gives

$$w_2(t)\|F_x(\cdot,t)\|^2 + \int_0^t w_2(\tau)\|F_{xx}(\cdot,\tau)\|^2\,d\tau \leqslant C. \tag{3.30}$$

Hence, (3.24) and (3.30) complete the proof of this lemma. □

Now, Theorem 3.1 has been proved. The following lemma contains the decay rates for the derivatives of F, which will be useful in the next section.

Lemma 3.6. *The solution F to* (3.2), *obtained in Theorem* 3.1, *satisfies*

$$w_3(t)\|F_t(t)\|^2 + \int_0^t w_3(\tau)\|F_{tx}(\tau)\|^2\,d\tau \leqslant C\delta.$$

Proof. Differentiation on $(3.2)_1$ in t leads to

$$F_{tt} + a\psi(\bar{w})F_{txx} + a\psi'(\bar{w})\bar{w}_t F_{xx} - [\psi(\bar{w})(a-a_1)(\bar{w}^{-\gamma})_{xx}]_t$$

$$-(\psi_1(\bar{w})F\bar{w}_t)_t - [a\psi(\bar{w})(fF^2)_{xx}]_t = 0. \tag{3.31}$$

Multiplying (3.31) by $a^{-1}w_3(t)F_t$, we have

$$\begin{aligned}&\left(\frac{1}{2}a^{-1}w_3(t)F_t^2\right)_t-\psi(\bar{w})w_3(t)F_{tx}^2+\frac{1}{2}F_t^2\psi(\bar{w})_{xx}w_3(t)-\frac{1}{2}F_t^2a^{-1}w_3'(t)\\&\quad+\psi'(\bar{w})\bar{w}_tF_{xx}w_3(t)F_t-a^{-1}[\psi(\bar{w})(a-a_1)(\bar{w}^{-\gamma})_{xx}]_tw_3(t)F_t\\&\quad-a^{-1}(\psi_1(\bar{w})F\bar{w}_t)_tw_3(t)F_t-[\psi(\bar{w})(fF^2)_{xx}]_tw_3(t)F_t+\{\cdots\}_x=0.\end{aligned}\tag{3.32}$$

From the proof of Lemma 2.9 and $(3.2)_1$, it is clear that

$$\begin{aligned}&\int_0^t w_2(\tau)\|F_t(\cdot,\tau)\|^2\,d\tau\\&\quad\leqslant C\left(\int_0^t w_2(t)\|F_{xx}(\cdot,\tau)\|^2\,d\tau+\int_0^t\int_{-\infty}^{+\infty}(a-a_1)^2\bar{w}_t^2w_2(\tau)\,dx\,d\tau\right.\\&\qquad\left.+\int_0^t\int_{-\infty}^{+\infty}F^2\bar{w}_t^2w_2(\tau)\,dx\,d\tau+\int_0^t\int_{-\infty}^{+\infty}(fF^2)_{xx}^2w_2(\tau)\,dx\,d\tau\right)\\&\quad\leqslant C.\end{aligned}\tag{3.33}$$

Moreover one has

$$a^{-1}(\psi_1(\bar{w})F\bar{w}_t)_tw_3(t)F_t=O(1)[\bar{w}_tw_3(t)F_t^2+(\bar{w}_t^2+\bar{w}_{tt})w_3(t)FF_t],$$

$$a^{-1}[\psi(\bar{w})(a-a_1)(\bar{w}^{-\gamma})_{xx}]_tw_3(t)F_t=O(1)(a-a_1)(\bar{w}_t^2+\bar{w}_{tt})w_3(t)F_t,$$

$$[\psi(\bar{w})(fF^2)_{xx}]_tw_3(t)F_t=O(1)\bar{w}_t(fF^2)_{xx}w_3(t)F_t-\psi(\bar{w})(fF^2)_{xxt}w_3(t)F_t.$$

Now, we can use a similar argument as used in deriving (3.24) to obtain

$$w_3(t)\|F_t(\cdot,t)\|^2+\int_0^t w_3(\tau)\|F_{tx}(\cdot,\tau)\|^2\,d\tau\leqslant C,\tag{3.34}$$

which completes the proof. □

Corollary 3.7. *The solution F to (3.2) satisfies*

$$w_3(t)\|F_{xx}(\cdot,t)\|^2\leqslant C,\ \|F_x(\cdot,t)\|_{L^\infty}^2\leqslant Cw_3(t)^{-\frac{1}{2}}w_2(t)^{-\frac{1}{2}}.$$

Proof. From (3.2), we see that

$$\begin{aligned}F_{xx}\ =\ &O(1)(F_t+(a-a_1)\bar{w}_t+F\bar{w}_t+F_x^2\\&+FF_x\bar{w}_x+(\bar{w}_{xx}+\bar{w}_x^2)F^2).\end{aligned}\tag{3.35}$$

Taking the L^2 norm in (3.35), we have

$$\begin{aligned} w_3(t)\|F_{xx}\|^2 &\leqslant Cw_3(t)(\|F_t\|^2 + \|(a-a_1)\bar{w}_t\|^2 + \|F\bar{w}_t\|^2 + \|F_x^2\|^2 \\ &\quad + \|FF_x\bar{w}_x\|^2 + \|(\bar{w}_{xx} + \bar{w}_x^2)F^2\|^2) \\ &\leqslant C(1 + w_3(t)\|F_x^2\|^2) \\ &\leqslant C(1 + w_3(t)\|F_x\|^2(\|F_x\|^2 + \|F_{xx}\|^2)) \\ &\leqslant C + Cw_3(t)\|F_x\|^2\|F_{xx}\|^2 \end{aligned}$$

which implies

$$w_3(t)\|F_{xx}\|^2 \leqslant C.$$

Then

$$\|F_x\|_{L^\infty}^2 \leqslant Cw_3(t)^{-\frac{1}{2}}\, w_2(t)^{-\frac{1}{2}}. \qquad \square$$

Now, from F, it is easy to obtain the solution ϕ of (3.1) and from ϕ the unique smooth solution w of (2.2). We note that the regularity of F is stated as in Theorem 3.1, one cannot expect better regularity due to discontinuity of coefficients. However, the solution $w(x,t)$ of (2.2) is smooth for $t > 0$. Indeed, we concluded from Theorem 3.1 that $w(x,t)$ has positive lower bound and finite upper bound, then the regularity results of uniform parabolic equations applies to (2.2). Thus, we are safe to use the regularity of $w(x,t)$ in the next section. By defining $\tilde{v} = a^{-1}(x)w$ and $\tilde{u} = -(w^{-\gamma})_x$, we obtain the unique smooth solution of (1.5). Theorem 2.1 then follows from Theorem 3.1, the decay estimates follow from the interpolation inequality and (3.21).

4. Hyperbolic problems

In this section, we will study (1.4). Since the result for $s(x,t)$ is clear, in the following part, we only deal with $(v,u)(x,t)$.

Let $(\tilde{v}, \tilde{u}, s(x))$ be the solution of (1.5) with the initial data $(\tilde{v}_0(x), s_0(x))$. As mentioned in introduction, we will only prove Theorem 2.2 for the case where $u_- = u_+ = 0$ and thus (2.10) turns into

$$\int_{-\infty}^{+\infty} (v_0(x) - \tilde{v}_0(x))\, dx = 0. \tag{4.1}$$

Let us denote by

$$v_e = v - \tilde{v}, \quad u_e = u - \tilde{u}, \tag{4.2}$$

it follows from (1.4) and (1.5)

$$\begin{cases} v_{et} - u_{ex} = 0, \\ u_{et} + [p(\tilde{v} + v_e, s) - p(\tilde{v}, s)]_x = -u_e + p(\tilde{v}, s)_{xt}. \end{cases} \tag{4.3}$$

As usual let us consider

$$y(x,t) = \int_{-\infty}^{x} v_e(\xi)\, d\xi, \tag{4.4}$$

which satisfies the following nonlinear wave equation:

$$\begin{cases} y_{tt} + [p(y_x + \tilde{v}, s) - p(\tilde{v}, s)]_x + y_t = p(\tilde{v}, s)_{xt}, \\ y(x,0) = y_0(x) = \int_{-\infty}^{x} (v_0 - \tilde{v}_0)(\xi)\, d\xi, \\ y_t(x,0) = y_1(x) = u_0(x) - \tilde{u}(x,0), \end{cases} \tag{4.5}$$

since $y_x = v_e$ and $y_t = u_e$. Therefore,

$$\begin{cases} y_{tt} + (p_v(\tilde{v}, s) y_x)_x + y_t = p(\tilde{v}, s)_{xt} - (F_1(\tilde{v}, y_x, s) y_x^2)_x, \\ y(x,0) = y_0(x) = \int_{-\infty}^{x} (v_0 - \tilde{v}_0)(\xi)\, d\xi, \\ y_t(x,0) = y_1(x) = u_0(x) - \tilde{u}(x,0), \end{cases} \tag{4.5$'$}$$

where

$$p(y_x + \tilde{v}, s) - p(\tilde{v}, s) = p_v(\tilde{v}, s) y_x + F_1(\tilde{v}, y_x, s) y_x^2.$$

The main result of this section is the following.

Theorem 4.1. *There exists* $\delta_0 > 0$ *such that if* $0 < \delta < \delta_0$ *and*

$$\|y_0\|_{H^3} + \|y_1\|_{H^2} + |v_+ - v_-| \leqslant \delta,$$

then (4.5) *has a unique smooth solution* $y \in H^3$ *and* $y_t \in H^2$, *satisfying*

$$\|y(\cdot,t)\|_{H^3}^2 + \|y_t(\cdot,t)\|_{H^2}^2 + \int_0^t \|(y_x, y_t)(\cdot,\tau)\|_{H^2}\, d\tau \leqslant C\delta^2.$$

Moreover,

$$(1+t)\|y_x(\cdot,t)\|^2 + (1+t)^2\|y_t(\cdot,t)\|^2 \leqslant C, \tag{4.6}$$

and

$$\|y_x(\cdot,t)\|_{L^\infty} \leqslant C(1+t)^{-\frac{3}{4}}, \quad \|y_t(\cdot,t)\|_{L^\infty} \leqslant C(1+t)^{-\frac{5}{4}}. \tag{4.7}$$

We now prove Theorem 4.1. First of all, we have

Theorem 4.2. *Let $(v,u,s)(x,t)$ be any C^1 solutions of (1.4) with uniform C^1 bounds. If $\|y_0\|_{H^1}+\|y_1\|_{L^2} \leqslant M$, for some positive constant M, then there are positive constants C and $0<k<\frac{1}{4}$ such that*

$$\|(y,y_t,y_x)(\cdot,t)\|_{L^2}^2 + \int_0^t \|(y_t,y_x)(\cdot,\tau)\|_{L^2}^2\,d\tau \leqslant C, \tag{4.8}$$

$$(1+t)^k\|(y_t,y_x)(\cdot,t)\|^2 + \int_0^t (1+\tau)\|y_t(\cdot,t)\|^2\,d\tau \leqslant C. \tag{4.9}$$

Remark 4. (a) The global existence of C^1 solutions of (1.4) under small smooth initial data has been proved in [22] by characteristic method. In particular, one has the global existence of unique smooth solution for (1.4) under conditions of Theorem 4.1. A modified version of characteristic method of [22] can be found in [8] for initial boundary value problems.

(b) It is worth to remark that Theorem 4.2 itself does not require any smallness on initial data or wave strength. This is achieved by the dissipative nature of the problem and entropy analysis.

Proof. Multiplying $(4.5)_1$ by y, we have

$$\begin{aligned}&\left[\frac{1}{2}y^2 + yy_t\right]_t - y_t^2 - [p(y_x+\tilde{v},s) - p(\tilde{v},s)]y_x\\ &\quad = p(\tilde{v},s)_t y_x + \{\cdots\}_x,\end{aligned}$$

where $\{\cdots\}_x$ denote the terms which vanish after integration with respect to x. Since $-[p(y_x+\tilde{v},s)-p(\tilde{v},s)]y_x \geqslant c_1 y_x^2$ for some positive constant c_1, integrating over $[0,t]\times(-\infty,+\infty)$, we get

$$\begin{aligned}&\int_{-\infty}^{+\infty}\left(\frac{1}{2}y^2 + yy_t\right)dx - \int_0^t\int_{-\infty}^{+\infty} y_t^2\,dx\,dt + c_1\int_0^t\int_{-\infty}^{+\infty} y_x^2\,dx\,d\tau\\ &\quad\leqslant C + C\int_0^t\int_{-\infty}^{+\infty} p(\tilde{v},s)_t^2\,dx\,d\tau + \frac{1}{2}c_1\int_0^t\int_{-\infty}^{+\infty} y_x^2\,dx\,d\tau\\ &\quad\leqslant C + \frac{1}{2}c_1\int_0^t\int_{-\infty}^{+\infty} y_x^2\,dx\,d\tau,\end{aligned}$$

which implies

$$\int_{-\infty}^{+\infty}\left(\frac{1}{2}y^2+yy_t\right)dx-\int_0^t\int_{-\infty}^{+\infty}y_t^2\,dx\,dt+\frac{1}{2}c_1\int_0^t\int_{-\infty}^{+\infty}y_x^2\,dx\,d\tau\leqslant C. \tag{4.10}$$

We now define $\eta(v,u,s)=\frac{1}{2}u^2+e^{s(x)}v^{-(\gamma-1)}$ the mechanical energy, which serves as an entropy of system (1.4) with $q(v,u,s)$ the corresponding flux. It is easy to see that

$$\eta_t+q_x+u^2=0 \tag{4.11}$$

for smooth solutions of (1.4). Define

$$\eta_*=\frac{1}{2}u^2+e^s[v^{-(\gamma-1)}-\tilde{v}^{-(\gamma-1)}+(\gamma-1)\tilde{v}^{-\gamma}(v-\tilde{v})],$$

it is easy to see from (1.4) that

$$\eta_{*t}+q_x-[(p(\tilde{v},s)(v-\tilde{v})]_t+(u^2-\tilde{u}^2)=0. \tag{4.12}$$

Since

$$u^2=\tilde{u}^2+2\tilde{u}y_t+y_t^2,$$

(4.12) is equivalent to

$$\begin{aligned}0&=\eta_{*t}+q_x+[(p(\tilde{v},s)_xy]_t+y_t^2+2(p(\tilde{v},s)_xy_t+\{\cdots\}_x\\&=\eta_{*t}+3[(p(\tilde{v},s)_xy]_t+y_t^2+2(p(\tilde{v},s)_ty_x+\{\cdots\}_x.\end{aligned} \tag{4.13}$$

Integrating (4.13) over $[0,t]\times(-\infty,+\infty)$, one has

$$\begin{aligned}&\int_{-\infty}^{+\infty}\eta_*(x,t)\,dx+\int_{-\infty}^{+\infty}p(\tilde{v},s)_xy\,dx+\int_0^t\int_{-\infty}^{+\infty}y_t^2\,dx\,d\tau\\&\quad\leqslant C+2\left|\int_0^t\int_{-\infty}^{+\infty}p(\tilde{v},s)_ty_x\,dx\,d\tau\right|.\end{aligned} \tag{4.14}$$

By Cauchy–Schwartz inequality, for any positive constant ε, we have

$$\left|\int_{-\infty}^{+\infty}p(\tilde{v},s)_xy\,dx\right|\leqslant C(\varepsilon)+\varepsilon\int_{-\infty}^{+\infty}y^2\,dx$$

and

$$2\left|\int_0^t\int_{-\infty}^{+\infty} p(\tilde{v},s)_t y_x \, dx\, d\tau\right| \leqslant C(\varepsilon) + \varepsilon\int_0^t\int_{-\infty}^{+\infty} y_x^2\, dx\, d\tau.$$

Furthermore, it is easy to see that there exist positive constants c_2 and c_3 such that

$$\eta_* \geqslant c_2(y_t^2 + y_x^2) - c_3\tilde{u}^2. \tag{4.15}$$

Hence, (4.14) implies that

$$\begin{aligned} c_2\int_{-\infty}^{\infty}(y_t^2+y_x^2)\,dx + \int_0^t\int_{-\infty}^{+\infty} y_t^2\,dx\,d\tau \\ \leqslant C(\varepsilon) + \varepsilon\int_0^t\int_{-\infty}^{+\infty} y_x^2\,dx\,d\tau + \varepsilon\int_{-\infty}^{+\infty} y^2\,dx. \end{aligned}\tag{4.16}$$

We now combine (4.10) with (4.16) by (4.10) + λ(4.16) for some positive constant λ to be determined. One thus obtains

$$\begin{aligned} \int_{-\infty}^{+\infty}\left[\left(\frac{1}{2}-\varepsilon\lambda\right)y^2 + yy_t + \lambda c_2 y_t^2) + c_2 y_x^2\right]dx \\ + \int_0^t\int_{-\infty}^{+\infty}\left[(\lambda-1)y_t^2 + \left(\frac{1}{2}c_1 - \lambda\varepsilon\right)y_x^2\right]dx\,d\tau \leqslant C. \end{aligned}\tag{4.17}$$

We choose ε and λ such that

$$\lambda\varepsilon < \frac{1}{4}\min\{1, c_1\}, \quad \lambda > 2, \quad \text{and} \quad \lambda c_2 > 16,$$

which can be achieved by choosing λ big and then choosing ε small. Hence, (4.17) leads to the following uniform estimates:

$$\|(y, y_t, y_x)(\cdot, t)\|_{L^2}^2 + \int_0^t \|(y_t, y_x)(\cdot,\tau)\|_{L^2}^2\,d\tau \leqslant C. \tag{4.18}$$

In order to achieve decay estimates, we multiply (4.13) by $(1+t)^k$ for $0 < k < \frac{1}{4}$. It turns out

$$\begin{aligned} [(1+t)^k\eta_* + 3(1+t)^k y p(\tilde{v},s)_x]_t + (1+t)^k y_t^2 \\ = 2(1+t)^k p(\tilde{v},s)_t y_x + k(1+t)^{k-1}(\eta_* - yp(\tilde{v},s)_x) + \{\cdots\}_x. \end{aligned}\tag{4.19}$$

Integrating (4.19) over $[0,t]\times(-\infty,\infty)$, one has

$$(1+t)^k\int_{-\infty}^{+\infty}[\eta_*+3yp(\tilde{v},s)_x]\,dx+\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^k y_t^2\,dx\,d\tau$$
$$\leqslant C+C\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^{2k}(p(\tilde{v},s)_t)^2\,dx\,d\tau+|I_1|+|I_2|. \tag{4.20}$$

On the other hand, we observe that

$$\left|(1+t)^k\int_{-\infty}^{+\infty}yp(\tilde{v},s)_x\,dx\right|$$
$$\leqslant (1+t)^{2k}\|p(\tilde{v},s)_t\|_{L^2}^2+\|y\|_{L^2}^2$$
$$\leqslant C. \tag{4.21}$$

Choosing $0<\mu<\frac{1}{4}-k$, one has

$$|I_2|=\left|\int_0^t\int_{-\infty}^{+\infty}k(1+\tau)^{k-1}yp(\tilde{v},s)_x\,dx\,d\tau\right|$$
$$\leqslant C\int_0^t\int_{-\infty}(1+\tau)^{2k-2+1+\mu}p(\tilde{v},s)_x^2+(1+\tau)^{-(1+\mu)}y^2\,dx\,d\tau$$
$$\leqslant C. \tag{4.22}$$

Furthermore, I_1 can be bounded by

$$|I_1|=\int_0^t\int_{-\infty}^{+\infty}k(1+\tau)^{k-1}\eta_*\,dx\,d\tau$$
$$\leqslant C\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^{(k-1)}(y_t^2+y_x^2+p(\tilde{v},s)_x^2)\,dx\,d\tau$$
$$\leqslant C. \tag{4.23}$$

Thus, (4.20)–(4.23), together with (4.15), give the following estimates:

$$(1+t)^k\|(y_t,y_x)\|^2+\int_0^t(1+\tau)\|y_t\|^2\,d\tau\leqslant C. \tag{4.24}$$

This ends the proof of Theorem 4.2. □

With the uniform estimates (4.8)–(4.9) in hand, standard energy estimates and weighted energy estimates give the proof of Theorem 4.1. For details, we refer the readers to the proof of Theorem 3.1 of [15], which consists Theorem 3.2, Lemmas 3.4–3.8 and Theorem 3.9 in [15]. We remark that the arguments of [15] go through just fine with the help of Theorem 4.2, since the solution of (1.5) is smooth although the asymptotic profile is singular.

Theorem 2.2 is then concluded from Theorems 4.1 and 2.1.

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References

[1] C.M. Dafermos, A system of hyperbolic conservation laws with frictional damping, Z. Angew. Math. Phys. 46 (1995) 294–307.
[2] C.J. Van Duyn, L.A. Peletier, Asymptotic behavior of solutions of a nonlinear diffusion equations, Arch. Rational Mech. Anal. 65 (1977) 363–377.
[3] C.J. Van Duyn, L.A. Peletier, A class of similarity solutions of the nonlinear diffusion equations, Nonlinear Anal. TMA 1 (1977) 223–233.
[4] L. Hsiao, Quasilinear Hyperbolic Systems and Dissipative Mechanisms, World Scientific, Singapore, 1997.
[5] L. Hsiao, T.P. Liu, Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping, Comm. Math. Phys. 143 (1992) 599–605.
[6] L. Hsiao, T.P. Liu, Nonlinear diffusive phenomena of nonlinear hyperbolic systems, Chinese Ann. Math. 14B (4) (1993) 465–480.
[7] L. Hsiao, T. Luo, Nonlinear diffusive phenomena of solutions for the system of compressible adiabatic flow through porous media, J. Differential Equations 125 (1996) 329–365.
[8] L. Hsiao, R. Pan, Initial boundary value problem for the system of compressible adiabatic flow through porous media, J. Differential Equations 159 (1999) 280–305.
[9] L. Hsiao, D. Serre, Global existence of solutions for the system of compressible adiabatic flow through porous media, SIAM J. Math. Anal. 27 (1) (1996) 70–77.
[10] L. Hsiao, D. Serre, Large-time behavior of solutions for the system of compressible adiabatic flow through porous media, Chinese Ann. Math. 16B (4) (1995) 1–14.
[11] F. Huang, R. Pan, Asymptotic behavior of the solutions to the damped compressible Euler equations, preprint.
[12] F. Huang, R. Pan, Convergence rate for compressible Euler equations with damping and vacuum, Arch. Rational Mech. Anal. 166 (2003) 359–376.
[13] F. Huang, P. Marcati, R. Pan, Convergence to Barenblatt's solutions for the compressible Euler equations with damping and vacuum, Arch. Rational Mech. Anal., in press.
[14] P. Marcati, M. Mei, Convergence to nonlinear diffusion waves for solutions of the initial boundary problem to the hyperbolic conservation laws with damping, Q. Appl. Math. 58 (2000) 763–784.
[15] P. Marcati, R. Pan, On the diffusive profiles for the system of compressible adiabatic flow through porous media, SIAM J. Math. Anal. 33 (2001) 790–826.
[16] P. Marcati, B. Rubino, Hyperbolic to parabolic relaxation theory for quasilinear first order systems, J. Differential Equations 162 (2000) 359–399.
[17] K. Nishihara, Convergence rates to nonlinear diffusion waves for solutions of system of hyperbolic conservation laws with damping, J. Differential Equations 131 (1996) 171–188.
[18] K. Nishihara, M. Nishikwa, Asymptotic behavior of solutions to the system of compressible adiabatic flow through porous media, SIAM J. Math. Anal. 33 (2001) 216–239.

[19] K. Nishihara, T. Yang, Boundary effect on asymptotic behavior of solutions to the p-system with damping, J. Differential Equations 161 (2000) 191–218.
[20] R. Pan, Boundary effects and large time behavior for the system of compressible adiabatic flow through porous media, Michigan Math. J. 49 (2001) 519–540.
[21] H. Zhao, Convergence to strong nonlinear diffusion waves for solutions of p-system with damping, J. Differential Equations 174 (2001) 200–236.
[22] Y. Zheng, Global smooth solutions to the adiabatic gas dynamics system with dissipation terms, Chinese Ann. Math. A 17 (1996) 155–162.
[23] C. Zhu, Convergence rates to nonlinear diffusion waves for weak entropy solutions to p-system with damping, Sci. China Ser. A 46 (2003) 562–575.