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Asymptotic behavior of the solutions to the damped compressible Euler equations with vacuum

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Abstract

The asymptotic behavior of solutions of the damped compressible Euler equations is conjectured to obey to the famous *porous media equations* (PMES). The previous works on this topic concern the case away from vacuum where the system is strictly hyperbolic. In present paper, we prove that the L^∞ entropy weak solution with vacuum, obtained by the compensated compactness theory, converges strongly in L^p_{loc} $(1 \leqslant p < \infty)$ space to the unique similarity solution of the related PME, as time goes to infinity.

MSC: 35L65; 76S05; 35K65

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1. Introduction

We study the asymptotic behavior of compressible isentropic flow through porous media when vacuum occurs. The model system is the compressible Euler equation with frictional damping and the density is conjectured to obey to the famous porous media equation (PME) asymptotically, as $t \to \infty$. Although, some results concerning

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the small smooth solutions or piecewise smooth Riemann solutions away from vacuum have been obtained, most problems in this topic remain open. Among them, the large time asymptotic behavior for the solutions with vacuum has been long-standing open problem. The main difficulties come from the interaction of three mechanism: nonlinear convection, lower order dissipation of damping and the resonance from vacuum. Since any result on this problem will help us to understand the interaction of the effects of these three mechanism, the evolution of vacuum boundary, singularity development and other complicated phenomena caused by vacuum, it is of mathematical significance and physical importance, in view of the strong physical background of vacuum. Besides, this study may present useful information, say, proper ansatz for approximation, for the design of effective numerical schemes to capture the vacuum boundary. In this paper, we prove that the L^∞ weak entropy solution with vacuum, selected by the physical (energy) entropy–flux pairs, converge strongly in $L^p_{\mathrm{loc}}(1 \leqslant p < \infty)$ to the similarity solution of the PME, determined uniquely by the end–states of the initial data. A direct consequence of our results implies the following simple fact: If the vacuum is surrounded by gas initially, then there is no vacuum almost everywhere as time goes to infinity.

We now formulate our results. In one-dimensional porous media, the motion of the isentropic compressible flow is described by the damped compressible Euler equations which express the conservation of mass and the balance momentum as follows:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = -\alpha \rho u. \end{cases} \tag{1.1}$$

Here ρ, u and P denote, respectively, the density, velocity, and pressure, $m = \rho u$ is the momentum and $\alpha > 0$ is a frictional constant. For convenience, we consider the polytropic perfect gas where

$$P(\rho) = P_0 \rho^\gamma, \quad 1 \leqslant \gamma < 3 \tag{1.2}$$

with P_0 a positive constant. Without loss of generality, α and P_0 are normalized to be 1 throughout this paper.

It is known that (1.1) is of the hyperbolic type with two characteristic speeds $\lambda_1 = u - \sqrt{P'(\rho)}$ and $\lambda_2 = u + \sqrt{P'(\rho)}$. Furthermore, (1.1) is strictly hyperbolic at the point away from vacuum where two characteristics coincide. We will consider the Cauchy problem of (1.1) with the following initial data

$$\rho(x, 0) = \rho_0(x) \geqslant 0, \quad m(x, 0) = m_0(x). \tag{1.3}$$

Namely, the vacuum may present initially. In what following, we will use m as unknown instead of u since u may be singular at vacuum for $\gamma = 1$.

For the case away from vacuum, (1.1) is already very complicated under the interaction between the nonlinear convection and damping. By changing into Lagrangian coordinates, (1.1) becomes into the p-system with damping:

$$\begin{cases} v_t - u_x = 0, \\ u_t + P(v)_x = -\alpha u \end{cases} \tag{1.4}$$

with v the specific volume. Its Cauchy problem has globally defined classical solutions if the C^1 norm of the initial data are small, see [29]; while shock will form in finite time if the data are large, see [38]. Since the damping term breaks the self-similarity of the system, it is observed by Hsiao and Tang [14,15] that the shock curve of the Riemann solutions to (1.4) is not straight line. On the other hand, damping is dissipative effects so that the wave strength of such shock decreases exponentially in time. Although BV solutions have been constructed by Luskin and Temple [25] when $\gamma = 1$ and by Dafermos in [4] where P is allowed to be any smooth, decreasing function for the initial data with small oscillation about some fixed equilibrium state, the question on global BV entropy weak solution to (1.4) for general BV initial data remains open!

When vacuum occurs in the solutions, the difficulty of the problem increases a lot. The vacuum here is a kind of resonance phenomenon, since two characteristics of (1.1) coincide each other. It is known that the resonance is a mechanism for singularity. Thus the presence of vacuum produces huge complexity in both theoretical and numerical analysis. In particularly, the velocity may become singular at vacuum state when $\gamma = 1$. This is why we know so little in vacuum case. The local existence of smooth solution to (1.1) under smooth initial data with vacuum is only known under some restriction on sound speed, see [22,23]. Liu and Yang [22] also observed that their smooth solutions blow up in finite time *before shock formation*. This implies the moving of the interface between vacuum and gas. And thus the vacuum is the dominant reason of singularity. In turn, Riemann problem of (1.1) still remains open and it seems very difficult to obtain the piecewise smooth solutions or BV solutions in the presence of vacuum, although some special global solutions are constructed by Liu [21]. Thus, we turn to consider the L^∞ weak solution.

Definition 1. We call $(\rho, m)(x, t) \in L^\infty$ an entropy weak solution of (1.1)–(1.3), if it holds, for any non-negative test function $\phi \in \mathcal{D}(\mathbf{R}^2_+)$, that

$$\begin{cases} \iint_{t>0} (\rho\phi_t + m\phi_x)\, dx\, dt + \int_{\mathbf{R}} \rho_0(x)\phi(x, 0)\, dx = 0, \\ \iint_{t>0} \left[m\phi_t + \left(\frac{m^2}{\rho} + P(\rho) \right) \phi_x - m\phi \right] dx\, dt + \int_{\mathbf{R}} m_0(x)\phi(x, 0)\, dx = 0, \\ \iint_{t>0} (\eta_{\mathrm{e}}\phi_t + q_{\mathrm{e}}\phi_x - \rho u^2\phi)\, dx\, dt + \int_{\mathbf{R}} \eta_{\mathrm{e}}(x, 0)\phi(x, 0)\, dx \geqslant 0. \end{cases}$$

Here, the entropy–flux pair $(\eta_{\mathrm{e}}, q_{\mathrm{e}})$ is associating with physical energy:

$$\begin{cases} \eta_{\mathrm{e}} = \frac{1}{2}\rho u^2 + \frac{1}{(\gamma-1)}\rho^\gamma, \\ q_{\mathrm{e}} = \frac{1}{2}\rho u^3 + \frac{\gamma}{\gamma-1}\rho^\gamma u, \quad \gamma > 1; \end{cases}$$

$$\begin{cases} \eta_{\mathrm{e}} = \frac{1}{2}\rho u^2 + \int_0^\rho \log s\, ds, \\ q_{\mathrm{e}} = \frac{1}{2}\rho u^3 + \rho u \log\rho, \quad \gamma = 1. \end{cases}$$

Fortunately, our system (1.1) is 2×2, so that the compensated compactness theory [36] is applicable to obtain the L^∞ weak solution. This was done by Ding et al. [6] in the case of $1 < \gamma \leqslant \frac{5}{3}$ by means of the frictional step Lax–Friedrichs scheme. We will prove here the existence of the global L^∞ weak entropy solutions, for $1 \leqslant \gamma < 3$. It is remarkable for the case $\gamma = 1$, where the classical compensated compactness frameworks fail. This is due to the fact that the velocity may tend to infinity at vacuum. We adopt a new approach in the entropy analysis to make it possible.

For the large time asymptotic behavior, it is conjectured that the system (1.1) is time-asymptotically equivalent to the PME,

$$\begin{cases} \rho_t = P(\rho)_{xx}, \\ m = -P(\rho)_x. \end{cases} \tag{1.5}$$

This was justified first by Hsiao and Liu [10,11] for smooth solutions away from vacuum. Since then, this problem attracts considerable attentions, see [9,12,13,24,30–32,37]. These results are based on the energy estimates on the derivatives of the solutions. However, as the compensated compactness theory gives no information on the regularity of the solutions, their methods are not applicable here. Recently, a new method has been developed by Serre and Xiao [35] for elastic equations with damping, which takes the same form as (1.4) while $P \in C^2$ satisfies $P'(v) < 0$ and $(v - \hat{v})P''(v) < 0$ for $v \neq \hat{v}$. Where, the large time behavior of L^∞ weak solution has been proved by combining the natural energy estimates to compensated compactness theory. However, the vacuum here breaks not only the strict hyperbolicity of system (1.1) but also the parabolicity of (1.5), i.e., the solutions to (1.5) may be singular at vacuum. This leads to new difficulties since the case in [35] has no degeneracy. Fortunately again, the detail information on the self-similar solution of PME as well as smart entropy analysis help us to improve the argument of [35] for the desired results.

Our main results state as follows:

Theorem 1 (*Existence*). *If* $1 < \gamma < 3$, $(\rho_0, m_0) \in L^\infty$ *such that*

$$0 \leqslant \rho_0(x) \leqslant C, \quad |m_0(x)| \leqslant C\rho_0(x),$$

then (1.1)–(1.3) *has a global entropy weak solution* $(\rho, m)(x,t) \in L^\infty$ *such that*

$$0 \leqslant \rho(x,t) \leqslant C, \quad |m(x,t)| \leqslant C\rho(x,t).$$

If $\gamma = 1$, $(\rho_0, m_0) \in L^\infty$ *such that*

$$0 \leqslant \rho_0(x) \leqslant C, \quad |m_0(x)| \leqslant C\rho_0(x)|\log\rho_0(x)|,$$

then (1.1)–(1.3) *has a global entropy weak solution* $(\rho, m)(x,t) \in L^\infty$ *such that*

$$0 \leqslant \rho(x,t) \leqslant C, \quad |m(x,t)| \leqslant C\rho(x,t)|\log \rho(x,t)|.$$

Theorem 2 (*Asymptotic behavior*). *Suppose* $\rho_- \geqslant 0$, $\rho_+ \geqslant 0$ *and* $\max\{\rho_-, \rho_+\} > 0$. *Assume that the sequence* $\{\rho_0(\lambda x)\}_{\lambda>0}$ *converges as* $\lambda \to \infty$, *in the weak-star topology of* L^∞, *to* $\rho_-\chi(x<0) + \rho_+\chi(x>0)$ *with* χ *the characteristic function. Let* (ρ, m) *be an* L^∞ *entropy weak solution of the Cauchy problem* (1.1)–(1.3), *satisfying the following estimates*

$$\begin{cases} 0 \leqslant \rho(x,t) \leqslant C, \quad |m(x,t)| \leqslant C\rho(x,t), \ \ if \ \ 1 < \gamma < 3, \\ 0 \leqslant \rho(x,t) \leqslant C, \ \ |m(x,t)| \leqslant C\rho(x,t)|\log\rho(x,t)|, \ \ if \ \ \gamma = 1, \end{cases}$$

and let $\bar{\rho}(z)$ $\left(z = \frac{x}{\sqrt{t}}\right)$ *be the similarity solution of* (1.5) *with boundary condition* $\bar{\rho}(\pm\infty) = \rho_\pm$. *Then* ρ *approaches to* $\bar{\rho}$ *in the sense that*

$$\lim_{t\to\infty} \int_{-L}^{L} |\rho\left(z\sqrt{t}, t\right) - \bar{\rho}(z)|^q \, dz = 0$$

for all $L > 0$ *and* $q \geqslant 1$.

Remark 1. (1) Theorem 2 implies that the function $w(t) : z \to \rho\left(z\sqrt{t}, t\right)$ converges strongly in L^q_{loc} towards the diffusive profile $\bar{\rho}$ as $t \to \infty$.

(2) This kind of measurement of convergence is appropriate for this problem, since $z = const.$ is the level curve of $\bar{\rho}$ in the (x,t)-plane. Similar measurement can be found in the recent works by Chen and Frid on the large time behavior for L^∞ weak solutions to hyperbolic conservation laws; see for instance [2].

(3) Theorem 2 claims the uniqueness of the asymptotic behavior for the solutions to (1.1)–(1.3), if the initial data has the same end-states. Hence, the asymptotic behavior of the solutions to (1.1)–(1.3) are uniquely determined by the end-states of initial data.

(4) Our results implies the following simple fact: If the vacuum is surrounded by gas; then there is not vacuum almost everywhere, time asymptotically. This is clear by the properties of $\bar{\rho}$ in Proposition 2.5 below in the case $\rho_- > 0$ and $\rho_+ > 0$.

(5) The case where $\rho_- = \rho_+ = 0$ is out of the scope of Theorem 2. This is because the scaling we used in (1.6) below destroys the conservation of mass in this case. It is obvious that the total mass of non-trivial $\rho_0(x)$ is uniformly away from zero, while the total mass of $\rho_0(\lambda x)$ tends to zero when λ goes to infinity if $\rho_- = \rho_+ = 0$. In the case of Theorem 2, the total mass of both ρ_0 and $\rho_0(\lambda x)$ remains infinite independent of λ. However, the case for compact support ρ_0 has particular interest for which the asymptotic behavior is expected to be the famous Barenblatt's solution of PME. Therefore, this case remains as an important open problem. We will treat this case in the future.

Let us explain the basic ideas of this paper. By using the compensated compactness theory, we first establish the global existence of L^∞ entropy weak solution. The case for $1 < \gamma < 3$ are rather standard by using the convergence framework of [19,20] to frictional step Lax–Friedrichs scheme with the invariant region theory [3]. In the case of $\gamma = 1$, the classical framework fails due to the singularity of velocity. As pointed out in [19,20], the strong entropy is useless for the case $\gamma > 1$, however, we will make use of both the strong and weak entropies. Based on the observation in [16], we carefully studied a class of entropies, parametrized by a complex number, and proved the commutation relations for some weak entropies. This commutation relations are shown to be valid for strong entropies by analytic extension upon the complex parameter. Hence, the support of the corresponding Young measure is proved to be either vacuum or one point. This shows the strong convergence of the viscosity approximation solutions.

Based on the existence results, we prove that any L^∞ entropy weak solutions of (1.1)–(1.3) converge strongly towards the unique self-similar solution of the PME. To this end, we first rescale the problem (1.1)–(1.3) by the diffusive scales:

$$\rho_\lambda(x,t) = \rho(\lambda x, \lambda^2 t), \;\; m_\lambda(x,t) = \lambda m(\lambda x, \lambda^2 t). \tag{1.6}$$

This produces a sequence of the functions $(\rho_\lambda, m_\lambda)$, satisfying the rescaled problem,

$$\begin{cases} \rho_{\lambda t} + m_{\lambda x} = 0, \\ \lambda^{-2} m_{\lambda t} + \left(\dfrac{m_\lambda^2}{\lambda^2 \rho_\lambda} + P(\rho_\lambda) \right)_x = -m_\lambda, \\ \rho_\lambda(x,0) = \rho_0(\lambda x), m_\lambda(x,0) = \lambda m_0(\lambda x). \end{cases} \tag{1.7}$$

Formally, (1.7) converges to the related PME (1.5) with self-similar singular initial data as $\lambda \to \infty$. The natural energy estimates through physical entropy inequality enables us to apply the compensated compactness upon such a sequence $(\rho_\lambda, m_\lambda)$. This compactness, together with the uniqueness established for the L^∞ weak solution for PME, implies the rigorous justification of the previous formal convergence. The convergence of $(\rho_\lambda, m_\lambda)$ is then translated to the weak justification of the large time asymptotic behavior of the solution (ρ, m) to (1.1)–(1.3) in the sense of the "convergence mean in time", which, combined with some energy estimates by entropy analysis, implies the strong convergence in large time and thus close the argument.

It should be mentioned that an analogous convergence result on the sequence $(\rho_\lambda, m_\lambda)$ of (1.7) to the solution of PME (1.5) has been obtained by Marcati and Milani [26]. Where, they studied the behavior of $(\rho^\varepsilon, u^\varepsilon)$ of the following problem

$$\begin{cases} \rho_t^\varepsilon + (\rho^\varepsilon u^\varepsilon)_x = 0, \\ \varepsilon(\rho^\varepsilon u^\varepsilon)_t + (\varepsilon \rho^\varepsilon (u^\varepsilon)^2 + P(\rho^\varepsilon))_x = -k u^\varepsilon, \\ \rho^\varepsilon(x,0) = \rho_0(x) \geqslant 0, \;\; u^\varepsilon(x,0) = u_0(x), \end{cases} \tag{1.8}$$

as $\varepsilon \to 0+$, and showed $\rho^\varepsilon \to \rho$ in $L^p_{\rm loc}$, $u^\varepsilon \to u$ in L^2 weak for all $p \in [1, \infty)$. Moreover, it was shown that ρ satisfies, in the sense of distributions, Darcy's law, so that ρ is a weak solution of the PME

$$\begin{cases} \rho_t = \frac{\gamma}{k(\gamma+1)}(\rho^{\gamma+1})_{xx}, \\ \rho(x,0) = \rho_0(x) \geqslant 0. \end{cases} \tag{1.9}$$

However, our case is different from theirs. First, the source term in (1.7) is different from (1.8) and the limit equations are different (see (1.5) and (1.9)). Next, the initial data in (1.8) is uniformly bounded while our initial data depends on the parameter λ which is more singular in the convergence. Indeed, $m_\lambda(x, 0)$ is not uniformly bounded in any L^q space for $q \in (1, +\infty]$ as $\lambda \to \infty$. Finally, our limit function of density is necessary to be self-similar and theirs may be not.

Another interesting paper related to the same issue of [26] is [27], where the singular limit problems are discussed for hyperbolic relaxation to parabolic equations.

The arrangement of the present paper is as follows. In Section 2, some knowledge on the PME are prepared carefully. A uniqueness for the L^∞ weak solution is established. The global existence of L^∞ weak entropy solution for the problem (1.1)–(1.3) is given in Section 3 by the compensated compactness theory. The large time asymptotic behavior is then presented in Section 4 with the help of the idea of [35].

2. Porous media equation

Consider the following Cauchy problem for PME:

$$\begin{cases} \rho_t = (\rho^\gamma)_{xx}, \quad \gamma \geqslant 1, t > 0, \\ \rho(x,0) = \rho_0(x) = \rho_-\chi(x<0) + \rho_+\chi(x>0), \end{cases} \tag{2.1}$$

where, χ is the characteristic function and $\rho_\pm$ are non-negative constants. Since the case $\rho_- = \rho_+ = 0$ is out of the interest of this paper, we assume that $\rho_- \geqslant \rho_+ \geqslant 0$ and $\rho_- > 0$ if $\rho_+ = 0$ in the discussion of this section without loss of generality.

If $\gamma = 1$, Eq. (2.1) becomes heat equation. The global existence, uniqueness and regularity of the non-negative solution to (2.1) are clear. Due to the similarity of the initial data, the solution is necessary to be self-similar.

Lemma 2.1. *For* $\gamma = 1$, (2.1) *has a unique solution* $\rho(x, t)$ *satisfying*:

(1) $\rho_+ \leqslant \rho \leqslant \rho_-$ *and* $\rho(x, t) > 0$ *for any* $t > 0$;
(2) $\rho(x, t) = \bar\rho\left(\frac{x}{\sqrt{t}}\right)$;
(3) $(|(\log\rho)_x|, |(\log\rho)_t|) \leqslant C(t_1)\left(t^{-\frac{1}{2}}, t^{-1}\right)$ *for any* $t \geqslant t_1 > 0$.

In the following of this section, we assume $\gamma > 1$. It is clear that the PME is parabolic if $\rho > 0$ and is not if $\rho = 0$. Furthermore, the derivatives of ρ is not necessary to be continuous across the interface which differs the state of vacuum from $\rho > 0$. Thus it is necessary to introduce weak solution.

Definition 2.2. A function $\rho(x,t) \in L^\infty(Q_T)$, $Q_T := [0,T] \times \mathbf{R}$, is called the weak solution of the problem (2.1), if it satisfies the following:

(1) $\rho(x,t) \geqslant 0$ for any $(x,t) \in Q_T$;
(2) $\iint_{Q_T} (\rho\phi_t + \rho^\gamma \phi_{xx})(x,t)\,dx\,dt + \int_{\mathbf{R}} \rho_0(x)\phi(x,0)\,dx = 0$ for smooth function $\phi(x,t)$ which vanishing for $|x|$ large and $t = T$.

The study on PME is extensive in literature. The global existence and uniqueness for suitable generalized solution (different from the Definition 2.2) of PME under the bounded continuous non-negative initial datum are proved in [33]. We refer to [8,17,34] for some further existence and/or uniqueness results of PME. However, these results need either the regularity or L^1 requirement on initial datum. For a good review on the existence, uniqueness and regularity of the solutions to PME as well as the behavior of interface, we refer to [1].

By modifying the approach of [17], we can prove the uniqueness of the weak solution to (2.1) defined above.

Theorem 2.3. *Eq.* (2.1) *has at most one weak solution.*

Proof. Suppose ρ_1 and ρ_2 are two weak solutions of (2.1) such that $0 \leqslant \rho_1, \rho_2 \leqslant M$. It turns out

$$\int_0^T \int_{\mathbf{R}} (\rho_1 - \rho_2)(\phi_t + a(x,t)\phi_{xx})(x,t)\,dx\,dt = 0, \tag{2.2}$$

where $a(x,t) = \int_0^1 \gamma(\theta\rho_1 + (1-\theta)\rho_2)^{\gamma-1}\,d\theta \geqslant 0$. For any given test function $F(x,t) \in C^\infty$ such that *supp* $F(x,t) \subset [0,T) \times [-\alpha_0, \alpha_0]$, with some number $\alpha_0 > 0$, we are going to prove

$$\int_0^T \int_{\mathbf{R}} (\rho_1 - \rho_2) F(x,t)\,dx\,dt = 0, \tag{2.3}$$

that implies Theorem 2.3.

We choose $a_\varepsilon(x,t) \in C^\infty$ to be a sequence of functions with the property $0 < \frac{1}{2}\varepsilon \leqslant a_\varepsilon(x,t) - a(x,t) \leqslant \varepsilon$ for suitable small positive constant ε. This is possible by first smoothing out $a(x,t)$ then adding $\frac{1}{2}\varepsilon$. It is known that, [18], for any $\alpha = \max\left\{\frac{1}{\sqrt{\varepsilon}}, \alpha_0 + 1\right\}$, there is a C^∞ solution $f(x,t;\alpha,\varepsilon)$ to the following backward problem:

$$\begin{cases} f_t + a_\varepsilon f_{xx} = F(x,t), \ \ (x,t) \in Q_T, \\ f(x,T) = 0, \ \ f(\mp\alpha, t) = 0. \end{cases} \tag{2.4}$$

Due to the maximum principle, we have for any α and ε,

$$|f(x,t;\alpha,\varepsilon)| \leqslant T\|F(x,t)\|_{L^\infty} \equiv C_1. \tag{2.5}$$

Multiply (2.4) by f_{xx}, then integrate the resulting equation on Q_T, we yield

$$\frac{1}{2}\int_\alpha^\alpha f_x^2(x,t)\,dx + \int_t^T \int_\alpha^\alpha a_\varepsilon f_{xx}^2\,dx\,dt \leqslant C_2, \tag{2.6}$$

for any $t \in [0,T]$. We remark that C_1 and C_2 depend on T and $F(x,t)$ but neither on α or ε.

Now, we set $\phi(x,t;\alpha,\varepsilon) = f(x,t;\alpha,\varepsilon)\xi_\alpha(x)$. Where, $\xi_\alpha(x) \in C_0^\infty(R)$, $0 \leqslant \xi_\alpha(x) \leqslant 1$, and $\xi_\alpha(x) = 1$, for $|x| \leqslant \alpha_0$, $\xi_\alpha(x) = 0$ for $|x| \geqslant \alpha$. Furthermore, it is possible to require $|\xi'_\alpha(x)| \leqslant \frac{2}{\alpha-\alpha_0}$. We thus extend $\phi(x,t;\alpha,\varepsilon)$ to be a test function in Q_T by putting $\phi(x,t;\alpha,\varepsilon) = 0$ for $|x| \geqslant \alpha$. Substituting such a ϕ into (2.2), we arrive at

$$\begin{aligned} &\left|\int_0^T \int_{-\infty}^\infty (\rho_1 - \rho_2) F(x,t)\,dx\,dt\right| \\ &\quad \leqslant \left|\int_0^T \int_{-\infty}^\infty (\rho_1 - \rho_2) a (2\xi'_\alpha f_x + f\xi''_\alpha)\,dx\,dt\right| \\ &\qquad + \left|\int_0^T \int_{-\infty}^\infty (\rho_1 - \rho_2)(a - a_\varepsilon) f_{xx}\xi_\alpha(x)\,dx\,dt\right| \\ &\quad \equiv I_1 + I_2. \end{aligned} \tag{2.7}$$

Since $\rho_i (i = 1,2)$ are uniformly bounded, we have

$$\begin{aligned} I_1 &\leqslant C\left[\left(\int_0^T \int_{-\alpha}^\alpha {\xi'_\alpha}^2\,dx\,dt\right)^{\frac{1}{2}} \left(\int_0^T \int_{-\alpha}^\alpha f_x^2\,dx\,dt\right)^{\frac{1}{2}} + \int_0^T \int_{-\alpha}^\alpha |\xi''_\alpha|\,dx\,dt\right] \\ &\leqslant C\left[(\alpha-\alpha_0)^{-\frac{1}{2}} (T.V.\{\xi_\alpha(x)\})^{\frac{1}{2}} + T.V.\{\xi'_\alpha(x)\}\right] \\ &\leqslant C\left[(\alpha-\alpha_0)^{-\frac{1}{2}} + (\alpha-\alpha_0)^{-1}\right], \end{aligned} \tag{2.8}$$

and

$$
\begin{aligned}
I_2 &\leqslant \|(\rho_1-\rho_2)\|_{L^\infty}\left(\int_0^T\int_{-\alpha}^{\alpha} a_\varepsilon f_{xx}^2\,dx\,dt\right)^{\frac{1}{2}}\left(\int_0^T\int_{-\alpha}^{\alpha}\frac{(a-a_\varepsilon)^2}{a_\varepsilon}\,dx\,dt\right)^{\frac{1}{2}}\\
&\leqslant C\alpha^{\frac{1}{2}}\|a-a_\varepsilon\|_{L^\infty}^{\frac{1}{2}}\\
&\leqslant C\varepsilon^{\frac{1}{4}}. \qquad (2.9)
\end{aligned}
$$

Eq. (2.3) follows from (2.7)–(2.9) by letting $\varepsilon \to 0$. This completes the proof of Theorem 2.3. □

Remark 2.4. From the proof of Theorem 2.3, it is clear that we have proved the uniqueness of the L^∞ weak solution to the following problem

$$
\begin{cases}
\rho_t = (\rho^\gamma)_{xx}, & \gamma\geqslant 1, t>0,\\
\rho(x,0)=\rho_0(x)\geqslant 0,
\end{cases}
$$

if $\rho_0(x)\in L^\infty$.

Due to the similarity of ρ_0, the above uniqueness implies that ρ itself is self-similar, i.e., $\rho(x,t)=\bar{\rho}(z)$ $\left(z=\frac{x}{\sqrt{t}}\right)$, which satisfies

$$
\begin{cases}
\left(\gamma\bar{\rho}^{\gamma-1}\bar{\rho}_z\right)_z+\frac{1}{2}z\bar{\rho}_z=0,\\
\bar{\rho}(-\infty)=\rho_-, \quad \bar{\rho}(+\infty)=\rho_+.
\end{cases}
\qquad (2.10)
$$

The problem (2.10) has been investigated clearly by [8]. The following propositions are due to [1,8].

Proposition 2.5. *If $\rho_+>0$, then there is one and only one solution $\bar{\rho}(z)\in C^2$ to* (2.10) *satisfying the following*:

(1) $\rho_+\leqslant\bar{\rho}(z)\leqslant\rho_-$ *is monotone decreasing on* $\mathbf{R}$.
(2) $(|\bar{\rho}_x|,|\bar{\rho}_t|)\leqslant C(t_1)(t^{-\frac{1}{2}},t^{-1})$ *for any* $t\geqslant t_1>0$.

Proposition 2.6. *If $\rho_+=0$ and $\rho_->0$, then there is one and only one solution $\bar{\rho}(z)$ to* (2.10). *Furthermore*, *the follows hold.*

(1) $0\leqslant\bar{\rho}(z)\leqslant\rho_-$ *is continuous and monotone decreasing on* $\mathbf{R}$.
(2) $\bar{\rho}^\gamma(z)$ *is smooth on* $\mathbf{R}$.
(3) *There is a number $b>0$, such that $\bar{\rho}(z)>0$ if $z<b$ and $\bar{\rho}(z)=0$ if $z\geqslant b$.*
(4) $\bar{\rho}(z)$ *is smooth if* $z<b$.

(5) $(\bar{\rho}^{\gamma})'(z) \to 0$ *as* $z \to b-0$.
(6) $(|\partial_x(\bar{\rho}^{\gamma-1})|, |\partial_t(\bar{\rho}^{\gamma-1})|) \leqslant C(t_1)(t^{-\frac{1}{2}}, t^{-1})$ *for any* $t \geqslant t_1 > 0$.

Proof. Proposition 2.5 and the first five statements of Proposition 2.6 are from [8] where the estimates in Proposition 2.5 are easily obtained from the explicit formula of the solution $\bar{\rho}(z)$. The estimates in Proposition 2.6 can be proved as follows.

For any $t \geqslant t_1 > 0$, $\bar{\rho}(z)$ satisfies the definition of the solution in [1]. The velocity estimate in [1] implies that $|\partial_x(\bar{\rho}^{\gamma-1})| \leqslant C(t_1)t^{-\frac{1}{2}}$. We see $\partial_x(\bar{\rho}^{\gamma-1}) = t^{-\frac{1}{2}}(\bar{\rho}^{\gamma-1})_z$ and thus $|(\bar{\rho}^{\gamma-1})_z| \leqslant C(t_1)$. Again, by [8] one can prove that $|\partial_t(\bar{\rho}^{\gamma-1})| \leqslant C(t_1)t^{-1}$ for $z \leqslant 0$, and $|\partial_t(\bar{\rho}^{\gamma-1})| = 0$ if $z > b$. Thus we have

$$\begin{aligned} |\partial_t(\bar{\rho}^{\gamma-1})| &= \tfrac{1}{2}t^{-1}z(\bar{\rho}^{\gamma-1})_z \\ &\leqslant C(t_1)t^{-1}b, \quad for\ 0 \leqslant z \leqslant b. \qquad \square \end{aligned}$$

3. Entropy weak solutions

We will study the global existence of the weak entropy solution to the Cauchy problem of (1.1). Namely,

$$\begin{cases} \rho_t + m_t = 0, \\ m_t + \left(\frac{m^2}{\rho} + P(\rho)\right)_x = -m, \\ \rho(x,0) = \rho_0(x), m(x,0) = m_0(x). \end{cases} \tag{3.1}$$

For the homogeneous case, i.e., the system of isentropic gas dynamics:

$$\begin{cases} \rho_t + m_t = 0, \\ m_t + \left(\frac{m^2}{\rho} + P(\rho)\right)_x = 0, \\ \rho(x,0) = \rho_0(x), m(x,0) = m_0(x), \end{cases} \tag{3.1$'$}$$

the global existence of entropy weak solution of Cauchy problem has been obtained by DiPerna [7] for $\gamma = 1 + \frac{2}{2n+1}$, $n \geqslant 2$, Ding et al. [5] for $1 < \gamma \leqslant \frac{5}{3}$, and Lions, Perthame et al. [19,20] for $\gamma > \frac{5}{3}$ by the theory of compensated compactness. Recently, the case for $\gamma = 1$ was solved by Huang and Wang [16] with a new convergence theorem.

In order to deal with source term, Ding et al. [6] proved the convergence of the frictional step Lax–Friedrichs scheme for the isentropic gas dynamic system with a class of source, if $1 < \gamma \leqslant \frac{5}{3}$. In particularly, their results give the global existence of weak entropy solutions for (3.1). Since the entropy is governed by Euler–Poisson–Douboux equation when $\gamma > 1$ and the compactness frameworks have been established by Lions, Perthame et al. [19,20], the global existence of weak solutions to (3.1) for $\frac{5}{3} < \gamma < 3$ can be proved in a quite similar analysis as the case of [6] by the approximation generated by frictional step Lax–Friedrichs scheme. We will omit the details of the proof for $1 < \gamma < 3$ and focus to the case $\gamma = 1$, where the above compactness frameworks fail. Here, we will sketch a proof based on the approach of [16] by viscosity approximation.

As usual, the proof of the global existence of weak solutions to (3.1) follows two steps:

- To build an approximate solution $(\rho_\varepsilon, m_\varepsilon)$ by artificial viscosity method and establish an uniformly L^∞ estimate by positively invariant region due to [3].
- To apply the div-curl Lemma [28] to infinite entropy–flux pairs.

As remarked in [19,20], the strong entropy–flux pairs are not useful for convergence in the case $\gamma > 1$, however, they are crucial for our proof when $\gamma = 1$.

Consider the viscous perturbation of (3.1)

$$\begin{cases} \rho_{\varepsilon t} + m_{\varepsilon x} = \varepsilon \rho_{\varepsilon xx}, \\ m_{\varepsilon t} + \left(\frac{m_\varepsilon^2}{\rho_\varepsilon} + P(\rho_\varepsilon) \right)_x = -m_\varepsilon + \varepsilon m_{\varepsilon xx}, \\ \rho_\varepsilon(x,0) = \rho_{\varepsilon 0}(x), m_\varepsilon(x,0) = m_{\varepsilon 0}(x), \end{cases} \tag{3.2}$$

where

$$\rho_{\varepsilon 0}(x) = (\rho_0 + \varepsilon) * j_\varepsilon, \;\; m_{\varepsilon 0}(x) = \rho_{\varepsilon 0}(u_0 * j_\varepsilon) \tag{3.3}$$

with j_ε a standard mollifier and $*$ the convolution product.

The corresponding Riemann invariants of (3.1) reads

$$w = \rho e^u, \;\; z = \rho e^{-u} \tag{3.4}$$

and the invariant region is

$$\Omega = \{(w, z) : 0 \leqslant w \leqslant const., \;\; 0 \leqslant z \leqslant const.\}. \tag{3.5}$$

Since the origin is a singular point of Ω, we cannot use the invariant region theory of [3] directly. However, Marcati and Milani [26] gave an important approach to recover this case. By their results, together with $\rho_\varepsilon(x,t) > 0$ due to DiPerna [7], we have

$$0 \leqslant w \leqslant const., \;\; 0 \leqslant z \leqslant const, \tag{3.6}$$

which implies

$$0 < \rho_\varepsilon \leqslant const., \;\; |m_\varepsilon| \leqslant const. \tag{3.7}$$

We remark that u_ε may tend to infinity as ρ_ε goes to zero. However, one infers from (3.6) that $|u_\varepsilon| \leqslant C|\log \rho_\varepsilon|$. It is easy to see that there exists a global solution to (3.2) due to the local existence results and the a priori estimate (3.7). This completes the first step.

In order to obtain the global solution of (3.1), it suffices to establish the strong convergence of the sequence $(\rho_\varepsilon, m_\varepsilon)$, extracting to the subsequence if necessary. However, the uniform bound of $(\rho_\varepsilon, m_\varepsilon)$ can only give the convergence in weak-star topology. We thus have to use the theory of compensated compactness.

We now recall the entropy–flux pairs. The functions (η, q) is an entropy–flux pair if they satisfy

$$\eta_t(\rho, u) + q_x(\rho, u) = 0$$

for any smooth solutions of (3.1′). Thus, (η, q) satisfy $\mathbf{D}q = \mathbf{D}\eta\mathbf{D}f$, with $\mathbf{D}$ the gradient operator, i.e.,

$$q_\rho = u\eta_\rho + \frac{1}{\rho}\eta_u, \quad q_u = \rho\eta_\rho + u\eta_u, \tag{3.8}$$

which indicates

$$\eta_{\rho\rho} = \frac{1}{\rho^2}\eta_{uu}. \tag{3.9}$$

A typical choice of the entropy is the physical energy pair:

$$\eta_{\mathrm{e}} = \frac{1}{2}\rho u^2 + \int_0^\rho \log s\, ds, \quad q_{\mathrm{e}} = \frac{1}{2}\rho u^3 + \rho u \log \rho. \tag{3.10}$$

By *weak entropies*, we call the entropy η vanishing at vacuum state.

Since the entropy equation for $\gamma = 1$ is completely different from the case for $\gamma > 1$, the compactness frameworks due to [5–7] and [19,20] fail here. We shall explore the idea developed in [16] to study the entropy equation (3.9).

To this end, we choose

$$\eta = \rho^{\frac{1}{1-\xi^2}} \exp\left\{\frac{\xi}{1-\xi^2}u\right\} = w^{\frac{1}{2(1-\xi)}} z^{\frac{1}{2(1+\xi)}}, \quad q = (u+\xi)\eta \tag{3.11}$$

with a complex number $\xi \in \mathbf{C}$. It is easy to see that such an (η, q) solves the equation (3.9). Furthermore, the formula (3.11) includes both *weak* and *strong* entropy pairs. We expect the strong entropy will be also useful in our analysis.

From (3.11), we see η is a weak entropy if and only if

$$\xi \in \Omega_w := \{\xi \in \mathbf{C}; -1 < Re\xi < 1\}. \tag{3.12}$$

To apply the div-curl lemma, we need to prove that $\eta_t(\rho_\varepsilon, m_\varepsilon)+q_x(\rho_\varepsilon, m_\varepsilon)$ lie in a compact set of $H^{-1}_{\rm loc}$ for weak entropy pairs. We compute that

$$\eta_{{\rm e}t}+q_{{\rm e}x}=(\mathbf{D}\eta_{\rm e})(0,-m_\varepsilon)^t+\varepsilon\eta_{{\rm e}xx}-\varepsilon(\rho_{\varepsilon x}, m_{\varepsilon x})(\mathbf{D}^2\eta_{\rm e})(\rho_{\varepsilon x}, m_{\varepsilon x})^t. \tag{3.13}$$

Choosing any $\phi\in C_0^\infty(\mathbf{R}^2_+)$ satisfying $\frac{\phi_x^2}{\phi}\leqslant C$, $\phi\geqslant 0$, we have from (3.13) that

$$\begin{aligned}&\iint\frac{\varepsilon}{\rho_\varepsilon}\left[\rho_{\varepsilon x}^2+\left(m_{\varepsilon x}-\frac{m_\varepsilon}{\rho_\varepsilon}\,\rho_{\varepsilon x}\right)^2\right]\phi\,dx\,dt\\
&\quad=\iint\eta_{\rm e}\phi_t+q_{\rm e}\phi_x\,dx\,dt-\iint\eta_{{\rm e}m}m_\varepsilon\phi\,dx\,dt\\
&\qquad-\varepsilon\iint\left[\left(-\frac{m_\varepsilon^2}{2\rho_\varepsilon^2}+\log\rho_\varepsilon\right)\rho_{\varepsilon x}+\frac{m_\varepsilon}{\rho_\varepsilon}\,m_{\varepsilon x}\right]\phi_x\,dx\,dt,\end{aligned}$$

where,

$$\begin{aligned}&\left|\varepsilon\iint\left[\left(-\frac{m_\varepsilon^2}{2\rho_\varepsilon^2}+\log\rho_\varepsilon\right)\rho_{\varepsilon x}+\frac{m_\varepsilon}{\rho_\varepsilon}m_{\varepsilon x}\right]\phi_x\,dx\,dt\right|\\
&\quad\leqslant\frac{1}{2}\varepsilon\iint\phi\rho_\varepsilon^{-1+k}\left[\frac{m_\varepsilon^2}{\rho_\varepsilon^2}\left(m_{\varepsilon x}-\frac{m_\varepsilon}{\rho_\varepsilon}\rho_{\varepsilon x}\right)^2+\left(\log\rho_\varepsilon+\frac{m_\varepsilon^2}{2\rho_\varepsilon^2}\right)\rho_{\varepsilon x}^2\right]dx\,dt\\
&\qquad+C\iint\rho_\varepsilon^{1-k}\,\frac{\phi_x^2}{\phi}\,dx\,dt,\quad for\ \ 0<k<1.\end{aligned}$$

Therefore, one has

$$\varepsilon\iint(\rho_{\varepsilon x}, m_{\varepsilon x})(\mathbf{D}^2\eta_{\rm e})(\rho_{\varepsilon x}, m_{\varepsilon x})^t\phi\,dx\,dt\leqslant C(\phi). \tag{3.14}$$

For any weak entropy $\eta=w^{\frac{1}{2(1-\xi)}}z^{\frac{1}{2(1+\xi)}}$, $\xi\in\Omega_w$, it is easy to show that $\eta_t(\rho_\varepsilon, m_\varepsilon)+q_x(\rho_\varepsilon, m_\varepsilon)$ lie in a compact set of $H^{-1}_{\rm loc}$, if

$$|\mathbf{D}^2\eta|\leqslant C|\mathbf{D}^2\eta_{\rm e}|. \tag{3.15}$$

However, (3.15) is not true due to the singularity of u. Hence, we have to study the behavior of Hessian matrix of η near the vacuum carefully. By f, we denote the flux function $f=\left(m, \frac{m^2}{\rho}+\rho\right)$. And $r_1=(1,u-1)^t$, $r_2=(1,u+1)^t$ are two eigenvectors of the Jacobi matrix $\mathbf{D}f$ of f. Since $\mathbf{D}^2\eta\mathbf{D}f$ is symmetric, we have

$$\mathbf{D}^2\eta(r_i, r_j) = 0, \ \ i \neq j,$$

and

$$\begin{cases} \mathbf{D}^2\eta(r_1, r_1) = \frac{2\xi^2}{(1-\xi^2)^2}\left(1 - \xi - 2u + 2u^2\right)\rho^{\frac{\xi^2}{1-\xi^2}-1}\exp\left\{\frac{\xi}{1-\xi^2}u\right\}, \\ \mathbf{D}^2\eta(r_2, r_2) = \frac{2\xi^2}{\left(1-\xi^2 r\right)^2}(1 + \xi + 2u + 2u^2)\rho^{\frac{\xi^2}{1-\xi^2}-1}\exp\left\{\frac{\xi}{1-\xi^2}u\right\}. \end{cases} \tag{3.16}$$

It is clear that

$$\mathbf{D}^2\eta(r_1, r_1) \geqslant 0, \quad \mathbf{D}^2\eta(r_2, r_2) \geqslant 0,$$

if $\xi \in (-1, 1)$ and $|u| > 2$. If $|u| \leqslant 2$, we have

$$|\mathbf{D}^2\eta| \leqslant C|\mathbf{D}^2\eta_{\mathrm{e}}|, \ \ if \ \ |u| \leqslant 2.$$

Now, in the same version of (3.14), by separating the value of u for discussion, one has

$$\varepsilon \iint (\rho_{\varepsilon x}, m_{\varepsilon x})\mathbf{D}^2\eta(\rho_{\varepsilon x}, m_{\varepsilon x})^t \phi \, dx \, dt \leqslant C(\phi). \tag{3.17}$$

for $\xi \in (-1, 1)$.

Since $(\mathbf{D}\eta)(0, -m_\varepsilon)^t$ is bounded in Radon measure space, $\varepsilon\eta_{xx}$ is compact in $H^{-1,2}_{\mathrm{loc}}$, and $\eta_t + q_x$ is bounded in $H^{-1,\infty}_{\mathrm{loc}}$, we conclude by Murat's Lemma that

Lemma 3.1. *If* $\xi \in (-1, 1)$, *then*

$$\eta_t(\rho_\varepsilon, m_\varepsilon) + q_x(\rho_\varepsilon, m_\varepsilon) \ \ is \ \ compact \ \ in \ \ H^{-1}_{\mathrm{loc}}, \tag{3.18}$$

where (η, q) *defined as in* (3.11).

Due to div-curl lemma, for any two entropy pairs $(\eta_1, q_1)(\xi_1)$ and $(\eta_2, q_2)(\xi_2)$, $\xi_1, \xi_2 \in (-1, 1)$, chosen as in (3.11), we have the so-called commutation relations

$$\langle v, q_1\eta_2 - q_2\eta_1\rangle = \langle v, q_1\rangle\langle v, \eta_2\rangle - \langle v, q_2\rangle\langle v, \eta_1\rangle \tag{3.19}$$

with v the corresponding Young measure. It is observed by Huang and Wang [16], Eq. (3.19) is valid for not only the weak entropies but the strong ones. In fact, Eq. (3.19) holds for $\xi \in \mathbf{C}$ except two points $(-1, 0)$ and $(1, 0)$. This can be verified by analytic extension on ξ. Then they can show the Young measure $\nu_{x,t}$ is either point mass or supported in vacuum. Thus we have proved

Theorem 3.2. *If* $\gamma = 1$ *and*

$$0 \leqslant \rho_0(x) \leqslant C, \quad |m_0(x)| \leqslant C\rho_0(x)|\log \rho_0(x)|, \quad a.e. \ x \in \mathbf{R},$$

then, *Eq.* (3.1) *has a global entropy weak solution* $(\rho, m)(x, t)$ *satisfying*

$$0 \leqslant \rho(x, t) \leqslant C, \quad |m(x, t)| \leqslant C\rho(x, t)|\log \rho(x, t)|, \quad a.e. \ x \in R.$$

4. Large time asymptotic behavior

This section is devoted to the proof of Theorem 2.

First of all, we prove the following simple lemma which will be useful in our argument below.

Lemma 4.1. *Let* $0 \leqslant a, b \leqslant M < \infty$, *there are positive constants* C_1 *and* C_2 *such that*

(1) $|a - b|^{\gamma+1} \leqslant (a - b)(P(a) - P(b))$,
(2) $C_1|a - b|^2 \leqslant [P(a) - P(b) - P'(b)(a - b)] \leqslant C_2|a - b|$, if $1 < \gamma \leqslant 2$,
(3) $C_1|a - b|^\gamma \leqslant [P(a) - P(b) - P'(b)(a - b)] \leqslant C_2|a - b|$, if $\gamma > 2$.

Proof. It is clear that $(a - b)(P(a) - P(b)) \geqslant 0$, $P(a) - P(b) - P'(b)(a - b) \geqslant 0$ and Lemma 4.1 is true if $a = b$. We assume that $a > b$ in the following discussion. The case for $a < b$ can be treated similarly, and will be omitted.

The first inequality is obvious, since $(b + (a - b))^\gamma - b^\gamma - (a - b)^\gamma \geqslant 0$.

For $\gamma > 1$, $P'(b)$ is bounded, thus $[P(a) - P(b) - P'(b)(a - b)] \leqslant C_2|a - b|$. Then, we notice that

$$\begin{aligned} [P(a) - P(b) - P'(b)(a - b)] &= P''(b + \theta(a - b))(a - b)^2 \\ &\geqslant \gamma(\gamma - 1)M^{\gamma-2}(a - b)^2 \\ &\geqslant C_1(a - b)^2, \end{aligned}$$

if $1 < \gamma \leqslant 2$. The second relations are proved.

We turn to prove the left side of the third inequality. It is easy to check that $f(x) = P(b + x) - P(b) - P'(b)x - P(x)$ is convex when $x \geqslant 0$ and $f(0) = f'(0) = 0$. Thus, $f(x) \geqslant 0$ and the proof is completed. $\square$

In order to study the time asymptotic behavior of the entropy weak solutions (ρ, m) of (1.1)–(1.3), we introduce the following rescaled problem

$$\begin{cases} \rho_{\lambda t} + m_{\lambda x} = 0, \\ \lambda^{-2} m_{\lambda t} + \left(\frac{m_\lambda^2}{\lambda^2 \rho_\lambda} + P(\rho_\lambda) \right)_x = -m_\lambda, \\ \rho_\lambda(x,0) = \rho_0(\lambda x), m_\lambda(x,0) = \lambda m_0(\lambda x), \end{cases} \tag{4.1}$$

where, $(\rho_\lambda, m_\lambda)(x,t)$ is defined as following

$$\rho_\lambda(x,t) = \rho(\lambda x, \lambda^2 t), \;\; m_\lambda(x,t) = \lambda m(\lambda x, \lambda^2 t). \tag{4.2}$$

Let (ρ, m) be an L^∞ entropy weak solution to (1.1)–(1.3) satisfying the conditions in Theorem 2. For any fixed λ, $(\rho_\lambda, m_\lambda)(x,t) \in L^\infty$ is an entropy weak solution to (4.1). Therefore, we have, for any non-negative test function $\phi \in \mathcal{D}(\mathbf{R}_+^2)$, that

$$\begin{cases} \iint_{t>0} (\rho_\lambda \phi_t + m_\lambda \phi_x)\, dx\, dt + \int_{\mathbf{R}} \rho_0(\lambda x)\phi(x,0)\, dx = 0 \\ \iint_{t>0} \left[\lambda^{-2} \left(m_\lambda \phi_t + \frac{m_\lambda^2}{\rho_\lambda} \phi_x \right) + P(\rho_\lambda)\phi_x - m_\lambda \phi \right] dx\, dt, \\ \quad + \int_{\mathbf{R}} \lambda^{-1} m_0(\lambda x) \phi(x,0)\, dx = 0, \\ \iint_{t>0} \left(\eta_{\lambda \mathrm{e}} \phi_t + q_{\lambda \mathrm{e}} \phi_x - \rho_\lambda u_\lambda^2 \phi \right) dx\, dt + \int_{\mathbf{R}} \eta_{\lambda \mathrm{e}}(x,0)\phi(x,0)\, dx \geqslant 0, \end{cases} \tag{4.3}$$

where, $\eta_{\lambda \mathrm{e}} = \eta_{\mathrm{e}} \left(\rho_\lambda, \frac{m_\lambda}{\lambda} \right)$, $q_{\lambda \mathrm{e}} = \lambda q_{\mathrm{e}} \left(\rho_\lambda, \frac{m_\lambda}{\lambda} \right)$. The following theorem claimed the convergence of the whole sequence $(\rho_\lambda, m_\lambda)(x,t)$ towards the unique similarity solution $(\bar\rho, \bar m)(x,t)$ of

$$\begin{cases} \bar\rho_t = P(\bar\rho)_{xx}, \\ \bar\rho(x,0) = \rho_- \chi(x < 0) + \rho_+(x > 0), \end{cases} \tag{4.4}$$

and

$$\bar m(x,t) = -P(\bar\rho)_x. \tag{4.5}$$

Theorem 4.1. *The uniformly bounded sequence ρ_λ converges strongly in $L^{\gamma+1}_{\mathrm{loc}}$ towards the unique similarity solution $\bar\rho$ of* (4.4) *and m_λ converges weakly towards $-P(\bar\rho)_x$ as $\lambda \to \infty$.*

Proof. We choose test function $\phi(x,t) = \psi(t)\theta(x)$ such that $0 \leqslant \theta(x) \in C^2(\mathbf{R})$ with compact support and $\frac{\theta'^2}{\theta} \leqslant M_1$ for some constant M_1 while $0 \leqslant \psi(t) \in C^2(\mathbf{R}_+)$ vanishing for $t \geqslant T$. Substituting such test function into the entropy inequality in (4.3), we get

$$\begin{aligned}
&\iint_{t>0} \left(\eta_{\lambda e} \psi' \theta + q_{\lambda e} \psi \theta' - \frac{m_\lambda^2}{\rho_\lambda} \psi \theta \right) dx\, dt \\
&\quad + \int_{\mathbf{R}} \left(\frac{m_0^2}{2\rho_0} + \frac{1}{(\gamma - 1)} \rho_0^\gamma \right) (\lambda x) \phi(x, 0)\, dx \geqslant 0, \quad (\gamma > 1),
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
&\iint_{t>0} \left[\left(\eta_{\lambda e} \psi' \theta + q_{\lambda e} \psi \theta' - \frac{m_\lambda^2}{\rho_\lambda} \psi \theta \right] dx\, dt \\
&\quad + \int_{\mathbf{R}} \left(\frac{m_0^2}{2\rho_0} + \int_0^{\rho_0} \log s\, ds \right) (\lambda x) \phi(x, 0)\, dx \geqslant 0, \quad (\gamma = 1).
\end{aligned} \tag{4.7}$$

These imply that

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbf{R}} \left(\frac{m_\lambda^2}{2\lambda^2 \rho_\lambda} + \frac{1}{(\gamma - 1)} \rho_\lambda^\gamma \right) \theta\, dx + \int_{\mathbf{R}} \frac{m_\lambda^2}{\rho_\lambda} \theta\, dx \\
&\quad \leqslant \left| \int_{\mathbf{R}} \left(\frac{m_\lambda^3}{2\lambda^2 \rho_\lambda^2} + \frac{\gamma}{\gamma - 1} {\rho_\lambda}^{\gamma-1} m_\lambda \right) \theta'\, dx \right| \\
&\quad \leqslant M_2 \int_{\mathbf{R}} |m_\lambda \theta'|\, dx \\
&\quad \leqslant \frac{1}{2} \int_{\mathbf{R}} \frac{m_\lambda^2}{\rho_\lambda} \theta\, dx + M_3 \int_{\mathbf{R}} \frac{\theta'^2}{\theta}\, dx, \quad (\gamma > 1),
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbf{R}} \left(\frac{m_\lambda^2}{2\lambda^2 \rho_\lambda} + \int_0^{\rho_\lambda} \log s\, ds \right) \theta\, dx + \int_{\mathbf{R}} \frac{m_\lambda^2}{\rho_\lambda} \theta\, dx \\
&\quad \leqslant \left| \int_{\mathbf{R}} \left(\frac{m_\lambda^3}{2\lambda^2 \rho_\lambda^2} + m_\lambda \log(\rho_\lambda) \right) \theta' \right| dx \\
&\quad \leqslant M_4 \int_{\mathbf{R}} \left| \frac{m_\lambda}{\sqrt{\rho_\lambda}} \theta' \right| dx \\
&\quad \leqslant \frac{1}{2} \int_{\mathbf{R}} \frac{m_\lambda^2}{\rho_\lambda} \theta\, dx + M_5 \int_{\mathbf{R}} \frac{\theta'^2}{\theta}\, dx, \quad (\gamma = 1),
\end{aligned} \tag{4.9}$$

and we get

$$\frac{d}{dt}\int_{\mathbf{R}}\left(\frac{m_\lambda^2}{2\lambda^2\rho_\lambda}+\frac{1}{(\gamma-1)}\rho_\lambda^\gamma\right)\theta\,dx+\frac{1}{2}\int_{\mathbf{R}}\frac{m_\lambda^2}{\rho_\lambda}\theta\,dx\leqslant M_6,\ \ \gamma>1,$$

$$\frac{d}{dt}\int_{\mathbf{R}}\left(\frac{m_\lambda^2}{2\lambda^2\rho_\lambda}+\int_0^{\rho_\lambda}\log s\,ds\right)\theta\,dx+\frac{1}{2}\int_{\mathbf{R}}\frac{m_\lambda^2}{\rho_\lambda}\theta\,dx\leqslant M_6,\ \ \gamma=1. \tag{4.10}$$

Hence, by integrating (4.10) on $[0,T]$, it holds that

$$\int_{\mathbf{R}}\left(\frac{m_\lambda^2}{2\lambda^2\rho_\lambda}+\frac{1}{(\gamma-1)}\rho_\lambda^\gamma\right)\theta(x,T)\,dx+\int_0^T\int_{\mathbf{R}}\frac{m_\lambda^2}{\rho_\lambda}\theta\,dx\,dt$$
$$\leqslant M_6(1+T),\ \ \gamma>1,$$

$$\int_{\mathbf{R}}\left(\frac{m_\lambda^2}{2\lambda^2\rho_\lambda}+\int_0^{\rho_\lambda}\log s\,ds\right)\theta(x,T)\,dx+\int_0^T\int_{\mathbf{R}}\frac{m_\lambda^2}{\rho_\lambda}\theta\,dx$$
$$\leqslant M_6(1+T),\ \ \gamma=1. \tag{4.11}$$

In fact, we have obtained the following estimates:

- ρ_λ, $\frac{m_\lambda}{\lambda\sqrt{\rho_\lambda}}$ and $\frac{m_\lambda}{\lambda}$ are uniformly bounded in L^∞;
- m_λ and $\frac{m_\lambda}{\sqrt{\rho_\lambda}}$ are uniformly bounded in L^2_{loc}

Thus, by passing to subsequence if necessary, we may claim the convergence of ρ_λ towards $\bar\rho$ in the weak-star topology of L^∞ and m_λ to $\bar m$ in the weak-star topology of L^2_{loc}. Since both $\rho_{\lambda t}+m_{\lambda x}$ and $\lambda^{-2}m_{\lambda t}+\left(\frac{m_\lambda^2}{\lambda^2\rho_\lambda}+P(\rho_\lambda)\right)_x$ lie in a compact set of H^{-1}_{loc}, and ρ_λ, m_λ, $\frac{m_\lambda}{\lambda^2}$, $\frac{m_\lambda^2}{\lambda^2\rho_\lambda}+P(\rho_\lambda)$ remain bounded in L^2_{loc}, we can use div-curl lemma to obtain that

$$\begin{aligned}
&\lim_{\lambda\to\infty}\left(\left(\frac{m_\lambda}{\lambda}\right)^2+\rho_\lambda P(\rho_\lambda)\right)-\lim_{\lambda\to\infty}\left(\frac{m_\lambda}{\lambda}\right)^2\\
&\quad=\lim_{\lambda\to\infty}\rho_\lambda P(\rho_\lambda)\\
&\quad=\lim_{\lambda\to\infty}\rho_\lambda\lim_{\lambda\to\infty}\left(\lambda^{-2}\frac{m_\lambda^2}{\rho_\lambda}+P(\rho_\lambda)\right)-\lim_{\lambda\to\infty}m_\lambda\lim_{\lambda\to\infty}\frac{m_\lambda}{\lambda^2}\\
&\quad=\lim_{\lambda\to\infty}\rho_\lambda\lim_{\lambda\to\infty}P(\rho_\lambda),
\end{aligned} \tag{4.12}$$

i.e.,

$$\lim_{\lambda\to\infty}(\rho_\lambda-\bar\rho)(P(\rho_\lambda)-P(\bar\rho))=0. \tag{4.13}$$

We observe from Lemma 4.1 that

$$|\rho_\lambda-\bar\rho|^{\gamma+1}\leqslant(\rho_\lambda-\bar\rho)(P(\rho_\lambda)-P(\bar\rho)),$$

which, together with (4.13), means that

$$\lim_{\lambda\to\infty}\iint_{t>0}\phi|\rho_\lambda-\bar\rho|^{\gamma+1}\,dx\,dt=0,\ \ \forall\phi\in\mathcal{D}^+(\mathbf{R}^2). \tag{4.14}$$

This shows that ρ_λ converges strongly in $L^{\gamma+1}_{\text{loc}}$ to $\bar\rho$, and then $P(\rho_\lambda)$ converges to $P(\bar\rho)$. Therefore, we may take the limits in (4.3) to get that

$$\iint_{t>0}(\bar\rho\phi_t+\bar m\phi_x)\,dx\,dt+\int_{\mathbf{R}}\bar\rho_0(x)\phi(x,0)\,dx=0,\ \ \forall\phi\in\mathcal{D}(\mathbf{R}^2) \tag{4.15}$$

and

$$\iint_{t>0}[P(\bar\rho)\phi_x-\bar m\phi]\,dx\,dt=0,\ \ \forall\phi\in\mathcal{D}(\mathbf{R}^2). \tag{4.16}$$

Namely, we have proved $\bar m=-P(\bar\rho)_x$, and thus $\bar\rho\in L^\infty$ satisfies

$$\begin{cases}\bar\rho_t=(\bar\rho^\gamma)_{xx},\ \ \gamma\geqslant1,\ \ t>0,\\ \bar\rho(x,0)=\bar\rho_0(x)=\rho_-\chi(x<0)+\rho_+\chi(x>0).\end{cases} \tag{4.17}$$

It is shown in Section 2, that (4.17) has a unique solution which is self-similar in the form of $\bar\rho(z)$ with $z=\frac{x}{\sqrt t}$. Its regularity and properties can be found in the propositions there. The uniqueness of solutions to (4.17) implies the convergence of the whole sequence (ρ_λ,m_λ). This completes the proof of Theorem 4.1. $\square$

It is observed in [35] that the convergence results in Theorem 4.1 implies some information on the asymptotic behavior of (ρ,m) to (1.1)(see Corollary 3.2 of [35]). However, we need a much stronger result.

Corollary 4.2. *Let $E_L^\beta(t)=\int_{-L}^{L}|\rho\left(z\sqrt t,t\right)-\bar\rho(z)|^\beta\,dz$. It holds for any $L>0,\beta>0$ that*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T E_L^\beta(t)\,dt=0.$$

Proof. In fact, the convergence of ρ_λ holds in L^q_{loc} for any finite $q \geqslant 1$, since both ρ_λ and $\bar\rho$ are uniformly bounded. By choosing $q > \max\{2,\ 2/\beta\}$, one has

$$
\begin{aligned}
&\lim_{\lambda\to\infty} \int_0^1 \int_{-L}^{L} \frac{1}{\sqrt{\tau}} \left|\rho_\lambda(x,\tau) - \bar\rho\left(\frac{x}{\sqrt{\tau}}\right)\right|^\beta dx\,d\tau \\
&\leqslant \lim_{\lambda\to\infty} \left(\int_0^1 \int_{-L}^{L} \left|\rho_\lambda(x,\tau) - \bar\rho\left(\frac{x}{\sqrt{\tau}}\right)\right|^{q\beta} dx\,d\tau\right)^{\frac{1}{q}} \left(2L\int_0^1 \tau^{-\frac{q'}{2}}\,d\tau\right)^{\frac{1}{q'}} \\
&\leqslant C(L,q') \lim_{\lambda\to\infty} \left(\int_0^1 \int_{-L}^{L} \left|\rho_\lambda(x,\tau) - \bar\rho\left(\frac{x}{\sqrt{\tau}}\right)\right|^{q\beta} dx\,d\tau\right)^{\frac{1}{q}} \\
&= 0,
\end{aligned}
\tag{4.18}
$$

where $q' = 1 + \frac{1}{q-1} < 2$. Taking $z = \frac{x}{\sqrt{\tau}}$ and $t = \lambda^2\tau$, (4.18) infers that

$$
\lim_{\lambda\to\infty} \lambda^{-2} \int_0^{\lambda^2} \int_{-\frac{\lambda L}{\sqrt{t}}}^{\frac{\lambda L}{\sqrt{t}}} |\rho\left(z\sqrt{t},t\right) - \bar\rho(z)|^\beta\,dz\,dt = 0. \tag{4.19}
$$

Denoting λ^2 by T, we have

$$
\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_{-L}^{L} |\rho\left(z\sqrt{t},t\right) - \bar\rho(z)|^\beta\,dz\,dt \\
&\leqslant \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_{-\frac{L\sqrt{T}}{\sqrt{t}}}^{\frac{L\sqrt{T}}{\sqrt{t}}} |\rho\left(z\sqrt{t},t\right) - \bar\rho(z)|^\beta\,dz\,dt \\
&= 0. \qquad \square
\end{aligned}
\tag{4.20}
$$

Based on this “convergence mean in time", we can prove the strong convergence with the help of the following lemma which is essentially the same as Lemma 4.1 in [35].

Lemma 4.3. *Let $F(t)\geqslant 0$, $G(t)\geqslant 0$ and $0\leqslant H(t)\leqslant C_2$ satisfy the following*:

$$
\begin{aligned}
&F(t) = G(t) + H(t),\quad \lim_{T\to\infty} \frac{1}{T}\int_{t_1}^T G(t)\,dt = 0, \\
&\frac{dF(t)}{dt} + H(t) \leqslant \frac{C_0}{t},
\end{aligned}
$$

for any $t_1 > 0$. Then, $\lim_{t\to\infty} F(t) = 0$.

Proof. Given $T > 0$, we define $\tau = T\exp\left\{-\frac{F(T)}{C_0}\right\} \leqslant T$ and $F_1(t) = F(t) - C_0\log t$. It follows that

$$\begin{aligned}(T-\tau)F_1(T) &= \int_\tau^T (t-\tau)F_1'(t)\,dt + \int_\tau^T F_1(t)\,dt \\ &= \int_\tau^T (t-\tau)\left(F'(t) - \frac{C_0}{t}\right)dt + \int_\tau^T (G(t)+H(t)-C_0\log t)\,dt \\ &\leqslant \int_\tau^T (1+\tau-t)H(t)\,dt + \int_\tau^T (G(t)-C_0\log t)\,dt \\ &\leqslant \int_\tau^T (G(t)-C_0\log t)\,dt + \int_\tau^{\tau+1} H(t)\,dt. \end{aligned} \tag{4.21}$$

It is easy to check the following:

$$\begin{aligned}(T-\tau)F_1(T) &+ \int_\tau^T C_0\log t\,dt \\ &= C_0(T-\tau)\left(\log\frac{T}{\tau} - \log T\right) + C_0(T\log T - T - \tau\log\tau + \tau) \\ &= C_0T\left(\log\frac{T}{\tau} - 1 + \frac{\tau}{T}\right) \\ &\equiv C_0TM\left(\frac{F(T)}{C_0}\right),\end{aligned}$$

where the strictly convex function $M(s)$ is defined by $M(s) = s + e^{-s} - 1$. Since $M(0) = M'(0) = 0$. Thus, $M(s) > 0$, if $s \neq 0$. Hence, it follows from (4.21) that

$$M\left(\frac{F(T)}{C_0}\right) \leqslant \frac{1}{C_0T}\left[\int_\tau^T G(t)\,dt + C_2\right] \to 0, \ \ as\ \ T \to \infty. \tag{4.22}$$

This implies $\lim_{t\to\infty} F(t) = 0$. $\square$

In the following, we will prove the strong convergence of $\rho - \bar\rho$ to zero as $t \to \infty$. The cases for $\gamma > 1$ and $\gamma = 1$ are separated.

4.1. Proof of Theorem 2: case $\gamma > 1$

Define

$$\Delta(x,t) = \eta_{\mathrm{e}}(\rho, m) - \frac{1}{\gamma - 1}(P(\bar\rho) + P'(\bar\rho)(\rho - \bar\rho)). \tag{4.21$'$}$$

For any given non-negative test function ϕ, we denote

$$E_{\phi}(t)=\int_{\mathbf{R}}\phi(z)\frac{1}{\gamma-1}(P(\rho)-P(\bar{\rho})-P'(\bar{\rho})\,(\rho-\bar{\rho}))(z\sqrt{t},t)\,dz. \tag{4.22$'$}$$

From Lemma 4.1, one has

$$\begin{cases} C_1\int_{\mathbf{R}}\phi(z)(\rho-\bar{\rho})^2\left(z\sqrt{t},t\right)\,dz\leqslant E_{\phi}(t)\\ \quad\leqslant C_2\int_{\mathbf{R}}\phi(z)|\rho-\bar{\rho}|\left(z\sqrt{t},t\right)\,dz,\ \ 1<\gamma\leqslant 2,\\ C_1\int_{\mathbf{R}}\phi(z)(\rho-\bar{\rho})^{\gamma}\left(z\sqrt{t},t\right)\,dz\leqslant E_{\phi}(t)\\ \quad\leqslant C_2\int_{\mathbf{R}}\phi(z)|\rho-\bar{\rho}|^2\left(z\sqrt{t},t\right)\,dz,\ \ 2<\gamma<3 \end{cases} \tag{4.23}$$

with two positive constants C_1 and C_2 independent of time. Thus, Corollary 4.2 implies, for any $\tau>0$ that

$$\lim_{T\to\infty}\frac{1}{T}\int_{\tau}^{T}E_{\phi}(t)\,dt=0. \tag{4.24}$$

Let

$$\begin{cases} Z_{\phi}(t)=\frac{1}{2}\int_{\mathbf{R}}\phi(z)\frac{m^2}{\rho}\left(z\sqrt{t},t\right)\,dz,\\ Y_{\phi}(t)=\int_{\mathbf{R}}\phi(z)\Delta\left(z\sqrt{t},t\right)\,dz=E_{\phi}(t)+Z_{\phi}(t). \end{cases} \tag{4.25}$$

For any $t>0$, we compute that

$$\begin{aligned}\partial_t\Delta &= \partial_t\eta_{\mathrm{e}}-\frac{\gamma}{\gamma-1}\bar{\rho}^{\gamma-1}\rho_t-\gamma\bar{\rho}^{\gamma-2}(\rho-\bar{\rho})\bar{\rho}_t\\ &= \partial_t\eta_{\mathrm{e}}+\frac{\gamma}{\gamma-1}\bar{\rho}^{\gamma-1}m_x-\gamma\bar{\rho}^{\gamma-2}(\rho-\bar{\rho})\bar{\rho}_t.\end{aligned}$$

Due to the entropy inequality, we have

$$\begin{aligned}&\partial_t\Delta+\partial_x q_{\mathrm{e}}(\rho,m)-\partial_x\left(\frac{\gamma}{\gamma-1}\bar{\rho}^{\gamma-1}m\right)+\frac{m^2}{\rho}\\ &\quad\leqslant-\gamma\bar{\rho}^{\gamma-2}m\bar{\rho}_x-\gamma\bar{\rho}^{\gamma-2}(\rho-\bar{\rho})\bar{\rho}_t\\ &\quad=-\frac{\gamma}{\gamma-1}m\partial_x(\bar{\rho}^{\gamma-1})-\frac{\gamma}{\gamma-1}(\rho-\bar{\rho})\partial_t(\bar{\rho}^{\gamma-1}),\end{aligned}$$

and thus

$$\partial_t \Delta + \partial_x q_{\mathrm{e}} - \partial_x \left(\frac{\gamma}{\gamma - 1} \bar{\rho}^{\gamma-1} m \right) + \frac{m^2}{\rho} \\ \leqslant C \left((\bar{\rho}^{\gamma-1})_x^2 + |\bar{\rho}_t^{\gamma-1}| \right). \tag{4.26}$$

By choosing $\psi(x,t) = \frac{1}{\sqrt{t}} \phi\left(\frac{x}{\sqrt{t}}\right)$ with $\frac{\phi'^2}{\phi}$ bounded, we have

$$\partial_t(\psi\Delta) + \partial_x \left(\frac{m^3}{2\rho^2}\psi + \frac{\gamma}{\gamma-1}\, \rho^{\gamma-1} m \right) \psi) - \partial_x \left(\frac{\gamma}{\gamma-1}\, \bar{\rho}^{\gamma-1} m\psi \right) + \frac{m^2}{\rho}\psi \\ \leqslant C \left(\psi(\bar{\rho}^{\gamma-1})_x^2 + \psi|\bar{\rho}_t^{\gamma-1}| + |\psi_t| + |m\psi_x| \right), \tag{4.27}$$

which implies that

$$\partial_t(\psi\Delta) + \partial_x \left(\left(\frac{m^3}{2\rho^2}\psi + \frac{\gamma}{\gamma-1} \rho^{\gamma-1} m \right) \psi \right) - \partial_x \left(\frac{\gamma}{\gamma-1} \bar{\rho}^{\gamma-1} m\psi \right) + \frac{m^2}{2\rho}\psi \\ \leqslant C \left(\psi(\bar{\rho}^{\gamma-1})_x^2 + \psi|\bar{\rho}_t^{\gamma-1}| + |\psi_t| + \frac{\phi'^2}{\phi} \right) \\ \leqslant t^{-\frac{3}{2}} f\left(\frac{x}{\sqrt{t}} \right), \tag{4.28}$$

where, f is a continuous function. By integrating the above inequality, we have

$$\frac{dY_\phi}{dt} + Z_\phi \leqslant C_0 t^{-1}, \tag{4.29}$$

which, together with (4.24), (4.25) and Lemma 4.3, implies that

$$\lim_{t\to\infty} E_\phi(t) = 0. \tag{4.30}$$

This proves the strong convergence of $(\rho - \bar{\rho})$ to zero in L^2_{loc} for $1 < \gamma \leqslant 2$ and in L^γ_{loc} for $2 < \gamma < 3$.

4.2. *Proof of Theorem 2: case $\gamma = 1$*

In this case, we have for any $t > 0$, $0 < \bar{\rho}(x,t) \in C^\infty$ due to the well-known properties of heat equation. The quantities Δ, E_ϕ and Y_ϕ are defined as follows:

$$\begin{cases} \Delta(x,t) = \eta_{\mathrm{e}}(\rho, m) - \int_0^{\bar\rho} \log s\, ds - (\rho - \bar\rho)\log\bar\rho, \\ E_\phi(t) = \int_{\mathbf{R}} \left(\int_0^{\rho} \log s\, ds - \int_0^{\bar\rho} \log s\, ds - (\log\bar\rho)(\rho-\bar\rho)\right)\left(z\sqrt{t}, t\right)\phi(z)\, dz, \\ Y_\phi(t) = \int_{\mathbf{R}} \Delta\left(z\sqrt{t}, t\right)\phi(z)\, dz, \\ Z_\phi(t) = \frac{1}{2}\int_{\mathbf{R}} \phi(z)\frac{m^2}{\rho}\left(z\sqrt{t}, t\right) dz, \end{cases} \tag{4.31}$$

where

$$\begin{aligned} C_1 \int_{\mathbf{R}} \phi(z)\,(\rho-\bar\rho)^2\left(z\sqrt{t}, t\right) dz &\leqslant E_\phi(t) \\ &\leqslant C_2 \int_{\mathbf{R}} \phi(z)|\rho-\bar\rho|^{\frac{1}{2}}\left(z\sqrt{t}, t\right) dz, \end{aligned} \tag{4.32}$$

and

$$Y_\phi(t) = E_\phi(t) + Z_\phi(t).$$

By means of the entropy inequality, it is easy to check that

$$\begin{aligned} &\partial_t(\psi\Delta) + \partial_x(q_{\mathrm{e}}\psi) - \partial_x(\psi m \log\bar\rho) + \psi\frac{m^2}{\rho} \\ &\quad\leqslant C\left(\psi|\partial_x \log\bar\rho|^2 + \psi|\partial_t\log\bar\rho| + |q_{\mathrm{e}} - m\log\bar\rho||\psi_x|\right), \end{aligned}$$

which implies

$$\begin{aligned} \partial_t(\psi\Delta) + \partial_x(q_{\mathrm{e}}\psi) - \partial_x(\psi m\log\bar\rho) + \psi\frac{m^2}{2\rho} &\leqslant C(\psi|\partial_x\log\bar\rho|^2 + \psi|\partial_t\log\bar\rho|) + \frac{\phi'^2}{\phi}\Bigg) \\ &\leqslant t^{-\frac{3}{2}} f\left(\frac{x}{\sqrt{t}}\right). \end{aligned} \tag{4.33}$$

We have again that

$$\frac{dY_\phi}{dt} + Z_\phi \leqslant C_0 t^{-1} \tag{4.34}$$

and then

$$\lim_{t\to\infty} E_\phi(t) = 0. \tag{4.35}$$

Thus, we have proved the desired strong convergence of $\rho - \bar\rho$ in L^2_{loc} for $\gamma = 1$.

Since ρ and $\bar\rho$ are uniformly bounded, the convergence obtained in Sections 4.1 and 4.2 are true in L^q_{loc} for any $q \geqslant 1$. Thus we completed the proof of Theorem 2.

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