

# Initial Boundary Value Problem for 2D Viscous Boussinesq Equations

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April 2, 2010

## Abstract

We study the initial boundary value problem of 2D viscous Boussinesq equations over a bounded domain with smooth boundary. We show that the equations have a unique classical solution for  $H^3$  initial data and no-slip boundary condition. In addition, we show that the kinetic energy is uniformly bounded in time.

**Keywords:** Viscous Boussinesq Equations, Global Regularity, Existence and Uniqueness

## 1 Introduction

Consider the 2D viscous Boussinesq equations

$$\begin{cases} U_t + U \cdot \nabla U + \nabla P = \nu \Delta U + \rho \mathbf{e}_2, \\ \rho_t + U \cdot \nabla \rho = 0, \\ \nabla \cdot U = 0, \end{cases} \quad (1)$$

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\*Email address: mjlai@math.uga.edu. Lai is partly supported by the National Science Foundation under grant DMS0713807.

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where  $U = (u, v)$  is the velocity vector field,  $P$  is the scalar pressure,  $\rho$  is the scalar density, the constant  $\nu > 0$  models viscous dissipation, and  $\mathbf{e}_2 = (0, 1)^T$ . In this paper, we consider (1) in a bounded domain  $\Omega \subset \mathbf{R}^2$  with smooth boundary  $\partial\Omega$ . The system is supplemented by the following initial and boundary conditions:

$$\begin{cases} (U, \rho)(\mathbf{x}, 0) = (U_0, \rho_0)(\mathbf{x}), & \mathbf{x} \in \Omega, \\ U|_{\partial\Omega} = 0. \end{cases} \quad (2)$$

The Boussinesq system is potentially relevant to the study of atmospheric and oceanographic turbulence, as well as other astrophysical situations where rotation and stratification play a dominant role (see e.g. [16] and [18]). In fluid mechanics, system (1) is used in the field of buoyancy-driven flow. It describes the motion of an incompressible inhomogeneous viscous fluid under the influence of gravitational force (c.f. [17]).

In addition to its own physical background, Boussinesq system was known by its close connection to the fundamental models, such as Euler and Navier-Stokes equations, for 3D incompressible flows. Due to the vortex stretching effect in 3D flows, the question of global existence/finite time blow-up of smooth solutions for the three-dimensional incompressible Euler or Navier-Stokes equations has been one of the most outstanding open problems in mathematics. Enormous efforts have been made during the last decades on this subject, yet the resolution is still elusive. There are a great amount of literatures concerning partial answers to this question. As part of the effort to understand the vortex stretching effect in 3D flows, various simplified model equations have been proposed. Among these models, the Boussinesq system is known to be one of the most commonly used because it is analogous to the 3D incompressible Euler or Navier-Stokes equations for axisymmetric swirling flow, and it shares a similar vortex stretching effect as that in the 3D incompressible flow. Better understanding of the 2D Boussinesq system will undoubtedly shed light on the understanding of 3D flows, at least for swirling flow (c.f. [17]).

In recent years, the 2D Boussinesq equations (1) have attracted significant attention. When  $\Omega = \mathbf{R}^2$ , the Cauchy problem for 2D Boussinesq equations with full viscosity has been well studied. In [3], Cannon and DiBenedetto studied the Cauchy problem for the Boussinesq equations with full viscosity

$$\begin{cases} U_t + U \cdot \nabla U + \nabla P = \nu \Delta U + \theta \mathbf{e}_2, \\ \theta_t + U \cdot \nabla \theta = \kappa \Delta \theta, \\ \nabla \cdot U = 0, \\ (U, \theta)(\mathbf{x}, 0) = (U_0, \theta_0)(\mathbf{x}), & \mathbf{x} \in \mathbf{R}^2, \end{cases} \quad (3)$$

which describe the flow of a viscous incompressible fluid subject to convective heat transfer, where  $\nu > 0, \kappa > 0$  are constants. They found a unique, global in time, weak solution. Furthermore, they improved the regularity of the solution when initial data is smooth. Recently, the result of global existence of smooth solutions to (3) is generalized to the cases of “partial viscosity” (i.e., either  $\nu > 0$  and  $\kappa = 0$ , or  $\nu = 0$  and  $\kappa > 0$ ) by Hou-Li [10] and Chae [4] independently. In [10], Hou and Li proved the global well-posedness of the Cauchy problem of the viscous Boussinesq equations. They showed that solutions with initial data in  $H^m$  ( $m \geq 3$ ) do not develop finite-time singularities. In [4], Chae considered the Boussinesq system for incompressible fluid in  $\mathbf{R}^2$  with either zero diffusion ( $\kappa = 0$ ) or zero viscosity ( $\nu = 0$ ). He proved global-in-time regularity in both cases. On the other hand, the global regularity/singularity question for the case of (3) with zero viscosity and zero diffusion ( $\kappa = \nu = 0$ ) still remains an outstanding open problem in mathematical fluid mechanics, and we refer the readers to [5], [6], [8], [9], [19] for studies in this direction.

In real world, the flows often move in bounded domains with constraints from boundaries, where the initial-boundary value problems appears. The solutions of the initial-boundary value problems usually exhibit different behaviors and much richer phenomena comparing with the Cauchy problem. In this direction, the case of  $\nu > 0$  and  $\kappa > 0$  has been analyzed in great extent (see e.g. [14] and references therein). The local existence and blow-up criterion of smooth solutions for the inviscid case ( $\nu = \kappa = 0$ ) is established very recently in [11], see also [7]. However, to the authors’ knowledge, the question of global regularity/finite time singularity for the cases of “partial viscosity”, such as (1), is still open. We will give a definite result to this problem in current paper.

Throughout this paper,  $\|\cdot\|_{L^p}$ ,  $\|\cdot\|_{W^{s,p}}$  and  $\|\cdot\|_{L^\infty}$  denote the norms of  $L^p(\Omega)$ ,  $W^{s,p}(\Omega)$  and  $L^\infty(\Omega)$  respectively, i.e.,

$$\begin{aligned} \|f\|_{L^p} &\equiv \|f\|_{L^p(\Omega)} = \left( \int_{\Omega} |f|^p d\mathbf{x} \right)^{1/p}, \quad \text{for } f \in L^p(\Omega), \\ \|f\|_{W^{s,p}} &\equiv \|f\|_{W^{s,p}(\Omega)} = \left( \sum_{|\alpha| \leq s} \int_{\Omega} |D^\alpha f|^p d\mathbf{x} \right)^{1/p}, \quad \text{for } f \in W^{s,p}(\Omega), \quad 1 \leq p < \infty, \\ \|f\|_{L^\infty} &\equiv \|f\|_{L^\infty(\Omega)} = \text{ess sup}_{\Omega} |f|, \quad \text{for } f \in L^\infty(\Omega), \end{aligned}$$

where  $\alpha = (\alpha_1, \alpha_2)$  is any multi-index with order  $|\alpha| = \alpha_1 + \alpha_2$  and  $D^\alpha = \partial_x^{\alpha_1} \partial_y^{\alpha_2}$ . For  $p = 2$ , we denote the norms  $\|\cdot\|_{L^2}$  and  $\|\cdot\|_{W^{s,2}}$  by  $\|\cdot\|$  and  $\|\cdot\|_{H^s}$  respectively. The function spaces under consideration are:

$$C([0, T]; H^3(\Omega)) \quad \text{and} \quad L^2([0, T]; H^4(\Omega)),$$

equipped with norms

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|\Psi(\cdot, t)\|_{H^3}, \quad \text{for } \Psi \in C([0, T]; H^3(\Omega)), \\ & \left( \int_0^T \|\Psi(\cdot, \tau)\|_{H^4}^2 d\tau \right)^{1/2}, \quad \text{for } \Psi \in L^2([0, T]; H^4(\Omega)). \end{aligned}$$

Unless specified,  $C$  will denote a generic constant which is independent of  $\rho$  and  $U$ , but may be dependent on  $\nu$  and the time  $T$  throughout the paper.

In this paper, we will generalize the study of [4] and [10] to bounded domains with typical physical boundary conditions (2)<sub>2</sub>. For the global existence of smooth solutions, we require the following compatibility conditions

$$\begin{cases} \nabla \cdot U_0 = 0, & U_0|_{\partial\Omega} = 0, \\ \nu \Delta U_0 + \rho_0 \mathbf{e}_2 - \nabla P_0 = 0, & \mathbf{x} \in \partial\Omega, \quad t = 0, \end{cases} \quad (4)$$

where  $P_0(\mathbf{x}) = P(\mathbf{x}, 0)$  is the solution to the Neumann boundary problem

$$\begin{cases} \Delta P_0 = \nabla \cdot [\rho_0 \mathbf{e}_2 - U_0 \cdot \nabla U_0], & \mathbf{x} \in \Omega, \\ \nabla P_0 \cdot \mathbf{n}|_{\partial\Omega} = [\nu \Delta U_0 + \rho_0 \mathbf{e}_2] \cdot \mathbf{n}|_{\partial\Omega}, \end{cases} \quad (5)$$

with  $\mathbf{n}$  the unit outward normal to  $\partial\Omega$ .

Our main results are stated in the following theorem.

**Theorem 1.1** *Let  $\Omega \subset \mathbf{R}^2$  be a bounded domain with smooth boundary. If  $(\rho_0(\mathbf{x}), U_0(\mathbf{x})) \in H^3(\Omega)$  satisfies the compatibility conditions (4)–(5), then there exists a unique solution  $(\rho, U)$  of (1)–(2) globally in time such that  $\rho(\mathbf{x}, t) \in C([0, T]; H^3(\Omega))$  and  $U(\mathbf{x}, t) \in C([0, T]; H^3(\Omega)) \cap L^2([0, T]; H^4(\Omega))$  for any  $T > 0$ . Moreover, there exists a constant  $\bar{C} > 0$  independent of  $t$  such that*

$$\|U(\cdot, t)\|_{L^2}^2 \leq \max \left\{ \|U(\cdot, 0)\|_{L^2}^2, \frac{\bar{C}^2}{\nu^2} \|\rho(\cdot, 0)\|_{L^2}^2 \right\}, \quad \forall t \geq 0. \quad (6)$$

The proof of Theorem 1.1 mainly consists of two parts. First, we show the global existence of weak solutions to (1)–(2), i.e., solutions satisfying the following definition:

**Definition 1.1**  *$(\rho, U)$  is said to be a global weak solution of (1)–(2), if for any  $T > 0$ ,  $U \in C([0, T]; L^2(\Omega)) \cap L^2([0, T]; H_0^1(\Omega))$ ,  $\rho \in C([0, T]; L^p(\Omega))$ ,  $\forall 1 \leq p < \infty$ , and it holds that*

$$\int_{\Omega} U_0 \cdot \Phi(\mathbf{x}, 0) d\mathbf{x} + \int_0^T \int_{\Omega} (U \cdot \Phi_t + U \cdot (U \cdot \nabla \Phi) + \rho \phi_2 - \nu \nabla \phi_1 \cdot \nabla u - \nu \nabla \phi_2 \cdot \nabla v) d\mathbf{x} dt = 0,$$

$$\int_{\Omega} \rho_0 \psi(\mathbf{x}, 0) d\mathbf{x} + \int_0^T \int_{\Omega} (\rho \psi_t + \rho U \cdot \nabla \psi) d\mathbf{x} dt = 0,$$

for any  $\Phi = (\phi_1, \phi_2) \in C_0^\infty(\Omega \times [0, T])^2$  satisfying  $\Phi(\mathbf{x}, T) = 0$  and  $\nabla \cdot \Phi = 0$ , and for any  $\psi \in C_0^\infty(\Omega \times [0, T])$  satisfying  $\psi(\mathbf{x}, T) = 0$ .

We then build up the regularity of the solution by energy estimate under the initial and boundary conditions (2). The energy estimate is somewhat delicate mainly due to the coupling between the velocity and density equations by convection and gravitational force and the boundary effects. Great efforts have been made to simplify the proof. Current proof involves intensive applications of Sobolev embeddings and we will see that the Ladyzhenskaya's inequalities, see Lemma 2.2, play a crucial role in the estimation of the velocity field. The results on Stokes equations by Temam [20], see lemma 2.1, are important in our energy framework. These are mainly due to the problem is set on the bounded domain, distinguishing itself from the Cauchy problem in [10] and [4]. Roughly speaking, because of the lack of the spatial derivatives of the solution at the boundary, our energy framework proceeds as follows: We first apply the standard energy estimate on the solution and the temporal derivatives of the solution. We then apply the Temam's results on Stokes equation to obtain the spatial derivatives. Such a process will be repeated up to third order, and then the carefully coupled estimates will be composed into a desired estimate leading to global regularity and uniqueness of solution. Finally, the uniform bound of the kinetic energy is shown in a very simple way. This result suggests that the viscous dissipation is strong enough to compensate the effects of gravitational force and nonlinear convection in order to prevent the development of singularity of the system. It should be pointed out that in the theorem obtained above, no smallness restriction is put upon the initial data.

The plan of the rest of this paper is as follows. In Section 2, we give some basic facts that will be used in this paper together with the global existence of weak solutions. Then we improve the regularity of the solution obtained in Section 2 by energy estimate in Section 3. We conclude the paper by proposing some future research problems in Section 4.

## 2 Preliminaries and Weak Solutions

In this section, we will list several facts which will be used in the proof of Theorem 1.1. Then we prove the global existence of weak solutions of (1)-(2). First we recall some useful results from [20].

**Lemma 2.1** *Let  $\Omega$  be any open bounded domain in  $\mathbf{R}^2$  with smooth boundary  $\partial\Omega$ . Consider the Stokes problem*

$$\begin{cases} -\nu\Delta U + \nabla P = f, & \text{in } \Omega, \\ \nabla \cdot U = 0, & \text{in } \Omega, \\ U = 0, & \text{on } \partial\Omega. \end{cases}$$

*If  $f \in W^{m,p}$ , then  $U \in W^{m+2,p}$ ,  $P \in W^{m+1,p}$  and there exists a constant  $c_0 = c_0(p, \nu, m, \Omega)$  such that*

$$\|U\|_{W^{m+2,p}} + \|P\|_{W^{m+1,p}} \leq c_0 \|f\|_{W^{m,p}},$$

*for any  $p \in (1, \infty)$  and the integer  $m \geq -1$ .*

We also need the following Sobolev Embeddings and Ladyzhenskaya's inequalities which are well-known and standard, (cf. [1] and [15]).

**Lemma 2.2** *Let  $\Omega \subset \mathbf{R}^2$  be any bounded domain with  $C^1$  smooth boundary. Then the following embeddings and inequalities hold:*

- (i)  $H^1(\Omega) \hookrightarrow L^p(\Omega)$ ,  $\forall 1 < p < \infty$ ;
- (ii)  $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ ,  $\forall 2 < p < \infty$ ;
- (iii)  $\|f\|_{L^4}^2 \leq 2\|f\|\|\nabla f\|$ ,  $\forall f : \Omega \rightarrow \mathbf{R}$  and  $f \in H_0^1(\Omega)$ ;
- (iv)  $\|f\|_{L^4}^2 \leq C(\|f\|\|\nabla f\| + \|f\|^2)$ ,  $\forall f : \Omega \rightarrow \mathbf{R}$  and  $f \in H^1(\Omega)$ .

Next, we establish the global existence of weak solutions of (1)-(2).

**Lemma 2.3** *Under the assumptions in Theorem 1.1, there exists a global weak solution  $(U, \rho)$  of (1)-(2) such that, for any  $T > 0$ ,  $U \in C([0, T]; L^2(\Omega)) \cap L^2([0, T]; H_0^1(\Omega))$ , and  $\rho \in C([0, T]; L^p(\Omega))$ ,  $\forall 1 \leq p < \infty$ .*

**Proof.** Following [13], we prove the lemma by a fixed point argument. To do so, we fix any  $T \in (0, \infty)$  and consider the problem (1)-(2) in  $\Omega \times [0, T]$ . Let  $B$  be the closed convex set in  $C([0, T]; L^2(\Omega)) \cap L^2([0, T]; H_0^1(\Omega))$  defined by

$$B = \left\{ V = (v_1, v_2) \in C([0, T]; L^2(\Omega)) \cap L^2([0, T]; H_0^1(\Omega)) \mid \begin{aligned} &\nabla \cdot V = 0, \text{ a.e. on } \Omega \times (0, T), \\ &\|V\|_{C([0, T]; L^2(\Omega))}^2 + \|V\|_{L^2([0, T]; H_0^1(\Omega))}^2 \leq R_0 \end{aligned} \right\}, \quad (7)$$

where  $R_0$  will be determined later. For fixed  $\varepsilon \in (0, 1)$  and any  $V \in B$ , we first mollify  $V$  using the standard procedure (c.f. [13]) to get

$$V_\varepsilon = \bar{V}_\varepsilon * \eta_{\varepsilon/2},$$

where  $\bar{V}_\varepsilon$  is the truncation of  $V$  in  $\Omega_\varepsilon = \{\mathbf{x} \in \Omega \mid \text{dist}(\mathbf{x}, \partial\Omega) > \varepsilon\}$  (extended by 0 to  $\Omega$ ), and  $\eta_{\varepsilon/2}$  is the standard mollifier. Then  $V_\varepsilon$  satisfies

$$\begin{aligned} V_\varepsilon &\in C([0, T]; C_0^\infty(\bar{\Omega})), \quad \nabla \cdot V_\varepsilon = 0, \\ \|V_\varepsilon\|_{C([0, T]; L^2(\Omega))} &\leq C \|V\|_{C([0, T]; L^2(\Omega))}, \\ \|V_\varepsilon\|_{L^2([0, T]; H_0^1(\Omega))} &\leq C \|V\|_{L^2([0, T]; H_0^1(\Omega))}, \end{aligned} \tag{8}$$

for some constant  $C > 0$  which is independent of  $\varepsilon$ . Similarly, we regularize the initial data to obtain the smooth approximation  $\rho_0^\varepsilon(\mathbf{x})$  for  $\rho_0(\mathbf{x})$  and  $U_0^\varepsilon(\mathbf{x})$  for  $U_0(\mathbf{x})$  respectively, such that

$$\begin{aligned} \rho_0^\varepsilon(\mathbf{x}) &\in C_0^\infty(\bar{\Omega}), \quad \|\rho_0^\varepsilon(\mathbf{x}) - \rho_0(\mathbf{x})\|_{H^1(\Omega)} < \varepsilon, \\ U_0^\varepsilon(\mathbf{x}) &\in C_0^\infty(\bar{\Omega}), \quad \nabla \cdot U_0^\varepsilon(\mathbf{x}) = 0 \text{ and } \|U_0^\varepsilon(\mathbf{x}) - U_0(\mathbf{x})\|_{H^1(\Omega)} < \varepsilon. \end{aligned}$$

Then we solve the transport equation with smooth initial data

$$\begin{cases} \rho_t + V_\varepsilon \cdot \nabla \rho = 0, \\ \rho(\mathbf{x}, 0) = \rho_0^\varepsilon(\mathbf{x}), \end{cases} \tag{9}$$

and we denote the solution by  $\rho^\varepsilon$ . Next, we solve the nonhomogeneous (linearized) Navier-Stokes equation with smooth initial data

$$\begin{cases} \nabla \cdot U = 0 \\ U_t + V_\varepsilon \cdot \nabla U + \nabla P = \nu \Delta U + \rho^\varepsilon \mathbf{e}_2, \\ U|_{\partial\Omega} = 0, \quad U(\mathbf{x}, 0) = U_0^\varepsilon(\mathbf{x}), \end{cases} \tag{10}$$

and denote the solution by  $U^\varepsilon$  and the corresponding pressure by  $P^\varepsilon$ . Then we define the mapping  $F_\varepsilon(V) = U^\varepsilon$ . The solvabilities of (9) and (10) follow easily from [13]. Next, we prove that  $F_\varepsilon$  satisfies the conditions of Schauder fixed point theorem, i.e.,  $F_\varepsilon : B \rightarrow B$  is continuous and compact. These will be achieved by the method of energy estimate.

We start from (9). For any  $2 \leq p < \infty$ , multiplying (9)<sub>1</sub> by  $\rho|\rho|^{p-2}$  and integrating the resulting equation over  $\Omega$  by parts, we get

$$\|\rho(\cdot, t)\|_{L^p} = \|\rho_0^\varepsilon\|_{L^p} \leq \|\rho_0\|_{L^p} + \varepsilon c(\Omega, p), \quad \forall 0 \leq t \leq T, \quad \forall 0 < \varepsilon < 1,$$

i.e.,

$$\|\rho^\varepsilon(\cdot, t)\|_{L^p} = \|\rho_0^\varepsilon\|_{L^p} \leq \|\rho_0\|_{L^p} + \varepsilon c(\Omega, p), \quad \forall 0 \leq t \leq T, \quad \forall 0 < \varepsilon < 1, \quad (11)$$

where  $c(\Omega, p)$  is a constant depending only on  $\Omega$  and  $p$ . We then estimate  $\|U^\varepsilon\|_{L^2([0, T]; H_0^1(\Omega))}^2$ . Taking  $L^2$  inner product of (10)<sub>2</sub> with  $U$ , after integrating by parts and using Young's inequality, we have

$$\frac{1}{2} \frac{d}{dt} \|U\|^2 + \nu \|\nabla U\|^2 \leq C(\delta) \|\rho^\varepsilon\|^2 + \delta \|U\|^2, \quad (12)$$

where  $\delta > 0$  is a constant to be determined. Since  $U$  satisfies the no-slip boundary condition, Poincaré's inequality implies that  $\|U\| \leq C \|\nabla U\|$  for some constant  $C$  depending only on  $\Omega$ . Choosing  $\delta = \nu/2C$  in (12) we obtain

$$\frac{1}{2} \frac{d}{dt} \|U\|^2 + \frac{\nu}{2} \|\nabla U\|^2 \leq C \|\rho^\varepsilon\|^2, \quad (13)$$

which together with (11) yields, after integration over  $[0, T]$ , that

$$\|U\|_{C([0, T]; L^2(\Omega))}^2 + \nu \|\nabla U\|_{L^2([0, T]; L^2(\Omega))}^2 \leq CT(\|\rho_0\|^2 + \varepsilon) + (\|U_0\|^2 + \varepsilon).$$

Since  $0 < \varepsilon < 1$ , we have

$$\|U\|_{C([0, T]; L^2(\Omega))}^2 + \|U\|_{L^2([0, T]; H_0^1(\Omega))}^2 \leq C(T, \rho_0, U_0, \nu, \Omega),$$

i.e.,

$$\|U^\varepsilon\|_{C([0, T]; L^2(\Omega))}^2 + \|U^\varepsilon\|_{L^2([0, T]; H_0^1(\Omega))}^2 \leq C(T, \rho_0, U_0, \nu, \Omega). \quad (14)$$

Choosing  $R_0$  such that  $R_0 \geq C(T, \rho_0, U_0, \nu, \Omega)$  we see that  $F_\varepsilon$  maps  $B$  into  $B$  for any  $0 < \varepsilon < 1$ . We remark that the constant  $C(T, \rho_0, U_0, \nu, \Omega)$  in (14) does not depend on  $\varepsilon$ .

Next we prove the compactness of  $F_\varepsilon$ . For this purpose, we continue to find estimates of  $\|\nabla U^\varepsilon\|_{C([0, T]; L^2(\Omega))}^2$  and  $\|U_t^\varepsilon\|_{L^2([0, T]; L^2(\Omega))}^2$ . Taking  $L^2$  inner product of (10)<sub>2</sub> with  $U_t$ , one has

$$\begin{aligned} \frac{\nu}{2} \frac{d}{dt} \|\nabla U\|^2 + \|U_t\|^2 &\leq \int_{\Omega} |V_\varepsilon| |U_t| |\nabla U| dx + \int_{\Omega} \rho \mathbf{e}_2 \cdot U_t dx \\ &\leq \frac{1}{4} \|U_t\|^2 + \|V_\varepsilon \nabla U\|^2 + \frac{1}{4} \|U_t\|^2 + \|\rho\|^2 \\ &\leq \frac{1}{2} \|U_t\|^2 + \|V_\varepsilon\|_{L^\infty}^2 \|\nabla U\|^2 + C \end{aligned}$$

which implies that

$$\frac{\nu}{2} \frac{d}{dt} \|\nabla U\|^2 + \frac{1}{2} \|U_t\|^2 \leq \|V_\varepsilon\|_{L^\infty}^2 \|\nabla U\|^2 + C. \quad (15)$$



Applying Gronwall's inequality to (15) and using (8) we have

$$\|\nabla U\|_{C([0,T];L^2(\Omega))}^2 + \|U_t\|_{L^2([0,T];L^2(\Omega))}^2 \leq C. \quad (16)$$

By Lemma 2.1 we know that

$$\begin{aligned} \|U\|_{H^2} &\leq C(\|U_t\| + \|\rho\| + \|V_\varepsilon \cdot \nabla U\|) \\ &\leq C(\|U_t\| + C + C\|V_\varepsilon\|_{L^\infty} \|\nabla U\|), \end{aligned} \quad (17)$$

which together with (16) yields

$$\|U^\varepsilon\|_{L^2([0,T];H^2(\Omega))}^2 \leq C. \quad (18)$$

From (16) and (18) we know that  $F_\varepsilon$  is compact by Sobolev embedding theorem.

Now we prove the continuity of  $F_\varepsilon$ . Let  $F_\varepsilon(V_i) = U_i^\varepsilon$ , by definition we know

$$\begin{cases} \rho_{it}^\varepsilon + V_{i\varepsilon} \cdot \nabla \rho_i^\varepsilon = 0, \\ U_{it}^\varepsilon + V_{i\varepsilon} \cdot \nabla U_i^\varepsilon + \nabla P_i^\varepsilon = \nu \Delta U_i^\varepsilon + \rho_i^\varepsilon \mathbf{e}_2, \\ \nabla \cdot U_i^\varepsilon = 0, \quad U_i^\varepsilon|_{\partial\Omega} = 0, \\ (\rho_i^\varepsilon, U_i^\varepsilon)(\mathbf{x}, 0) = (\rho_0^\varepsilon, U_0^\varepsilon)(\mathbf{x}), \quad i = 1, 2. \end{cases}$$

Subtracting the equation for  $i = 2$  from the one for  $i = 1$  we have

$$\begin{cases} \varrho_t^\varepsilon + V_{1\varepsilon} \cdot \nabla \varrho^\varepsilon + W_\varepsilon \cdot \nabla \rho_2^\varepsilon = 0, \\ \chi_t^\varepsilon + V_{1\varepsilon} \cdot \nabla \chi^\varepsilon + W_\varepsilon \cdot \nabla U_2^\varepsilon + \nabla Q^\varepsilon = \nu \Delta \chi^\varepsilon + \varrho^\varepsilon \mathbf{e}_2, \\ \nabla \cdot \chi^\varepsilon = 0, \quad \chi^\varepsilon|_{\partial\Omega} = 0, \\ (\varrho^\varepsilon, \chi^\varepsilon)(\mathbf{x}, 0) = \mathbf{0}, \end{cases} \quad (19)$$

where  $\varrho^\varepsilon = \rho_1^\varepsilon - \rho_2^\varepsilon$ ,  $W_\varepsilon = V_{1\varepsilon} - V_{2\varepsilon}$ ,  $\chi^\varepsilon = U_1^\varepsilon - U_2^\varepsilon$ , and  $Q^\varepsilon = P_1^\varepsilon - P_2^\varepsilon$ . Taking the  $L^2$  inner products of (19)<sub>1</sub> with  $\varrho^\varepsilon$  and (19)<sub>2</sub> with  $\chi^\varepsilon$  we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\varrho^\varepsilon\|^2 &= - \int_{\Omega} (W_\varepsilon \cdot \nabla \rho_2^\varepsilon) \varrho^\varepsilon dx, \\ \frac{1}{2} \frac{d}{dt} \|\chi^\varepsilon\|^2 + \nu \|\nabla \chi^\varepsilon\|^2 &= - \int_{\Omega} (W_\varepsilon \cdot \nabla U_2^\varepsilon) \chi^\varepsilon dx + \int_{\Omega} \varrho^\varepsilon \mathbf{e}_2 \cdot \chi^\varepsilon dx. \end{aligned} \quad (20)$$

Since  $\rho_2^\varepsilon \in C([0, T]; C^\infty(\bar{\Omega}))$ , we get from (20)<sub>1</sub> that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\varrho^\varepsilon\|^2 &\leq \|\nabla \rho_2^\varepsilon\|_{L^\infty} \|W_\varepsilon\| \|\varrho^\varepsilon\| \\ &\leq C(\|W_\varepsilon\|^2 + \|\varrho^\varepsilon\|^2), \end{aligned}$$

from which we get

$$\begin{aligned}\|\varrho^\varepsilon\|^2 &\leq e^{CT} \int_0^T \|W_\varepsilon\|^2 d\tau \\ &\leq C \|W_\varepsilon\|_{C([0,T];L^2(\Omega))}^2.\end{aligned}\tag{21}$$

Since  $U_2^\varepsilon \in L^2([0, T]; H^2(\Omega))$ , we derive from (20)<sub>2</sub>:

$$\begin{aligned}\frac{1}{2} \frac{d}{dt} \|\chi^\varepsilon\|^2 + \nu \|\nabla \chi^\varepsilon\|^2 &\leq \|W_\varepsilon\| \|\nabla U_2^\varepsilon\|_{L^4} \|\chi^\varepsilon\|_{L^4} + \|\varrho^\varepsilon\| \|\chi^\varepsilon\| \\ &\leq C \|W_\varepsilon\| \|U_2^\varepsilon\|_{H^2} \|\chi^\varepsilon\|_{H^1} + \|\varrho^\varepsilon\| \|\chi^\varepsilon\| \\ &\leq C \|W_\varepsilon\| \|U_2^\varepsilon\|_{H^2} \|\nabla \chi^\varepsilon\| + \|\varrho^\varepsilon\| \|\chi^\varepsilon\| \\ &\leq C \|W_\varepsilon\|^2 \|U_2^\varepsilon\|_{H^2}^2 + \frac{\nu}{2} \|\nabla \chi^\varepsilon\|^2 + \frac{1}{2} \|\varrho^\varepsilon\|^2 + \frac{1}{2} \|\chi^\varepsilon\|^2 \\ &\leq C(t) \|W_\varepsilon\|_{C([0,T];L^2(\Omega))}^2 + \frac{\nu}{2} \|\nabla \chi^\varepsilon\|^2 + \frac{1}{2} \|\chi^\varepsilon\|^2,\end{aligned}\tag{22}$$

where  $\int_0^T C(\tau) d\tau \leq C$  and we have used (21). From (22) we get

$$\frac{1}{2} \frac{d}{dt} \|\chi^\varepsilon\|^2 + \frac{\nu}{2} \|\nabla \chi^\varepsilon\|^2 \leq \frac{1}{2} \|\chi^\varepsilon\|^2 + C(t) \|W_\varepsilon\|_{C([0,T];L^2(\Omega))}^2,\tag{23}$$

which implies, after applying Gronwall's inequality, that

$$\|\chi^\varepsilon\|^2 \leq C \|W_\varepsilon\|_{C([0,T];L^2(\Omega))}^2.\tag{24}$$

Integrating (23) over  $[0, T]$  using (24) we have

$$\int_0^T \|\nabla \chi^\varepsilon\|^2 d\tau \leq C \|W_\varepsilon\|_{C([0,T];L^2(\Omega))}^2.\tag{25}$$

Combining (24) and (25) we get

$$\|\chi^\varepsilon\|_{C([0,T];L^2(\Omega))}^2 + \|\chi^\varepsilon\|_{L^2([0,T];H_0^1(\Omega))}^2 \leq C \|V_1 - V_2\|_{C([0,T];L^2(\Omega))}^2,$$

i.e.,

$$\|U_1^\varepsilon - U_2^\varepsilon\|_B^2 \leq C \|V_1 - V_2\|_B^2,$$

where  $\|\cdot\|_B^2 = \|\cdot\|_{C([0,T];L^2(\Omega))}^2 + \|\cdot\|_{L^2([0,T];H_0^1(\Omega))}^2$ . By definition we know

$$\|F_\varepsilon(V_1) - F_\varepsilon(V_2)\|_B^2 \leq C \|V_1 - V_2\|_B^2,$$

which implies that  $F_\varepsilon : B \rightarrow B$  is continuous.

Therefore, Schauder theorem implies that for any fixed  $\varepsilon \in (0, 1)$ , there exists  $U^\varepsilon \in B$  such that  $F_\varepsilon(U^\varepsilon) = U^\varepsilon$ , namely,

$$\begin{cases} \rho_t^\varepsilon + U_\varepsilon \cdot \nabla \rho^\varepsilon = 0 \\ U_t^\varepsilon + U_\varepsilon \cdot \nabla U^\varepsilon + \nabla P^\varepsilon = \nu \Delta U^\varepsilon + \rho^\varepsilon \mathbf{e}_2, \\ \nabla \cdot U^\varepsilon = 0, \\ U^\varepsilon|_{\partial\Omega} = 0, \quad (\rho^\varepsilon, U^\varepsilon)(\mathbf{x}, 0) = (\rho_0^\varepsilon, U_0^\varepsilon)(\mathbf{x}), \end{cases}$$

where  $U_\varepsilon$  is the regularization of  $U^\varepsilon$ . By a bootstrap argument (c.f. [13]) we know that  $(\rho^\varepsilon, U^\varepsilon) \in C^\infty(\bar{\Omega} \times [0, T])$ . Then it is obvious that  $(\rho^\varepsilon, U^\varepsilon)$  satisfy the integral identities, i.e.,

$$\begin{aligned} 0 &= \int_{\Omega} U_0^\varepsilon \cdot \Phi(\mathbf{x}, 0) d\mathbf{x} \\ &+ \int_0^T \int_{\Omega} (U^\varepsilon \cdot \Phi_t + U_\varepsilon \cdot (U^\varepsilon \cdot \nabla \Phi) + \rho^\varepsilon \mathbf{e}_2 \cdot \Phi - \nu \nabla \phi_1 \cdot \nabla u^\varepsilon - \nu \nabla \phi_2 \cdot \nabla v^\varepsilon) d\mathbf{x} dt, \quad (26) \\ 0 &= \int_{\Omega} \rho_0^\varepsilon \psi(\mathbf{x}, 0) d\mathbf{x} + \int_0^T \int_{\Omega} (\rho^\varepsilon \psi_t + \rho^\varepsilon U_\varepsilon \cdot \nabla \psi) d\mathbf{x} dt, \end{aligned}$$

for any  $\varepsilon > 0$ ,  $\Phi = (\phi_1, \phi_2) \in C_0^\infty(\bar{\Omega} \times [0, T])^2$  satisfying  $\Phi(\mathbf{x}, T) = 0$  and  $\nabla \cdot \Phi = 0$ , and for any  $\psi \in C^\infty(\bar{\Omega} \times [0, T])$  satisfying  $\psi(\mathbf{x}, T) = 0$ .

In view of (11), (14) and from the definition of  $U_\varepsilon$  we know that there exist functions  $U \in B$  and  $\rho \in C([0, T]; L^p(\Omega))$ ,  $\forall 2 \leq p < \infty$  such that as  $\varepsilon \rightarrow 0+$ ,

$$\begin{aligned} U_\varepsilon &\rightharpoonup U \text{ weakly in } C([0, T]; L^2(\Omega)) \cap L^2([0, T]; H_0^1(\Omega)), \\ U^\varepsilon &\rightharpoonup U \text{ weakly in } C([0, T]; L^2(\Omega)) \cap L^2([0, T]; H_0^1(\Omega)), \\ \rho^\varepsilon &\rightharpoonup \rho \text{ weakly in } C([0, T]; L^p(\Omega)), \quad \forall 2 \leq p < \infty, \end{aligned}$$

and

$$\begin{aligned} \|U\|_{C([0, T]; L^2(\Omega))}^2 + \|U\|_{L^2([0, T]; H_0^1(\Omega))}^2 &\leq C(T, \rho_0, U_0, \nu, \Omega), \\ \|\rho\|_{C([0, T]; L^p(\Omega))} &\leq \|\rho_0\|_{C([0, T]; L^p(\Omega))}, \quad \forall 2 \leq p < \infty. \end{aligned} \quad (27)$$

Since

$$U \cdot \nabla \psi \in C([0, T]; L^2(\Omega)),$$

we have

$$\begin{aligned}
& \left| \int_0^T \int_{\Omega} (\rho^\varepsilon U^\varepsilon \cdot \nabla \psi - \rho U \cdot \nabla \psi) \, d\mathbf{x} dt \right| \\
& \leq C \|\rho^\varepsilon\|_{L^2([0,T];L^2(\Omega))} \|U^\varepsilon - U\|_{L^2([0,T];L^2(\Omega))} + \left| \int_0^T \int_{\Omega} (\rho^\varepsilon U \cdot \nabla \psi - \rho U \cdot \nabla \psi) \, d\mathbf{x} dt \right| \\
& \leq C \|U^\varepsilon - U\|_{L^2([0,T];L^2(\Omega))} + \left| \int_0^T \int_{\Omega} (\rho^\varepsilon - \rho) U \cdot \nabla \psi \, d\mathbf{x} dt \right| \\
& \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0+.
\end{aligned}$$

Moreover, since

$$\begin{aligned}
& \left| \int_0^T \int_{\Omega} [U_\varepsilon \cdot (U^\varepsilon \cdot \nabla \Phi) - U \cdot (U \cdot \nabla \Phi)] \, d\mathbf{x} dt \right| \\
& = \left| \int_0^T \int_{\Omega} [U_\varepsilon \cdot (U^\varepsilon \cdot \nabla \Phi) - U_\varepsilon \cdot (U \cdot \nabla \Phi) + U_\varepsilon \cdot (U \cdot \nabla \Phi) - U \cdot (U \cdot \nabla \Phi)] \, d\mathbf{x} dt \right| \\
& \leq C \int_0^T \int_{\Omega} (|U_\varepsilon| |U^\varepsilon - U| + |U| |U_\varepsilon - U|) \, d\mathbf{x} dt \\
& \leq C (\|U_\varepsilon\|_{L^2([0,T];L^2(\Omega))} \|U_\varepsilon - U\|_{L^2([0,T];L^2(\Omega))} + \|U\|_{L^2([0,T];L^2(\Omega))} \|U_\varepsilon - U\|_{L^2([0,T];L^2(\Omega))}) \\
& \leq C \|U_\varepsilon - U\|_{L^2([0,T];L^2(\Omega))} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0+,
\end{aligned}$$

letting  $\varepsilon \rightarrow 0+$  in (26) we verified that  $(\rho, U)$  is a weak solution to (1)-(2) in  $\Omega \times [0, T]$ . We conclude the argument by noticing that  $T$  is arbitrary. This combining with (27) completes the proof of Lemma 2.3.

### 3 Global Regularity

In this section, we shall establish the regularity and uniqueness of the solution obtained in Lemma 2.3, and therefore give a proof of our main result, the Theorem 1.1. The following theorem gives the key estimates

**Theorem 3.1** *Under the assumption of Theorem 1.1, the solution obtained in Lemma 2.3 satisfies the following estimates:*

$$\|U\|_{C([0,T];H^3(\Omega))} + \|U\|_{L^2([0,T];H^4(\Omega))} + \|\rho\|_{C([0,T];H^3(\Omega))} \leq C,$$

for any  $T > 0$ . Moreover, there exists a constant  $\bar{C} > 0$  independent of  $t$  such that

$$\|U(\cdot, t)\|^2 \leq \max \left\{ \|U(\cdot, 0)\|^2, \frac{\bar{C}^2}{\nu^2} \|\rho(\cdot, 0)\|^2 \right\}, \quad \forall t \geq 0. \quad (28)$$

**Remark 3.1** *The constant  $\bar{C}$  in the Theorem is actually the constant of Poincaré's inequality on the domain  $\Omega$ . Therefore, it depends only on  $\Omega$ . See the proof of Lemma 3.9 below for details.*

The proof of Theorem 3.1 is based on several steps of careful energy estimates which are stated as a sequence of lemmas. First, we observe that the same method used to derive (11) can be applied to (1)<sub>2</sub> if  $V_\varepsilon$  is replaced by  $U$  in (9). Therefore, we have the conservation of  $L^p$  norm for  $\rho$ , i.e., for any  $p \in [2, \infty)$ , it holds that

$$\|\rho(\cdot, t)\|_{L^p} = \|\rho_0\|_{L^p}, \quad \forall t \geq 0.$$

Furthermore, by letting  $p \rightarrow \infty$  in the above estimate, one has

$$\|\rho(\cdot, t)\|_{L^\infty} = \|\rho_0\|_{L^\infty}, \quad \forall t \geq 0.$$

Fix any  $T > 0$ . In the rest part of this section, the time is restricted to be within the interval  $[0, T]$  until specified otherwise. Then we start with estimates of  $\|U\|_{C([0, T]; L^2(\Omega))}^2$  and  $\|\nabla U\|_{L^2([0, T]; L^2(\Omega))}^2$ .

**Lemma 3.1** *Under the assumptions of Theorem 1.1, it holds that*

$$\|U\|_{C([0, T]; L^2(\Omega))}^2 \leq C \quad \text{and} \quad \|\nabla U\|_{L^2([0, T]; L^2(\Omega))}^2 \leq C. \quad (29)$$

**Proof.** Taking  $L^2$  inner product of (1)<sub>1</sub> with  $U$ , we obtain, after integration by parts, that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|U\|^2 + \nu \|\nabla U\|^2 &= - \int_{\Omega} (U \cdot \nabla U) \cdot U \, d\mathbf{x} + \int_{\Omega} \rho \mathbf{e}_2 \cdot U \, d\mathbf{x} \\ &= - \frac{1}{2} \int_{\Omega} U \cdot \nabla (|U|^2) \, d\mathbf{x} + \int_{\Omega} \rho \mathbf{e}_2 \cdot U \, d\mathbf{x} \\ &= - \frac{1}{2} \int_{\Omega} \nabla \cdot (U |U|^2) \, d\mathbf{x} + \int_{\Omega} \rho \mathbf{e}_2 \cdot U \, d\mathbf{x} \\ &= \int_{\Omega} \rho \mathbf{e}_2 \cdot U \, d\mathbf{x}. \end{aligned}$$

Applying Cauchy-Schwarz inequality to the right hand side of the above equality, we get

$$\frac{1}{2} \frac{d}{dt} \|U\|^2 + \nu \|\nabla U\|^2 \leq \frac{1}{2} \|\rho\|^2 + \frac{1}{2} \|U\|^2. \quad (30)$$

By dropping  $\nu \|\nabla U\|^2$  from (30) and then applying Gronwall's inequality to the resulting inequality, we find that

$$\begin{aligned} \|U(\cdot, t)\|^2 &\leq e^t \left( \|U_0\|^2 + \int_0^t \|\rho_0\|^2 \, d\tau \right) \\ &\leq e^T (\|U_0\|^2 + T \|\rho_0\|^2) \leq C, \quad \forall t \in [0, T], \end{aligned}$$

which also implies, after integrating (30) over  $[0, T]$ , that

$$\nu \int_0^T \|\nabla U(\cdot, \tau)\|^2 d\tau \leq C.$$

This completes the proof of Lemma 3.1.

The next Lemma is dealing with  $\|\nabla U\|_{C([0,T];L^2(\Omega))}^2$  and  $\|U_t\|_{L^2([0,T];L^2(\Omega))}^2$ .

**Lemma 3.2** *Under the assumptions of Theorem 1.1, it holds that*

$$\|\nabla U\|_{C([0,T];L^2(\Omega))}^2 \leq C \quad \text{and} \quad \|U_t\|_{L^2([0,T];L^2(\Omega))}^2 \leq C.$$

**Proof.** Taking  $L^2$  inner product of (1)<sub>1</sub> with  $U_t$ , integrating the resulting equations over  $\Omega$  by parts, we get

$$\begin{aligned} \frac{\nu}{2} \frac{d}{dt} \|\nabla U\|^2 + \|U_t\|^2 &\leq \int_{\Omega} |U| |U_t| |\nabla U| \, d\mathbf{x} + \int_{\Omega} \rho v_t \, d\mathbf{x} \\ &\leq C \|U\|_{L^4}^2 \|\nabla U\|_{L^4}^2 + \frac{1}{4} \|U_t\|^2 + C \|\rho_0\|^2, \end{aligned} \quad (31)$$

where we have used Hölder's inequality and Cauchy-Schwarz inequality as follows:

$$\int_{\Omega} |U| |U_t| |\nabla U| \, d\mathbf{x} \leq C \|U\|_{L^4}^2 \|\nabla U\|_{L^4}^2 + \frac{1}{8} \|U_t\|^2,$$

and

$$\int_{\Omega} \rho v_t \, d\mathbf{x} \leq \frac{1}{8} \|U_t\|^2 + C \|\rho_0\|^2.$$

We now apply the Ladyzhenskaya's inequality to estimate  $\|U\|_{L^4}^2 \|\nabla U\|_{L^4}^2$ . Applying Lemma 2.2 (iii) on  $U$  and (iv) on  $\nabla U$ , we have

$$\begin{aligned} \|U\|_{L^4}^2 \|\nabla U\|_{L^4}^2 &\leq C(\|U\| \|\nabla U\|) (\|\nabla U\| \|\nabla^2 U\| + \|\nabla U\|^2) \\ &\leq C \|\nabla U\|^2 \|\nabla^2 U\| + C \|\nabla U\|^3 \\ &\leq C(\delta) \|\nabla U\|^4 + C \|\nabla U\|^3 + \delta \|U\|_{H^2}^2, \end{aligned} \quad (32)$$

where we have used Lemma 3.1 and  $\delta > 0$  is a small number to be determined. Therefore, we update (31) as

$$\frac{\nu}{2} \frac{d}{dt} \|\nabla U\|^2 + \frac{3}{4} \|U_t\|^2 \leq C + C(\delta) \|\nabla U\|^4 + C \|\nabla U\|^3 + \delta \|U\|_{H^2}^2. \quad (33)$$

We now rewrite the equation (1)<sub>1</sub> as

$$-\nu \Delta U + \nabla P = -U_t - U \cdot \nabla U + \rho \mathbf{e}_2.$$

Lemma 2.1 with  $m = 0$  and  $p = 2$  implies that

$$\begin{aligned}
\|U\|_{H^2}^2 &\leq C(\|U_t\|^2 + \|\rho\|^2 + \|U \cdot \nabla U\|^2) \\
&\leq C(\|U_t\|^2 + C) + C\|U\|_{L^4}^2 \|\nabla U\|_{L^4}^2 \\
&\leq \tilde{C}(C + \|U_t\|^2 + \|\nabla U\|^4 + \|\nabla U\|^3) + \frac{1}{2}\|U\|_{H^2}^2,
\end{aligned} \tag{34}$$

where we have used (32). Now, choosing  $\delta = 1/(4\tilde{C})$  and combining (33) and (34), we get

$$\frac{\nu}{2} \frac{d}{dt} \|\nabla U\|^2 + \frac{1}{2} \|U_t\|^2 \leq C(\|\nabla U\|^4 + \|\nabla U\|^3) + C.$$

Therefore, Young's inequality yields

$$\frac{\nu}{2} \frac{d}{dt} \|\nabla U\|^2 + \frac{1}{2} \|U_t\|^2 \leq C\|\nabla U\|^2 \|\nabla U\|^2 + C. \tag{35}$$

By dropping  $\frac{1}{2}\|U_t\|^2$  from (35) we obtain

$$\frac{\nu}{2} \frac{d}{dt} \|\nabla U\|^2 \leq C(\|\nabla U\|^2 \|\nabla U\|^2 + C). \tag{36}$$

Then using Lemma 3.1, Gronwall's inequality implies that

$$\|\nabla U(\cdot, t)\|^2 \leq C, \quad \forall t \in [0, T]. \tag{37}$$

Using (37), after integrating (35) over  $[0, T]$  we obtain

$$\int_0^T \|U_t(\cdot, \tau)\|^2 d\tau \leq C, \tag{38}$$

which completes the proof of Lemma 3.2.

Next, we estimate  $\|U_t\|_{C([0, T]; L^2(\Omega))}^2$  and  $\|\nabla U_t\|_{L^2([0, T]; L^2(\Omega))}^2$ .

**Lemma 3.3** *Under the assumptions of Theorem 1.1, it holds that*

$$\|U_t\|_{C([0, T]; L^2(\Omega))}^2 \leq C \quad \text{and} \quad \|\nabla U_t\|_{L^2([0, T]; L^2(\Omega))}^2 \leq C. \tag{39}$$

**Proof.** We take the temporal derivative of (1)<sub>1</sub> to get

$$U_{tt} + U_t \cdot \nabla U + U \cdot \nabla U_t + \nabla P_t = \nu \Delta U_t + \rho_t \vec{e}_2. \tag{40}$$

Taking  $L^2$  inner product of (40) with  $U_t$  we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|U_t\|^2 + \nu \|\nabla U_t\|^2 &= - \int_{\Omega} (U_t \cdot \nabla U) \cdot U_t d\mathbf{x} + \int_{\Omega} \rho_t v_t d\mathbf{x} \\
&= - \int_{\Omega} (U_t \cdot \nabla U) \cdot U_t d\mathbf{x} - \int_{\Omega} (U \cdot \nabla \rho) v_t d\mathbf{x} \\
&\leq \|U_t\|_{L^4}^2 \|\nabla U\| + \int_{\Omega} \rho (U \cdot \nabla v_t) d\mathbf{x}.
\end{aligned} \tag{41}$$

With the help of Lemma 3.1 and 3.2, and Lemma 2.2 (iii) on  $U_t$ , we note that

$$\begin{aligned} \|U_t\|_{L^4}^2 \|\nabla U\| &\leq C \|U_t\|_{L^4}^2 \\ &\leq C \|U_t\| \|\nabla U_t\| \\ &\leq \frac{\nu}{4} \|\nabla U_t\|^2 + C \|U_t\|^2. \end{aligned} \quad (42)$$

On the other hand, we have

$$\begin{aligned} \int_{\Omega} \rho(U \cdot \nabla v_t) d\mathbf{x} &\leq \|\rho\|_{L^\infty} \|U\| \|\nabla U_t\| \\ &\leq \frac{\nu}{4} \|\nabla U_t\|^2 + C. \end{aligned} \quad (43)$$

Therefore, combining (41)–(43), we arrive at

$$\frac{1}{2} \frac{d}{dt} \|U_t\|^2 + \frac{\nu}{2} \|\nabla U_t\|^2 \leq C(\|U_t\|^2 + 1). \quad (44)$$

Using Gronwall's inequality, and Lemma 3.2, we obtain (39). This completes the proof of Lemma 3.3.

As an immediate consequence of Lemma 3.3 and Lemma 2.2 (i), one has

**Lemma 3.4** *Under the assumptions of Theorem 1.1, it holds that*

$$\int_0^T \|U_t(\cdot, \tau)\|_{L^p}^2 d\tau \leq C, \quad \forall 1 \leq p < \infty. \quad (45)$$

This lemma will play an important role on the estimations of the maximum norms of  $U$  and  $\nabla U$  in the following Lemma.

**Lemma 3.5** *Under the assumptions of Theorem 1.1, it holds that*

$$\|U\|_{C([0,T];L^\infty(\Omega))}^2 \leq C \quad \text{and} \quad \|\nabla U\|_{L^2([0,T];L^\infty(\Omega))}^2 \leq C. \quad (46)$$

**Proof.** We see that  $\|U_t\|$  and  $\|\nabla U\|$  are bounded by Lemmas 3.2 and 3.3. Therefore, one reads from (34) that

$$\|U\|_{H^2}^2 \leq C(\|U_t\|^2 + \|\nabla U\|^3 + \|\nabla U\|^4 + C) \leq C, \quad (47)$$

which implies, by Sobolev embedding,

$$\|U(\cdot, t)\|_{L^\infty}^2 \leq C, \quad \forall t \in [0, T]. \quad (48)$$



As an immediate consequence of (47)-(48) we see that

$$\|U \cdot \nabla U\|_{H^1}^2 \leq C(\|U\|_{L^\infty}^2 + \|U\|_{H^2}^2)\|U\|_{H^2}^2 \leq C, \quad \forall t \in [0, T], \quad (49)$$

which implies by Lemma 2.2 (i) that

$$\|U \cdot \nabla U\|_{L^p}^2 \leq C, \quad \forall 1 \leq p < \infty, \quad \forall t \in [0, T]. \quad (50)$$

Therefore, using Lemma 2.1, (45) and (50) we obtain

$$\begin{aligned} \int_0^T \|U\|_{W^{2,p}}^2 d\tau &\leq C \int_0^T (\|U_t\|_{L^p}^2 + \|U \cdot \nabla U\|_{L^p}^2 + \|\rho\|_{L^p}^2) d\tau \\ &\leq C, \quad \forall 1 \leq p < \infty. \end{aligned} \quad (51)$$

Applying Lemma 2.2 (ii) to  $\nabla U$  we get the second half of (46) from (51) immediately. This completes the proof of Lemma 3.5.

In order to improve the regularity of  $U$ , the problem will involve the spatial derivatives of  $\rho$ . We now establish the following lemma to estimate  $\nabla \rho$ .

**Lemma 3.6** *Under the assumptions of Theorem 1.1, it holds that*

$$\|\nabla \rho(\cdot, t)\|_{L^\infty} \leq C, \quad \forall t \in [0, T]. \quad (52)$$

**Proof.** For any  $p \geq 2$ , taking  $\nabla$  of (1)<sub>2</sub>, dot multiplying the resulting equation with  $|\nabla \rho|^{p-2} \nabla \rho$ , after integration by parts we get

$$\frac{1}{p} \frac{d}{dt} \left( \|\nabla \rho\|_{L^p}^p \right) \leq \|\nabla U\|_{L^\infty} \|\nabla \rho\|_{L^p}^p, \quad (53)$$

which yields

$$\frac{d}{dt} \left( \|\nabla \rho\|_{L^p} \right) \leq \|\nabla U\|_{L^\infty} \|\nabla \rho\|_{L^p}. \quad (54)$$

Gronwall's inequality yields

$$\begin{aligned} \|\nabla \rho(\cdot, t)\|_{L^p} &\leq \|\nabla \rho_0\|_{L^p} \exp \left\{ \int_0^t \|\nabla U\|_{L^\infty} d\tau \right\} \\ &\leq C, \quad \forall p \geq 2, \text{ and } \forall t \in [0, T]. \end{aligned} \quad (55)$$

Letting  $p \rightarrow \infty$  we obtain (52). This completes the proof of Lemma 3.6.

The estimates of  $\|\nabla U_t\|_{C([0,T];L^2(\Omega))}^2$  and  $\|U_{tt}\|_{L^2([0,T];L^2(\Omega))}^2$  will be given in the next lemma, based on which we will establish the desired regularity stated in Theorem 3.1.

**Lemma 3.7** *Under the assumptions of Theorem 1.1, it holds that*

$$\|\nabla U_t\|_{C([0,T];L^2(\Omega))}^2 \leq C, \quad \text{and} \quad \|U_{tt}\|_{L^2([0,T];L^2(\Omega))}^2 \leq C. \quad (56)$$

**Proof.** Taking  $L^2$  inner product of (40) with  $U_{tt}$  we get

$$\frac{\nu}{2} \frac{d}{dt} \|\nabla U_t\|^2 + \|U_{tt}\|^2 \leq \int_{\Omega} (|U_{tt}| |U_t| |\nabla U| + |U_{tt}| |U| |\nabla U_t| + \rho_t v_{tt}) \, d\mathbf{x}. \quad (57)$$

We now estimate the right hand side term by term. First of all, we apply the Hölder inequality and Lemma 3.3 to obtain

$$\begin{aligned} \int_{\Omega} |U_{tt}| |U_t| |\nabla U| \, d\mathbf{x} &\leq \frac{1}{6} \|U_{tt}\|^2 + C \|\nabla U\|_{L^\infty}^2 \|U_t\|^2 \\ &\leq \frac{1}{6} \|U_{tt}\|^2 + C \|\nabla U\|_{L^\infty}^2, \quad \forall t \in [0, T]. \end{aligned} \quad (58)$$

Similarly, using Hölder inequality and Lemmas 3.5 and 3.6, we have the following estimates

$$\begin{aligned} \int_{\Omega} |U_{tt}| |U| |\nabla U_t| \, d\mathbf{x} &\leq \frac{1}{6} \|U_{tt}\|^2 + C \|U\|_{L^\infty}^2 \|\nabla U_t\|^2 \\ &\leq \frac{1}{6} \|U_{tt}\|^2 + C \|\nabla U_t\|^2, \quad \forall t \in [0, T], \end{aligned} \quad (59)$$

and

$$\begin{aligned} \int_{\Omega} |\rho_t v_{tt}| \, d\mathbf{x} &\leq \frac{1}{6} \|U_{tt}\|^2 + C \|\rho_t\|^2 \\ &\leq \frac{1}{6} \|U_{tt}\|^2 + C \|U \cdot \nabla \rho\|^2 \\ &\leq \frac{1}{6} \|U_{tt}\|^2 + C \|\nabla \rho\|_{L^\infty}^2 \|U\|^2 \\ &\leq \frac{1}{6} \|U_{tt}\|^2 + C, \quad \forall t \in [0, T]. \end{aligned} \quad (60)$$

Substituting (58)–(60) into (57), one has

$$\frac{\nu}{2} \frac{d}{dt} \|\nabla U_t\|^2 + \frac{1}{2} \|U_{tt}\|^2 \leq C + C \|\nabla U\|_{L^\infty}^2 + C \|\nabla U_t\|^2. \quad (61)$$

We note that all the terms on the right hand side of (61) are integrable in time due to Lemmas 3.3 and 3.5. Therefore, we integrate (61) in time over  $[0, T]$  to obtain the estimates in (56). This completes the proof of Lemma 3.7.

We are now ready to complete the regularity stated in Theorem 3.1.

**Lemma 3.8** *Under the assumptions of Theorem 1.1, it holds that*

$$\|(\rho, U)\|_{C([0,T];H^3(\Omega))}^2 \leq C, \quad \text{and} \quad \|U\|_{L^2([0,T];H^4(\Omega))}^2 \leq C. \quad (62)$$

**Proof.** Based on (49), (55) and (56), we see from Lemma 2.1 that,

$$\|U(\cdot, t)\|_{H^3}^2 \leq C(\|\rho\|_{H^1}^2 + \|U \cdot \nabla U\|_{H^1}^2 + \|U_t\|_{H^1}^2) \leq C, \quad \forall t \in [0, T], \quad (63)$$

which implies by Sobolev inequality (c.f. Lemma 2.2) that

$$\|U(\cdot, t)\|_{W^{2,p}}^2 \leq C\|U(\cdot, t)\|_{H^3}^2 \leq C, \quad \forall t \in [0, T], \quad \forall 1 \leq p < \infty, \quad (64)$$

and thus

$$\|\nabla U(\cdot, t)\|_{L^\infty} \leq C, \quad \forall t \in [0, T]. \quad (65)$$

Furthermore, for  $t \in [0, T]$ , it is easy to see that

$$\begin{aligned} \|U_t \cdot \nabla U\|^2 &\leq \|U_t\|^2 \|\nabla U\|_{L^\infty}^2 \leq C, \\ \|U \cdot \nabla U_t\|^2 &\leq \|U\|_{L^\infty}^2 \|\nabla U_t\|^2 \leq C, \\ \|\rho_t\|^2 &= \|U \cdot \nabla \rho\|^2 \leq \|U\|_{L^\infty}^2 \|\nabla \rho\|^2 \leq C. \end{aligned} \quad (66)$$

From (40) and Lemma 2.1, we know

$$\int_0^T \|U_t\|_{H^2}^2 d\tau \leq C \int_0^T (\|U_{tt}\|^2 + \|U_t \cdot \nabla U\|^2 + \|U \cdot \nabla U_t\|^2 + \|\rho_t\|^2) d\tau, \quad (67)$$

which, together with (56) and (66), gives

$$\int_0^T \|U_t(\cdot, \tau)\|_{H^2}^2 d\tau \leq C. \quad (68)$$

In addition, Sobolev inequality and (63) yield

$$\begin{aligned} \|U \cdot \nabla U\|_{H^2}^2 &\leq C(\|U\|_{L^\infty}^2 \|U\|_{H^3}^2 + \|\nabla U\|_{L^\infty}^2 \|U\|_{H^2}^2) \\ &\leq C\|U\|_{H^2}^2 \|U\|_{H^3}^2 \leq C, \quad \forall t \in [0, T]. \end{aligned} \quad (69)$$

Now, it is clear that one needs higher order estimate on  $\rho$  to complete the proof of this lemma. For this purpose, taking  $\partial_{xx}$  of (1)<sub>2</sub>, we get

$$\rho_{xxt} + u_{xx}\rho_x + 2u_x\rho_{xx} + v_{xx}\rho_y + 2v_x\rho_{xy} + U \cdot \nabla \rho_{xx} = 0. \quad (70)$$

For any  $p \geq 2$ , multiplying (70) by  $|\rho_{xx}|^{p-2}\rho_{xx}$ , integrating over  $\Omega$ , and using Hölder's inequality, we obtain

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_{\Omega} |\rho_{xx}|^p d\mathbf{x} &= - \int_{\Omega} (u_{xx}\rho_x + v_{xx}\rho_y + 2u_x\rho_{xx} + 2v_x\rho_{xy}) |\rho_{xx}|^{p-2} \rho_{xx} d\mathbf{x} \\ &\leq \|\nabla \rho\|_{L^\infty} \|\nabla^2 U\|_{L^p} \|\nabla^2 \rho\|_{L^p}^{p-1} + 2\|\nabla U\|_{L^\infty} \|\nabla^2 \rho\|_{L^p}^p \\ &\leq C(\|\nabla^2 \rho\|_{L^p}^{p-1} + \|\nabla^2 \rho\|_{L^p}^p), \end{aligned} \quad (71)$$

where we have used (52), (64) and (65). Similarly, one can show

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} |\rho_{xy}|^p dx \leq C (\|\nabla^2 \rho\|_{L^p}^{p-1} + \|\nabla^2 \rho\|_{L^p}^p), \quad (72)$$

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} |\rho_{yy}|^p dx \leq C (\|\nabla^2 \rho\|_{L^p}^{p-1} + \|\nabla^2 \rho\|_{L^p}^p). \quad (73)$$

Summing (71)–(73) together, we obtain

$$\frac{1}{p} \frac{d}{dt} (\|\nabla^2 \rho\|_{L^p}^p) \leq C (\|\nabla^2 \rho\|_{L^p}^{p-1} + \|\nabla^2 \rho\|_{L^p}^p). \quad (74)$$

It follows that

$$\frac{d}{dt} (\|\nabla^2 \rho\|_{L^p}) \leq C (1 + \|\nabla^2 \rho\|_{L^p}), \quad (75)$$

Applying Gronwall inequality to (75), one has

$$\|\nabla^2 \rho(\cdot, t)\|_{L^p} \leq C, \quad \forall 2 \leq p < \infty, \quad \forall t \in [0, T]. \quad (76)$$

In a quite similar manner as in the derivation of (76), further estimates show that

$$\begin{aligned} \frac{d}{dt} \|\nabla^3 \rho\|^2 &\leq C \left( \|\nabla U\|_{L^\infty} \|\nabla^3 \rho\|^2 + \|\nabla \rho\|_{L^\infty} \|\nabla^3 U\| \|\nabla^3 \rho\| + \|\nabla^2 U\|_{L^4} \|\nabla^2 \rho\|_{L^4} \|\nabla^3 \rho\| \right) \\ &\leq C (\|\nabla^3 \rho\|^2 + \|\nabla^3 \rho\|) \\ &\leq C (\|\nabla^3 \rho\|^2 + 1), \end{aligned} \quad (77)$$

which implies

$$\|\rho(\cdot, t)\|_{H^3}^2 \leq C, \quad \forall t \in [0, T]. \quad (78)$$

Now, by Lemma 2.1, combining (68), (69) and (78), one has

$$\int_0^T \|U(\cdot, \tau)\|_{H^4}^2 d\tau \leq C \int_0^T (\|U_t\|_{H^2}^2 + \|U \cdot \nabla U\|_{H^2}^2 + \|\rho\|_{H^2}^2) d\tau \leq C,$$

which completes the proof of Lemma 3.8.

For the proof of Theorem 3.1, it remains to prove the uniform bound of the kinetic energy (28).

**Lemma 3.9** *Under the assumptions of Theorem 1.1, there is a uniform constant  $\bar{C}$  independent of  $t$ , such that*

$$\|U(\cdot, t)\|^2 \leq \max \left\{ \|U(\cdot, 0)\|^2, \frac{\bar{C}^2}{\nu^2} \|\rho(\cdot, 0)\|^2 \right\}, \quad \forall t \geq 0. \quad (79)$$

**Proof.** From the proof of Lemma 3.1, we observe that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|U\|^2 + \nu \|\nabla U\|^2 &= \int_{\Omega} \rho \mathbf{e}_2 \cdot U \, d\mathbf{x} \\ &\leq \frac{1}{2\delta\nu} \|\rho\|^2 + \delta \frac{\nu}{2} \|U\|^2, \end{aligned} \quad (80)$$

for any positive  $\delta$ . Poincaré's inequality says that there is a constant  $\bar{C} = \bar{C}(\Omega)$  such that

$$\|U\| \leq \bar{C} \|\nabla U\|.$$

Choosing  $\delta = \frac{1}{\bar{C}}$ , we know from (80) that

$$\frac{d}{dt} \|U\|^2 + \frac{\nu}{\bar{C}} \|U\|^2 \leq \frac{\bar{C}}{\nu} \|\rho\|^2. \quad (81)$$

Solving the above differential inequality we get

$$\exp \left\{ \frac{\nu}{\bar{C}} t \right\} \|U(\cdot, t)\|^2 - \|U(\cdot, 0)\|^2 \leq \frac{\bar{C}^2}{\nu^2} \|\rho_0\|^2 (\exp \left\{ \frac{\nu}{\bar{C}} t \right\} - 1), \quad (82)$$

which implies

$$\|U(\cdot, t)\|^2 \leq \exp \left\{ -\frac{\nu}{\bar{C}} t \right\} \left( \|U(\cdot, 0)\|^2 - \frac{\bar{C}^2}{\nu^2} \|\rho_0\|^2 \right) + \frac{\bar{C}^2}{\nu^2} \|\rho_0\|^2, \quad \forall t > 0. \quad (83)$$

Therefore, (79) follows immediately from (83). This completes the proof of Lemma 3.9.

Lemmas 3.8-3.9 conclude Theorem 3.1. With the global regularity established in Lemmas 3.1–3.8, we are able to prove the uniqueness of the solution.

**Theorem 3.2** *Under the assumptions of Theorem 1.1, the solution of (1)–(2) is unique.*

**Proof.** Suppose there are two solutions  $(\rho_1, U_1, P_1)$  and  $(\rho_2, U_2, P_2)$  to (1)–(2). Setting  $\tilde{\rho} = \rho_1 - \rho_2$ ,  $\tilde{U} = U_1 - U_2$ , and  $\tilde{P} = P_1 - P_2$ , then  $(\tilde{\rho}, \tilde{U}, \tilde{P})$  satisfy

$$\begin{cases} \tilde{U}_t + U_1 \cdot \nabla \tilde{U} + \tilde{U} \cdot \nabla U_2 + \nabla \tilde{P} = \nu \Delta \tilde{U} + \tilde{\rho} \mathbf{e}_2 \\ \tilde{\rho}_t + U_1 \cdot \nabla \tilde{\rho} + \tilde{U} \cdot \nabla \rho_2 = 0, \\ \nabla \cdot \tilde{U} = 0, \\ \tilde{U}|_{\partial\Omega} = 0, \\ \tilde{U}(\mathbf{x}, 0) = 0, \tilde{\rho}(\mathbf{x}, 0) = 0, \mathbf{x} \in \Omega. \end{cases} \quad (84)$$

Since  $\nabla \cdot U_1 = 0$  and  $U_1|_{\partial\Omega} = 0$ , taking the  $L^2$  inner products of (84)<sub>1</sub> with  $\tilde{U}$  and (84)<sub>2</sub> with  $\tilde{\rho}$ , one has

$$\frac{1}{2} \frac{d}{dt} (\|\tilde{\rho}\|^2 + \|\tilde{U}\|^2) + \nu \|\nabla \tilde{U}\|^2 = - \int_{\Omega} \tilde{\rho} (\tilde{U} \cdot \nabla \rho_2) \, d\mathbf{x} - \int_{\Omega} \tilde{U} \cdot (\tilde{U} \cdot \nabla U_2) \, d\mathbf{x} + \int_{\Omega} \tilde{\rho} \tilde{v} \, d\mathbf{x},$$

where  $\tilde{v}$  is the second component of  $\tilde{U}$ . Using the estimates for  $\rho_2$  and  $U_2$ , standard calculations give that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \|\tilde{\rho}\|^2 + \|\tilde{U}\|^2 \right) + \nu \|\nabla \tilde{U}\|^2 \\ & \leq \|\nabla \rho_2\|_{L^\infty} (\|\tilde{\rho}\|^2 + \|\tilde{U}\|^2) + \|\nabla U_2\|_{L^\infty} \|\tilde{U}\|^2 + (\|\tilde{\rho}\|^2 + \|\tilde{U}\|^2) \\ & \leq C(\|\tilde{\rho}\|^2 + \|\tilde{U}\|^2), \end{aligned}$$

which implies that

$$e^{-2Ct} (\|\tilde{\rho}\|^2 + \|\tilde{U}\|^2) \leq \|\tilde{\rho}(0)\|^2 + \|\tilde{U}(0)\|^2 = 0,$$

for any  $t \geq 0$ . So the solution of (1)–(2) is unique. This completes the proof of Theorem 3.2.

This theorem and Theorem 3.1 implies our main result, Theorem 1.1.

## 4 Remarks

We have the following remarks in order.

1. It is interesting to study the 2D Boussinesq equations over bounded domains with non-smooth boundary, e.g., any polygonal domain. In that case, we have to introduce a weak solution. Similar to Navier-Stokes equations, one could use several formulations, e.g., velocity and pressure formulation, vorticity and stream function formulation or stream function formulation. In particular, the regularity of the solutions is an interesting problem when the domain is a polygon. We leave the study in a future paper.
2. It is also interesting to study the 2D inviscid Boussinesq equations with density diffusion in bounded domains with smooth boundary. Due to the diffusion and boundary effect, the potential energy associated with the density is expected to converge exponentially to a constant, which is either the value of the density on the boundary of the domain in the case of the Dirichlet boundary condition, or the average of the density over the domain in the case of the Neumann boundary condition due to the conservation of total mass. The investigation will be carried out in a forthcoming paper.

## References

- [1] R.A. Adams, *Sobolev Spaces*, Academic, New York, 1975.
- [2] S.N. Antontsev, A.V. Kazhikhov and V.N. Monakhov, *Boundary Value Problems in Mechanics of Nonhomogeneous Fluids*, North-Holland, Amsterdam, 1990.
- [3] J. R. Cannon, and E. DiBenedetto, The initial value problem for the Boussinesq equations with data in  $L^p$ . Approximation methods for Navier-Stokes problems (Proc. Sympos., Univ. Paderborn, Paderborn, 1979), Lecture Notes in Math., 771, Springer, Berlin, 1980, pp. 129–144,
- [4] D. Chae, Global regularity for the 2D Boussinesq equations with partial viscosity terms. *Adv. Math.* 203 (2006), no. 2, 497–513.
- [5] D. Chae and H.-S. Nam, Local existence and blow-up criterion for the Boussinesq equations. *Proc. Roy. Soc. Edinburgh Sect. A* 127 (1997), no. 5, 935–946.
- [6] D. Chae, S.-K. Kim, and H.S. Nam, Local existence and blow-up criterion of Hölder continuous solutions of the Boussinesq equations. *Nagoya Math. J.* 155 (1999), 55–80.
- [7] D. Chae and O. Y. Imanuvilov, Generic solvability of the axisymmetric 3-D Euler equations and the 2-D Boussinesq equations. *J. Differential Equations* 156 (1999), no. 1, 1–17.
- [8] D. Córdoba, C. Fefferman and R. De La LLave, On squirt singularities in hydrodynamics. *SIAM J. Math. Anal.* **36** (2004), no. 1, 204-213.
- [9] W. E and C. -W. Shu, Small-scale structures in Boussinesq convection, *Phys. Fluids* 6 (1994), no. 1, 49–58.
- [10] T. Hou and C. Li, Global well-posedness of the viscous Boussinesq equations, *Discr. Cont. Dynam. Sys.*, 12(2005), 1–12.
- [11] L. Hu and H. Jian, Blow-up criterion for 2-D Boussinesq equations in bounded domain, *Front. Math. China* 2 (2007), no. 4, 559–581.
- [12] T. Hmidi and S. Keraani, On the global well-posedness of the two-dimensional Boussinesq system with a zero diffusivity. *Adv. Differential Equations* 12 (2007), no. 4, 461–480.

- [13] P. L. Lions, *Mathematical Topics in Fluid Mechanics, Vol. I-II*, Oxford University Press, New York, 1996-1998.
- [14] S. A. Lorca and J. L. Boldrini, The initial value problem for a generalized Boussinesq model. *Nonlinear Analysis* **36** (1999), 457-480.
- [15] O.A. Ladyzhenskaya, A., V. A. Solonnikov, and N. N. Uraltseva, *Linear and Quasi-linear Equations of Parabolic Type*, American Math. Society, 1968.
- [16] A. Majda, *Introduction to PDEs and Waves for the Atmosphere and Ocean*, Courant Lecture Notes in Mathematics, no. 9, AMS/CIMS, 2003.
- [17] A. Majda and A. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge University Press, Cambridge, UK, 2002.
- [18] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer-Verlag, New York, 1987.
- [19] Y. Taniuchi, A note on the blow-up criterion for the inviscid 2-D Boussinesq equations, in: R. Salvi (Ed.), *The Navier-Stokes Equations: Theory and Numerical Methods*, Lecture Notes in Pure and Applied Mathematics, Vol. 223, 2002, pp. 131-140.
- [20] R. Temam, *Navier-Stokes Equations*, North Holland, 1977.