Zero dissipation limit with two interacting shocks of the 1D non–isentropic Navier–Stokes equations

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Abstract

We investigate the zero dissipation limit problem of the one dimensional compressible non–isentropic Navier–Stokes equations with Riemann initial data in the case of the composite wave of two shock waves. It is shown that the unique solution of the Navier–Stokes equations exists for all time, and converge to the Riemann solution of the corresponding Euler equations with the same Riemann initial data uniformly on the set away from the shocks, as both the viscosity and heat-conductivity tend to zero. In contrast to previous related works, where either shock waves are absent or the effects of initial layers are ignored, this gives the first mathematical justification of this limit for the compressible non–isentropic Navier–Stokes equations in the presence of both shocks and initial layers. Our method of proof consists of a scaling argument, the construction of the approximate solution and delicate energy estimates.

Keywords: Zero dissipation limit, compressible Navier–Stokes equations, shock waves, initial layers.

1. Introduction

The asymptotic behavior of viscous flows in vanishing dissipation limit process is one of the important, longstanding problems in the theory of compressible fluid flow. It is expected that the solution of viscous flows should converge strongly, when dissipation vanishes, to the solution of the corresponding inviscid flow. When the solution of the inviscid flow is smooth,

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this problem can be solved by classical Hilbert expansion along with energy method. However, the inviscid compressible flow usually contains discontinuities, such as shock waves and contact discontinuities, which have so far prevented solving the problem in the general setting by means of known analytic tools. Essential new ideas and methods are needed to tackle this open problem. Therefore, any attempt on this problem that involves the singularity in the inviscid solution can be viewed as progress to this general program.

In one space dimension, interesting progress has been made on system of hyperbolic conservation laws with artificial viscosity

$$u_t + f(u)_x = \epsilon u_{xx},$$

Using a matched asymptotic expansion method, Goodman and Xin [8] proved that, given any piecewise smooth entropy solution with finitely many non-interacting shock waves of the inviscid conservation laws, the above viscous problem admits a sequence of smooth solutions converging to the given invisid solutions in vanishing viscosity limit. Later, Yu [36] improved the results of [8] to allow initial layers by a detailed pointwise analysis. Recently, Zeng in [38] justifies the limit for the superposition of shock waves with contact discontinuities. In the context of small BV initial data, the seminal result of Bianchini and Bressan [1] proved the the vanishing visicosity limit of solutions of this viscous hyperbolic system by deriving the uniform BV estimates of solutions independent of the viscosity. This fully settled the problem in small BV case when viscosity matrix is ϵI. However, the problem is still unsolved for physical systems such as the Navier–Stokes equations.

In this paper, we study the existence and asymptotic behavior, as the viscosity and heat–conductivity coefficients go to zero, of solutions to the one dimensional compressible non–isentropic Navier–Stokes equations in Lagrangian coordinates

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \epsilon(\frac{u_x}{v})_x, \\ E_t + (pu)_x = \kappa(\frac{\theta_x}{v})_x + \epsilon(\frac{uu_x}{v})_x, \end{cases} \tag{1.1}$$

where the functions $v(x,t) > 0$, $u(x,t)$, $\theta(x,t) > 0$, $p = p(v,\theta)$ and $e = e(v,\theta)$ represent the specific volume, velocity, absolute temperature, pressure and internal energy respectively. $E = e + \frac{u^2}{2}$ is the total energy, while $\epsilon > 0$ and $\kappa > 0$ denote the viscosity constant, and the coefficient of the heat conduction respectively. Here we consider the perfect ideal gas, that is,

$$p = p(v,\theta) = \frac{R\theta}{v}, \qquad e = \frac{R\theta}{\gamma - 1}, \tag{1.2}$$

where $R > 0$ is the gas constant and $\gamma > 1$ is the adiabatic exponent. Using the specific entropy s (see [5]), one can write

$$p = p(v, s) = Av^{-\gamma} e^{\frac{\gamma-1}{R}s}, \tag{1.3}$$

for some positive constant A.
As in [19] and [32], we assume that the viscosity ϵ and heat–conductivity κ satisfy

$$\begin{cases} \kappa = O(\epsilon), & \text{as} \quad \epsilon \to 0; \\ \mu \doteq \kappa/\epsilon \geq c > 0, & \text{for some positive constant } c, \text{ as } \epsilon \to 0. \end{cases} \tag{1.4}$$

The assumption (1.4) stems from physical consideration. Indeed, when one derives the compressible Navier–Stokes system (1.1) as the first order approximation of Bolzmann equation through Chapman-Enskog expansion, (1.4) is verified.

We consider the Cauchy problem for (1.1) with Riemann initial data

$$(v, u, E)(x, 0) = \begin{cases} (v_-, u_-, E_-), & x < 0, \\ (v_+, u_+, E_+), & x > 0, \end{cases} \tag{1.5}$$

where $v_\pm > 0$, $u_\pm \in R$ and $E_\pm = \frac{R\theta_\pm}{\gamma-1} + \frac{u_\pm^2}{2}$ ($\theta_\pm > 0$) are given constants. We are especially interested in the relation between the Navier–Stokes solutions, $U^\epsilon(x,t)$ of (1.1) and (1.5), and the solutions, $U^0(x,t)$, of the corresponding Euler equations

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ E_t + (pu)_x = 0 \end{cases} \tag{1.6}$$

with the same Riemann initial data (1.5).

Although the zero dissipation limit for compressible Naiver-Stokes equation remains as an important open problem, many interesting results were achieved in the past. These results roughly fall into three categories. The first is to use theory of compensated compactness to establish the compactness of the sequence of solutions of the Navier-Stokes, and then to extract a subsequence to converge to a limit, which was later justified as a weak solution of corresponding compressible Euler equations. The representative results are obtained by DiPerna [7], and by Chen and Pereperitsa [4]. These results are valid for the 1D isentropic Navier–Stokes equations

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \epsilon(\frac{u_x}{v})_x \end{cases} \tag{1.7}$$

to the corresponding isentropic Euler equations

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = 0, \end{cases} \tag{1.8}$$

in the vanishing viscosity limit. However, the framework of compensated compactness is basicly limited to 2×2 systems so far, and the abstract analysis yields little information on the qualitative nature of the viscous solutions. The second kind of results utilize recent development of nonlinear stability analysis results on elementary waves for compressible Navier-Stokes equations. Motivated by early work of Xin [33] for rarefaction waves, and Goodman and Xin [8] for solution with shock waves, exciting advancement has been made in this direction. It was shown that, given a solution of the compressible Euler equations (1.6) which is piecewise smooth and contains simple wave patterns, there exists a sequence of solutions of the compressible Navier-Stokes equations that converge to the pre-fixed Euler solution in zero dissipation limit. The advantage of this approach is that it can be generalized to general system, and the explicit construction of the visous solutions gives detailed structure of solutions along with explicit convergence rate. The possible disadvantage of this approach is that this kind of results are often valid only for finite time when shock presents, and the convergence is good only for the preferred (or constructed) sequence of viscous solutions. We refer the readers to [2-3, 6, 11-13, 16-28, 30-3 37] for a partial list of results in this direction. Last but not least, Hoff and Liu [9] proposed a framework to study directly the compressible Navier-Stokes equations with Riemann data, establish sharp and uniform estimates, analyze the detailed behavior of the solutions in initial, intermediate, and large time regimes, and finally prove the zero dissipation limit to the Riemann solutions of compressible Euler equation. Comparing to the second category, this program is different in at least four aspects. First, rather than the preferred sequence with approxiamte initial data, this program shows uniform convergence of Navier-Stokes system with fixed same data as Euler. Second, the stability analysis component in this program has large initial perturbation. Third, this program takes care of both shock waves and initial layers. Finally, the convergence result of this program is globally, not only for a finite time. So far, except for Hoff and Liu [9] where the isentropic Naver-Stokes (1.7) with a signle shock wave initial data was solved, no much development appeared in the past two decades. In an early paper, we generalized this result to isentropic flow with two compsite shocks. In this paper, we will further extend this framework to the full Navier-Stokes system (1.1) with Riemann initial data (1.3) in the case of the composite two shock waves. In particular, we prove that the solution of the full compressible Navier–Stokes system exist for all time, and converge to the Riemann solution of the Euler equations with the same Riemann initial data that is a composite wave of two shock waves, as the viscosity and heat–conductivity tend to zero. This gives the first mathematical justification of this limit for the compressible non–isentropic Navier–Stokes equations in the presence of both shock waves and initial layers.

We now introduce some preliminary notations and give some background materials before stating the main theorem. It is known that the Euler system (1.6) has three eigenvalues: $\lambda_1 = -\sqrt{\gamma p/v} < 0$, $\lambda_2 = 0$, $\lambda_3 = \sqrt{\gamma p/v} > 0$, where the second characteristic field is linearly degenerate and other two are genuinely nonlinear. In the present paper, we focus our attention on the situation where the Riemann solution of (1.6) and (1.5) is a composite wave of two shock waves (and three constants states):

$$U^0(x,t) = (v^0, u^0, E^0)(x,t) = \begin{cases} (v_-, u_-, E_-), & x < s_1 t, \\ (v_m, u_m, E_m), & s_1 t < x < s_3 t, \\ (v_+, u_+, E_+), & x > s_3 t. \end{cases} \tag{1.9}$$

Here, (v_m, u_m, E_m) is the intermediate state and the shock speeds s_1 and s_3 are constants determined by the Rankine-Hugoniot condition and satisfy entropy conditions

$$\lambda_1(v_-, u_-, E_-) > s_1 > \lambda_1(v_m, u_m, E_m), \quad \lambda_3(v_m, u_m, E_m) > s_3 > \lambda_3(v_+, u_+, E_+). \tag{1.10}$$

To describe the wave strengths for later use, we set

$$\begin{aligned} &\delta_1 = |v_m - v_-| + |u_m - u_-| + |E_m - E_-|, \ \delta_3 = |v_m - v_+| + |u_m - u_+| + |E_m - E_+|; \\ &\delta = |v_+ - v_-| + |u_+ - u_-| + |E_+ - E_-|, \ \bar{\delta} = \min\{\delta_1, \delta_3\}. \end{aligned} \tag{1.11}$$

When δ is chosen small in our situation for the fixed (v_-, u_-, E_-), it holds

$$\delta_1 + \delta_3 \le C\delta, \tag{1.12}$$

where C is a positive constant depending only on (v_-, u_-, E_-). Then, if it holds

$$\delta_1 + \delta_3 \le C\bar{\delta}, \quad \text{as} \quad \delta_1 + \delta_3 \to 0, \tag{1.13}$$

for some positive constant C, we call the strengths of the shock waves "small with same order". In what follows, we always assume (1.13).

Next, we recall the definitions of viscous shock waves of (1.1) which correspond to the above shock waves. The 1-viscous shock wave associate to the 1-shock wave is a traveling wave solution of (1.1) with the formula $\bar{U}_1^\epsilon(x-s_1t)=(V_1^\epsilon,U_1^\epsilon,E_1^\epsilon)(x-s_1t)$ which is determined by

$$\begin{cases} -s_1(V_1^\epsilon)'-(U_1^\epsilon)'=0, \\ -s_1(U_1^\epsilon)'+p(V_1^\epsilon,\Theta_1^\epsilon)'=\epsilon[\frac{(U_1^\epsilon)'}{V_1^\epsilon}]', \\ -s_1(E_1^\epsilon)'+[p(V_1^\epsilon,\Theta_1^\epsilon)U_1^\epsilon]'=[\kappa\frac{(\Theta_1^\epsilon)'}{V_1^\epsilon}+\epsilon\frac{U_1^\epsilon(U_1^\epsilon)'}{V_1^\epsilon}]', \\ (V_1^\epsilon,U_1^\epsilon,E_1^\epsilon)(-\infty)=(v_-,u_-,E_-), \\ (V_1^\epsilon,U_1^\epsilon,E_1^\epsilon)(+\infty)=(v_m,u_m,E_m), \end{cases} \tag{1.14}$$

where $'=\frac{d}{d\xi}$, $\xi=x-s_1t$.

Similarly, the 3-viscous shock wave $\bar{U}_3^\epsilon(x-s_3t)=(V_3^\epsilon,U_3^\epsilon,E_3^\epsilon)(x-s_3t)$ is defined by

$$\begin{cases} -s_3(V_3^\epsilon)'-(U_3^\epsilon)'=0, \\ -s_3(U_3^\epsilon)'+p(V_3^\epsilon,\Theta_3^\epsilon)'=\epsilon[\frac{(U_3^\epsilon)'}{V_3^\epsilon}]', \\ -s_3(E_3^\epsilon)'+[p(V_3^\epsilon,\Theta_3^\epsilon)U_3^\epsilon]'=[\kappa\frac{(\Theta_3^\epsilon)'}{V_3^\epsilon}+\epsilon\frac{U_3^\epsilon(U_3^\epsilon)'}{V_3^\epsilon}]', \\ (V_3^\epsilon,U_3^\epsilon,E_3^\epsilon)(-\infty)=(v_m,u_m,E_m), \\ (V_3^\epsilon,U_3^\epsilon,E_3^\epsilon)(+\infty)=(v_+,u_+,E_+), \end{cases} \tag{1.15}$$

where $'=\frac{d}{d\xi}$, $\xi=x-s_3t$.

Let $E_m=\frac{R\theta_m}{\gamma-1}+\frac{u_m^2}{2}$ and $m^\epsilon(x,t)=(v^\epsilon,u^\epsilon,E^\epsilon)$, we define $\bar{m}^\epsilon(x,t)=(\bar{v}^\epsilon,\bar{u}^\epsilon,\bar{E}^\epsilon)$, by

$$\begin{cases} \bar{v}^\epsilon=V_1^\epsilon(x-s_1t)+V_3^\epsilon(x-s_3t)-v_m, \\ \bar{u}^\epsilon=U_1^\epsilon(x-s_1t)+U_3^\epsilon(x-s_3t)-u_m, \\ \bar{E}^\epsilon=E_1^\epsilon(x-s_1t)+E_3^\epsilon(x-s_3t)-E_m. \end{cases} \tag{1.16}$$

One of the main difficulties arises here, comparing with Hoff and Liu [9] of isentropic case where initial excess mass is zero up to a spatial shift in shock profile, the Riemann initial data (1.5) is not a perturbation with zero excess mass over viscous shock waves any more. Indeed, the integral $\int_{-\infty}^{\infty}(m^\epsilon-\bar{m}^\epsilon)(x,0)dx$ is in general not zero in the direction of second characteristic field, which cannot be handled through location shift of shock waves. To use the anti–derivative technique, we need to find a refined ansatz $\tilde{m}^\epsilon$ such that $\int_{-\infty}^{\infty}(m^\epsilon-\tilde{m}^\epsilon)(x,0)dx=0$ in addition to the requirement $|\tilde{m}^\epsilon-\bar{m}^\epsilon|\to 0$ as $t\to\infty$. Motivated by the stability result in [12], we introduce a linear diffusion wave to carry the initial excess mass and construct the new ansatz

$\tilde{m}^\epsilon = (\tilde{v}^\epsilon, \tilde{u}^\epsilon, \tilde{E}^\epsilon)$ by

$$\begin{aligned}\tilde{v}^\epsilon &= V_1^\epsilon(x - s_1 t + \alpha_1\epsilon) + V_3^\epsilon(x - s_3 t + \alpha_3\epsilon) - v_m + \Theta(\frac{x}{\epsilon}, \frac{t}{\epsilon}),\\ \tilde{u}^\epsilon &= U_1^\epsilon(x - s_1 t + \alpha_1\epsilon) + U_3^\epsilon(x - s_3 t + \alpha_3\epsilon) - u_m + \tilde{a}\Theta_x(\frac{x}{\epsilon}, \frac{t}{\epsilon}),\\ \tilde{E}^\epsilon &= E_1^\epsilon(x - s_1 t + \alpha_1\epsilon) + E_3^\epsilon(x - s_3 t + \alpha_3\epsilon) - E_m + \frac{p_m}{\gamma - 1}\Theta(\frac{x}{\epsilon}, \frac{t}{\epsilon}), + \tilde{a}u_m\Theta_x(\frac{x}{\epsilon}, \frac{t}{\epsilon}),\end{aligned} \tag{1.17}$$

where Θ and $\tilde{a}$ are defined in (5.15). From (5.19), it is easy to see that we can choose α_1 and α_3 and the total mass of Θ so that

$$\int_{-\infty}^{\infty} (m^\epsilon - \tilde{m}^\epsilon)(x, 0)dx = 0. \tag{1.18}$$

Define the composite wave consisting of two viscous shock waves:

$$\bar{U}^\epsilon_{\alpha_1^\epsilon, \alpha_3^\epsilon} = \bar{U}_1^\epsilon(x - s_1 t + \alpha_1\epsilon) + \bar{U}_3^\epsilon(x - s_3 t + \alpha_3\epsilon) - U_m. \tag{1.19}$$

Our main results are given in the following Theorem.

Theorem 1.1. Let the constant states $U_\pm$ (with $v_\pm > 0,\ \theta_\pm > 0$) be connected by a composite wave of two shock waves, defined by (1.9) above, and $\delta = |U_+ - U_-|$ be sufficiently small. α_1 and α_3 are choosen such that (1.18) holds. If (1.4) and (1.13) hold, then the Navier–Stokes equations (1.1) with Riemann initial data (1.5) have a unique, global, piecewise smooth solution $U^\epsilon(x, t) = (v^\epsilon, u^\epsilon, E^\epsilon)(x, t)$, for each $\epsilon > 0$, satisfying the following properties:
(i) $u^\epsilon(x, t),\ \theta^\epsilon(x, t)$ is continuous for $t > 0$; u_x^ϵ, θ_x^ϵ, v^ϵ, v_x^ϵ and v_t^ϵ are uniformly Hölder continuous in the sets $\{x < 0, t \geq \tau\}$ and $\{x > 0, t \geq \tau\}$ for any $\tau > 0$; u_t^ϵ, u_{xx}^ϵ, v_{xt}^ϵ, θ_t^ϵ and θ_{xx}^ϵ are Hölder continuous on compact set in $\{(x, t), x \neq 0, t > 0\}$. Moreover the jumps in $v^\epsilon(x, t)$, $u_x^\epsilon(x, t)$ and $\theta_x^\epsilon(x, t)$ at $x = 0$ satisfy

$$|[v^\epsilon(0, t)]|,\quad |[u_x^\epsilon(0, t)]|,\quad |[\theta_x^\epsilon(0, t)]| \leq c\exp\{-ct/\epsilon\}, \tag{1.20}$$

where c is a positive constant independently of t and ϵ.
(ii) For fixed viscosity $\epsilon > 0$, the solution $U^\epsilon(x, t)$ approaches the composite wave $\bar{U}^\epsilon_{\alpha_1^\epsilon, \alpha_3^\epsilon}$ defined in (1.19) uniformly as time t goes to infinity, i.e.,

$$\lim_{t\to\infty} \sup_{x\in\mathbf{R}} |U^\epsilon(x, t) - \bar{U}^\epsilon_{\alpha_1^\epsilon, \alpha_3^\epsilon}(x, t)| = 0. \tag{1.21}$$

(iii)The solutions $U^\epsilon(x, t)$ converge to the composite wave $U^0(x, t)$ defined in (1.9) uniformly as the viscosity $\epsilon \to 0$ on sets of the form $\{(x, t) : |x - s_1 t| \geq h$ and $|x - s_3 t| \geq h\}$, for any positive number h, i.e.,

$$\lim_{\epsilon\to 0}\sup_{|x-s_it|\geq h, i=1,3}|U^\epsilon(x,t)-U^0(x,t)|=0. \tag{1.22}$$

The convergence rate in L^p-distance is given by

$$\sup_{t\geq 0}\|U^\epsilon(\cdot,t)-U^0(\cdot,t)\|_{L^p}\leq C\epsilon^{\frac{1}{p}},\quad \text{for any } 2\leq p<\infty, \tag{1.23}$$

where the positive constant C is independent of ϵ and t.

Remark 1.2. It is interesting to make a comparison between Theorem 1.1 and those of Yu [36], where the author gives a sharp characterization of the zero dissipation limit process with shock and initial layer for the hyperbolic conservation laws with artificial viscosity. The main theorem of [36] is valid on the time interval $\delta^{-2-\alpha_0}\epsilon\leq t\leq O(1)\delta^3$ (here α_0 is a given positive constant, ϵ and δ denote the viscosity coefficient and the strength of the wave, respectively, see the main theorem on page 278 of [36] for details). The convergence rate in (1.22) and (1.23) is not as good as in [36], but they are valid for all the time $t>0$.

Now, we sketch the main idea of the proof and explain on some of the main difficulties and techniques involved in the process. Roughly speaking, we follow the framework of Hoff and Liu [9] on isentropic flows, and the proof involves the following four steps.

In first step, using the hyperbolic scaling property of the problem (1.1) and (1.5) and the Riemann problem (1.6) and (1.5), we perform the scaling argument to reduce the proof of Theorem 1.1 to the nonlinear stability problem in large time. Therefore, we encounter the problem to prove large time noninear asymptotic stability of a composite wave of two viscous shock waves for (1.1) under the Riemann data (1.5). It is worth mentioning here that for Riemann initial data (1.5), the L^2–norm of the spatial antiderivative of the initial perturbation, i.e., $\int_{-\infty}^x (U(y,0)-\bar{U}_{\alpha_1,\alpha_3}(y,0))dy$, is of the order $\delta^{-1/2}$, where $\delta=|U_+-U_-|$. Thus, if we take δ small, the H^2–norm of $\int_{-\infty}^x (U(y,0)-\bar{U}_{\alpha_1,\alpha_3}(y,0))dy$ becomes arbitrarily large. Therefore, the classic energy methods in [12, 21-24], depending essentially on the smallness of the H^2–norm of the spatial antiderivative of the initial perturbation, do not work here. Comparing to isentropic case in [9], where the excess mass is zero up to a shift of viscous shock wave, the new difficulty here is that the excess mass is not zero, and a linear diffusive waves was introduced to carry the excessive mass, as explained earlier in (1.17). Furthermore, we have two viscous shocks in the solution. Therefore, one expects the interaction between two shock waves and the linear diffusion wave in the solution of Navier-Stokes. These difficulties become more crucial in the last step, where the stability estimate is proved.

In second step, we develop sharp approximate solution of Navier-Stokes equations with Riemann data (1.5) in the initial time regime. It is well-known that viscous shock waves are leading asymptotic ansatz for shock wave in large time, but do not generate good approximation in short initial time. Therefore, both [9] and us encounter difficulty from initial layer. To overcome this difficulty, we construct approximate solutions through nonlinear Burgers' equation and the solution of a linear parabolic equation with contact discountinuity wave data, the latter appears as new difficulty in non-isentropic case. The key idea is that instead of viscous shock wave, we decompose the Riemann data in phase space and reconnect them through two rarefaction waves and a contact discountinuity. For rarefaction wave parts, we built the Navier-Stokes' correspondences through nonlinear Burgers' equation, while the contact discontuity is described by a linear diffusion wave. These give a much better approximation to the Navier-Stokes solution in its leading order and matches well the initial Riemann data. Therefore, detailed local information on the solution is obtained, and the solution is extended to the intermediate time regime of order $O(\delta^{-2-\vartheta})$, where δ denotes the strength of the initial jumps, and ϑ is a small positive constant.

In step 3, we establish the key property of the solution of Navier-Stokes in the critical intermediate time regime. By making full use of the nonlinearity of Burgers' equation, decay properties of the solution of the linear parabolic equation with contact discontinuity wave data and the delicated energy methods, we can show the difference between the solution of the Navier–Stokes equations and the approximate solution remains small, at least for times up to intermediate time of order $O(\delta^{-2-\vartheta})$. It is now we are able to deal with the problem caused by the fact that the L^2–norm of the spatial antiderivative of the initial perturbation is as large as $\delta^{-1/2}$. In fact, motivated by [9], one of key observations is that the square of the L^2–norm of $\int_{-\infty}^{x}(U(y,t)-\bar{U}_{\alpha_1,\alpha_3}(y,t))dy$ is of the order $\delta^a(t+1)^b$ (where a and b are nonnegative and $a-2b\geq 0$, see (5.21)), which may be arbitrarily large if the strength of the initial jumps δ is sufficiently small and $t=O(\delta^{-2-\vartheta})$. The estimate (5.20) will enable us to obtain that the square of the L^2–norms of higher-order derivatives are of the order $\delta^a(t+1)^b$ (where a and b are nonnegative and $a-2b\geq 1$), which may be arbitrarily small if the strength of the initial jumps δ is sufficiently small and $t=O(\delta^{-2-\vartheta})$. This, in turn, will lead to the desired smallness of the L^∞–norm of $\int_{-\infty}^{x}(U(y,t)-\bar{U}_{\alpha_1,\alpha_3}(y,t))dy$ and L^2–norms of higher-order derivatives which is exactly the a priori assumption of Theorem 5.1. The energy estimates thus can be closed. We remark that the smallness assumption on the strength of the initial jumps is essential here.

In the last step, we show that for very large time, the solution of the Navier–Stokes equations coalesces with the composite viscous traveling wave of the Navier–Stokes equations. This argument is proved by using the new linear diffusion wave firstly introduced in [12] and by means of energy estimates, using time $O(\delta^{-2-\vartheta})$ as initial time. The detailed estiamte of solution in intermediate time regime helps to soften the roughness of the initial data due to dissipation of Navier-Stokes. The resumed smallness in certain order norms gives the possibility to prove this stability result using energy method. Comparing to [12], the main novelty in this step of this paper is to overcome the difficulties arising from non-smooth initial perturbations and the careful energy estimate on the boundary integral terms. These can be easily seen from the new and very different energy estimates in, for instance, the proofs of (6.7) and (6.8), and the estimates on the boundary integral terms arising from the non–smooth initial perturbations (see (6.41)).

The rest of this paper is organized as follows. In the next section, we first perform scaling argument, and construct approximate solutions $\bar{U}_R$ and $\bar{U}_{TW}$. We also introduce the correction function to $\bar{\bar{U}}$ to deal with the major error caused by $\bar{U}_R$. Properties and estimates are collected in this section for later use. In Section 3, we study the difference between the exact solution U and our approximate solution $\bar{U}_R + \bar{\bar{U}}$. These estimates ensure the existence of U up to intermediate time. In Section 4, we collect some facts of viscous shock waves and some estimates on wave interactions. With the help of results established in sections 2–4, we construct the long time ansatz $\tilde{U}$ and prove the intermediate-time estimate for $U - \tilde{U}$. In Section 6, we make careful energy estimates to complete the proof of our main results–Theorem 1.1. For the readers's convenience, the frequently used symbols are also given in the Appendix at the end of the paper.

To conclude the introduction, we classify some notations. Throughout this paper, we will use the following

$$||\,.\,|| = ||\,.\,||_{L^2(\mathbf{R})}, \quad \|\,.\,\# = ||\,.\,||_{L^2(\mathbf{R}^-)} + ||\,.\,||_{L^2(\mathbf{R}^+)}, \qquad \fint dy = \int_{-\infty}^{0} dy + \int_{0}^{+\infty} dy.$$

2. Approximate solutions

In this section, we modify the technique developed in Hoff and Liu [9] to construct the approximate solution based on the self–similar solutions of the Burgers equation and the solution of a linear parabolic equation with contact discontinuity wave data, and collect some estimates needed in sections 3-6. To begin with, using scaling $(x,t) \to (\frac{x}{\epsilon}, \frac{t}{\epsilon})$, we scale the system (1.1) to the form

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = (\frac{u_x}{v})_x, \\ E_t + (pu)_x = \mu(\frac{\theta_x}{v})_x + (\frac{uu_x}{v})_x. \end{cases} \tag{2.1}$$

Normalize $\frac{\mu(\gamma-1)}{R} = 1$ and rewrite (2.1) as

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(B(U)\frac{\partial U}{\partial x}), \tag{2.2}$$

where

$$U = \begin{pmatrix} v \\ u \\ E \end{pmatrix}, \quad A(U) = \begin{pmatrix} 0 & -1 & 0 \\ -\frac{p}{v} & -\frac{(\gamma-1)u}{v} & \frac{\gamma-1}{v} \\ -\frac{pu}{v} & p - \frac{(\gamma-1)u}{v} & \frac{(\gamma-1)u}{v} \end{pmatrix}, \quad B(U) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{v} & 0 \\ 0 & 0 & \frac{1}{v} \end{pmatrix}.$$

The characteristic speeds λ_i and right eigenvectors r_i, $i = 1, 2, 3$, for A are

$$\begin{aligned} &\lambda_1 = -\sqrt{\frac{\gamma p}{v}}, \quad \lambda_2 = 0, \quad \lambda_3 = \sqrt{\frac{\gamma p}{v}}, \\ &r_1 = \frac{2v}{\gamma+1}\sqrt{\frac{v}{\gamma p}}(1, \sqrt{\frac{\gamma p}{v}}, u\sqrt{\frac{\gamma p}{v}} - p)^t, r_3 = \frac{2v}{\gamma+1}\sqrt{\frac{v}{\gamma p}}(-1, \sqrt{\frac{\gamma p}{v}}, u\sqrt{\frac{\gamma p}{v}} + p)^t, \\ &r_2 = (1, 0, \frac{p}{\gamma-1})^t, \quad \nabla\lambda_i \cdot r_i \equiv 1, i = 1, 3, \quad \nabla\lambda_2 \cdot r_2 \equiv 0. \end{aligned} \tag{2.3}$$

We can compute that:

$$\begin{aligned} &1- \text{ Riemann invariants } : \{s, u - \int^v \sqrt{-p_v(v,s)}dv\}, \\ &2- \text{ Riemann invariants } : \{u, p\}, \\ &3- \text{ Riemann invariants } : \{s, u + \int^v \sqrt{-p_v(v,s)}dv\}. \end{aligned}$$

Let R_1 and R_3 be integral curves of r_1 and r_3, passing through U_- and U_+ respectively, and $R_2 = \{(v, u, E)|u = \text{const}, p = \text{const}\}$, then the curve R_2 intersects R_1 and R_3 at U_{m_1} and U_{m_2} respectively. Here, the two constant states are given by

$$U_{m_1} = (v_{m_1}, u_{m_1}, e_{m_1} + \frac{u_{m_1}^2}{2})^t, \quad U_{m_2} = (v_{m_2}, u_{m_2}, e_{m_2} + \frac{u_{m_2}^2}{2})^t, \tag{2.4}$$

where $u_{m_1} = u_{m_2} = \bar{u}_m, \quad p_{m_1} = p_{m_2} = \bar{p}_m$.

Define

$$\tilde{\delta}_1 = \frac{1}{2}[\lambda_1(U_-) - \lambda_1(U_{m_1})] > 0, \tag{2.5}$$

and

$$\tilde{\delta}_3 = \frac{1}{2}[\lambda_3(U_{m_2}) - \lambda_3(U_+)] > 0. \tag{2.6}$$

Noting that

$$\delta/C \leq \tilde{\delta}_1, \quad \tilde{\delta}_3 \leq C\delta, \tag{2.7}$$

we have from Theorem 17.16 of Smoller [29] that

$$|U_{m_1} - U_{m_2}| \leq C\delta^3, \ |U_m - U_{m_1}| \leq C\delta^3, \ |U_m - U_{m_2}| \leq C\delta^3. \tag{2.8}$$

Now, we are ready to describe Riemann data solutions and traveling wave solutions of the Burgers equation, and use this information to construct the approximations $\bar{U}_R$ and $\bar{U}_{TW}$ required later on.

Let λ_R be the solution of the Burgers equation

$$\lambda_t + \lambda\lambda_x = \beta\lambda_{xx}, \quad t > 0, \quad x \in R, \tag{2.9}$$

with Riemann initial data

$$\lambda_R(x,0) = \begin{cases} 0, & x < 0, \\ -2\hat{\delta}, & x > 0, \end{cases} \tag{2.10}$$

where β and $\hat{\delta}$ are positive constants.

Using the well-known Hopf-Cole transform, we can solve the initial value problem (2.9)-(2.10) directly. The following lemma is given in Theorem 2.1 of [9].

Lemma 2.1. The solution of the initial value problem (2.9)-(2.10) is given by

$$\lambda_R = -2\hat{\delta}\frac{e^{\hat{\delta}(x+\hat{\delta}t)/\beta} f(-\frac{x+2\hat{\delta}t}{\sqrt{4\beta t}})}{e^{\hat{\delta}(x+\hat{\delta}t)/\beta} f(-\frac{x+2\hat{\delta}t}{\sqrt{4\beta t}}) + f(\frac{x}{\sqrt{4\beta t}})}, \tag{2.11}$$

where

$$f(a) = \pi^{-\frac{1}{2}} \int_a^{+\infty} e^{-\tau^2} d\tau. \tag{2.12}$$

Moreover, λ_R satisfies

$$-2\hat{\delta} \leq \lambda_R \leq 0, \tag{2.13}$$

$$\begin{cases} |\frac{\partial\lambda_R}{\partial x}(x,t)| \leq C[\hat{\delta}t^{-\frac{1}{2}}e^{-\frac{x^2}{4\beta t}} + \hat{\delta}^2 e^{-\hat{\delta}|x+\hat{\delta}t|/\beta}], \\ |\frac{\partial^2\lambda_R}{\partial x^2}(x,t)| \leq C[(\hat{\delta}|x|t^{-\frac{3}{2}} + \hat{\delta}^2 t^{-\frac{1}{2}})e^{-\frac{x^2}{4\beta t}} + \hat{\delta}^3 e^{-\hat{\delta}|x+\hat{\delta}t|/\beta}], \\ |\frac{\partial^3\lambda_R}{\partial x^3}(x,t)| \leq C\{[\hat{\delta}(x^2t^{-3} + |x|t^{-\frac{5}{2}}) + \hat{\delta}^2(|x|t^{-\frac{3}{2}} + t^{-1}) + \hat{\delta}^3 t^{-\frac{1}{2}}]e^{-\frac{x^2}{4\beta t}} + \hat{\delta}^4 e^{-\hat{\delta}|x+\hat{\delta}t|/\beta}]\}. \end{cases} \tag{2.14}$$

So, we let λ_R^i, $i = 1, 3$, be the solution of the Burgers equation (2.9) with initial data

$$\lambda_R^i(x,0) = \begin{cases} 0, & x < 0, \\ -2\tilde{\delta}_i, & x > 0, \end{cases} \tag{2.15}$$

and $\beta = b_{ii}(U_{m_1}) = \frac{2\gamma-1}{2\gamma v_{m_1}}$, $i = 1, 3$, and set

$$\begin{cases} \mu_R^1(x,t) = \lambda_R^1(x - \lambda_1(U_-)(t+1), t+1) + \lambda_1(U_-), \\ \mu_R^3(x,t) = \lambda_R^3(x - \lambda_3(U_{m_2})(t+1), t+1) + \lambda_3(U_{m_2}). \end{cases} \tag{2.16}$$

It is easy to see that μ_R^i are also solutions of (2.9) with "initial data"

$$\mu_R^1(x,-1) = \begin{cases} \lambda_1(U_-), & x < 0, \\ \lambda_1(U_{m_1}), & x > 0, \end{cases}$$

and

$$\mu_R^3(x,-1) = \begin{cases} \lambda_3(U_{m_2}), & x < 0, \\ \lambda_3(U_+), & x > 0. \end{cases}$$

So, we construct approximate solutions $\bar{U}_R^i$, $i = 1, 3$, by taking $\bar{U}_R^i \in R_i$ and $\lambda_i(\bar{U}_R^i(x,t)) = \mu_R^i(x,t)$. The 2-family approximate solution $\bar{U}_R^2 = (\bar{v}_R^2, \bar{u}_R^2, \bar{e}_R^2 + (\bar{u}_R^2)^2/2)^t$ is defined by

$$\begin{aligned} &\bar{u}_R^2 = \bar{u}_m, \ \bar{p}_R^2 = \bar{p}_m, \\ &(\bar{U}_R^2)_t = \beta_2(\bar{U}_R^2)_{xx}, \ \bar{U}_R^2(x,-1) = \begin{cases} U_{m_1}, & x < 0, \\ U_{m_2}, & x > 0, \end{cases} \end{aligned} \tag{2.17}$$

where $\beta_2 = \frac{1}{\gamma v_{m_1}}$. It is clear that $\bar{U}_R^2$ is selfsimiar in the form $\bar{U}_R^2(\frac{x}{\sqrt{t+1}})$ with total variation of order δ^3 due to (2.8). We finally arrive at

$$\bar{U}_R = \bar{U}_R^1 + \bar{U}_R^2 + \bar{U}_R^3 - U_{m_1} - U_{m_2}, \tag{2.18}$$

which satisfies

$$(\bar{U}_R)_t + F(\bar{U}_R)_x = B(U_{m_1})(\bar{U}_R)_{xx} - A_1 - A_2 - A_3 + [B(U_{m_2}) - B(U_{m_1})](\bar{U}_R^3)_{xx} + D_x, \tag{2.19}$$

where

$$\begin{cases} A_1 = \sum_{j=2}^3 b_{1j}(\bar{U}_R^1)(\mu_R^1)_{xx} r_j(\bar{U}_R^1) + (\mu_R^1)_x B(U_{m_1})(r_1(\bar{U}_R^1))_x, \\ A_3 = \sum_{j=1}^2 b_{3j}(\bar{U}_R^3)(\mu_R^3)_{xx}(r_j(\bar{U}_R^3) + \frac{\gamma-1}{2\gamma}(\frac{1}{v_{m_2}} - \frac{1}{v_{m_1}}) + (\mu_R^3)_x B(U_{m_2})(r_3(\bar{U}_R^3))_x, \\ A_2 = (\bar{v}_R^2)_{xx}(-\frac{1}{\gamma v_{m_1}}, 0, \frac{\bar{p}}{\gamma v_{m_1}})^t, \\ D = (0, \bar{p}_R - \bar{p}_R^1 - \bar{p}_R^3 + \bar{p}_m, \bar{p}_R\bar{u}_R - \bar{p}_R^1\bar{u}_R^1 - \bar{p}_R^3\bar{u}_R^3 + \bar{p}_m\bar{u}_m)^t \end{cases} \tag{2.20}$$

and

$$(b_{ij}(U))_{3\times 3} = \left(\begin{array}{ccc} \frac{2\gamma-1}{2\gamma}(v_{m_1})^{-1} & -\frac{\gamma-1}{\gamma}(v_{m_1}\frac{\gamma+1}{2v}\sqrt{\frac{\gamma p}{v}})^{-1} & \frac{1}{2\gamma}(v_{m_1})^{-1} \\ -\frac{1}{2\gamma}\frac{\gamma+1}{2v}\sqrt{\frac{\gamma p}{v}}(v_{m_1})^{-1} & \frac{1}{\gamma}(v_{m_1})^{-1} & \frac{1}{2\gamma}\frac{\gamma+1}{2v}\sqrt{\frac{\gamma p}{v}}(v_{m_1})^{-1} \\ \frac{1}{2\gamma}(v_{m_1})^{-1} & \frac{\gamma-1}{\gamma}(v_{m_1}\frac{\gamma+1}{2v}\sqrt{\frac{\gamma p}{v}})^{-1} & \frac{2\gamma-1}{2\gamma}(v_{m_1})^{-1} \end{array} \right). \tag{2.21}$$

At $t = -1$, $\bar{U}_R$ agrees with $U(\cdot, 0)$:

$$\bar{U}_R(x, -1) = \begin{cases} U_-, & x < 0, \\ U_+, & x > 0. \end{cases} \tag{2.22}$$

Another approximate solution $\bar{U}_{TW}$ is constructed in exactly the same way, except that, in place of the solutions λ^i_R of the Burgers equation with Riemann data, we substitute the corresponding traveling wave with the traveling wave data $\lambda^i_{TW}(x, 0) = (-2\tilde{\delta}_i)/(1 + e^{-\tilde{\delta}_i x/\beta})$, with $\beta = b_{ii}(U_{m_1}) = \frac{2\gamma-1}{2\gamma v_{m_1}}$. Thus,

$$\lambda^i_{TW}(x, t) = \frac{-2\tilde{\delta}_i}{1 + e^{-\tilde{\delta}_i(x-\tilde{s}_i t)/\beta}}, \ i = 1, 3, \tag{2.23}$$

where

$$\tilde{s}_1 = \frac{\lambda_1(U_-) + \lambda_1(U_{m_1})}{2} \quad \text{and} \quad \tilde{s}_3 = \frac{\lambda_3(U_{m_2}) + \lambda_3(U_+)}{2}, \tag{2.24}$$

and

$$\begin{cases} \mu^1_{TW}(x, t) = \lambda^1_{TW}(x, t + 1) + \lambda_1(U_-), \\ \mu^3_{TW}(x, t) = \lambda^3_{TW}(x, t + 1) + \lambda_3(U_{m_2}), \\ \bar{U}_{TW} = \bar{U}^1_{TW} + \bar{U}^2_{TW} + \bar{U}^3_{TW} - U_{m_1} - U_{m_2}, \quad \text{where,} \\ \lambda_i(\bar{U}^i_{TW}(x, t)) = \mu^i_{TW}(x, t), \quad \bar{U}^i_{TW} \in R_i, \ i = 1, 3, \quad \bar{U}^2_{TW} = \bar{U}^2_R. \end{cases} \tag{2.25}$$

The following lemma collects some basic properties of $\bar{U}_R$ needed in sections 3-6.

Lemma 2.2. Let $\bar{U}_R$ be defined in (2.18), then it holds that

$$\left\|\frac{\partial \bar{U}_R}{\partial x}(\cdot, t)\right\| \le C[\delta(t + 1)^{-1/4} + \delta^{3/2}], \tag{2.26}$$

$$\int_0^t \int_R \left(\frac{\partial \bar{U}_R}{\partial x}\right)^2 dxdt \le C[\delta^2(t + 1)^{1/2} + \delta^3 t], \tag{2.27}$$

$$\left\|\frac{\partial \bar{U}_R}{\partial t}(\cdot, t)\right\| \le C[\delta(t + 1)^{-3/4} + \delta^2], \tag{2.28}$$

$$\left\|\frac{\partial \bar{U}_R}{\partial x}(\cdot, t)\right\|_{L^\infty}, \quad \left\|\frac{\partial \bar{U}_R}{\partial t}(\cdot, t)\right\|_{L^\infty} \le C[\delta(t + 1)^{-1/2} + \delta^2], \tag{2.29}$$

$$\left|\left|\frac{\partial \bar{U}_R}{\partial x}(\cdot,t)\right|\right|_{L^1} \le C\delta, \tag{2.30}$$

$$\frac{\partial \bar{v}_R}{\partial t} = f_1 + f_2, \tag{2.31}$$

where $-C\delta \le f_1 < 0$, and $||f_2(\cdot,t)||_{L^\infty} \le C[\delta(t+1)^{-1} + \delta^2(t+1)^{-1/2} + \delta^3]$,

$$\int_0^{+\infty} \left| \int_x^{\infty} (\bar{U}_R^1 - U_{m_1})(y,t)dy \right|^2 dx \le C\delta^2 (t+1)^{3/2} e^{-(t+1)/C}, \tag{2.32}$$

$$\int_{-\infty}^{0} \left| \int_{-\infty}^{x} (\bar{U}_R^3 - U_{m_2})(y,t)dy \right|^2 dx \le C\delta^2 (t+1)^{3/2} e^{-(t+1)/C}, \tag{2.33}$$

$$\left|\left|\bar{U}_R(\cdot,0) - \bar{U}_R(\cdot,-1)\right|\right| \le C\delta, \tag{2.34}$$

$$\begin{cases} \int_0^{+\infty} \int_R |D| \, dxdt \le C\delta^2, \\ \int_0^{+\infty} \int_R |D|^2 \, dxdt \le C\delta^4. \end{cases} \tag{2.35}$$

Proof: The proof of (2.26)-(2.28), (2.30) and (2.32)-(2.35) can be found in Hoff and Liu [9], we focus our attentions on the proof of (2.29) and (2.31). From construction, we have

$$\begin{aligned} \left|\frac{\partial \bar{U}_R}{\partial x}(x,t)\right| &= \left|\frac{\partial \bar{U}_R^1}{\partial x}(x,t) + \frac{\partial \bar{U}_R^2}{\partial x}(x,t) + \frac{\partial \bar{U}_R^3}{\partial x}(x,t)\right| \\ &= \left|\frac{\partial \mu_R^1}{\partial x}(x,t) r_1(\bar{U}_R^1)\right| + \left|\frac{\partial \mu_R^3}{\partial x}(x,t) r_3(\bar{U}_R^3)\right| + \left|\frac{\partial \bar{U}_R^2}{\partial x}(x,t)\right| \\ &\le C[\delta(t+1)^{-\frac{1}{2}} + \delta^2], \end{aligned} \tag{2.36}$$

where we have used (2.7) and (2.14).
Similarly, we can obtain

$$\left|\frac{\partial \bar{U}_R}{\partial t}(x,t)\right| \le C[\delta(t+1)^{-\frac{1}{2}} + \delta^2]. \tag{2.37}$$

Then, (2.29) follows from (2.36) and (2.37) immediately.
For (2.31), one has from the construction that $\frac{\partial \bar{U}_R^i}{\partial t} = (\mu_R^i)_t r_i(\bar{U}_R^i)$, $i = 1, 3$. Therefore, with the help of (2.3), it is clear that

$$\frac{\partial \bar{v}_R}{\partial t} = a_1 \frac{\partial \mu_R^1}{\partial t} - a_3 \frac{\partial \mu_R^3}{\partial t} + \frac{\partial \bar{v}_R^2}{\partial t}, \tag{2.38}$$

where $a_i = O(1)(i = 1, 3)$, is positive. By virtue of (2.16), we have

$$\frac{\partial \bar{v}_R^1}{\partial t} = [\frac{\partial \lambda_R^1}{\partial t}(x - \lambda_1(U_-)(t+1), t+1) - \lambda_1(U_-)\frac{\partial \lambda_R^1}{\partial x}(x - \lambda_1(U_-)(t+1), t+1)] \tag{2.39}$$

and

$$\frac{\partial \bar{v}_R^3}{\partial t} = [\frac{\partial \lambda_R^3}{\partial t}(x-\lambda_3(U_{m_2})(t+1), t+1) - \lambda_3(U_{m_2})\frac{\partial \lambda_R^3}{\partial x}(x-\lambda_3(U_{m_2})(t+1), t+1)]. \qquad (2.40)$$

(2.38)-(2.40) gives that

$$\begin{aligned}\frac{\partial \bar{v}_R}{\partial t} = \; & [-a_1\lambda_1(U_-)\frac{\partial \lambda_R^1}{\partial x}(x-\lambda_1(U_-)(t+1), t+1) + a_3\lambda_3(U_{m_2})\frac{\partial \lambda_R^3}{\partial x}(x-\lambda_3(U_{m_2})(t+1), t+1)] \\ & +[a_1\frac{\partial \lambda_R^1}{\partial t}(x-\lambda_1(U_-)(t+1), t+1) - a_3\frac{\partial \lambda_R^3}{\partial t}(x-\lambda_3(U_{m_2})(t+1), t+1) + \frac{\partial \bar{v}_R^2}{\partial t}]. \quad (2.41)\end{aligned}$$

The first term here, which we take to be f_1, is negative, since the solution operator for (2.14) preserves monotonicity. It is clear that $-C\delta \leq f_1 \leq 0$. The second term above, which we define to be f_2, satisfies

$$\begin{aligned}||f_2(\cdot,t)||_{L^\infty} \; & \leq C[||\frac{\partial \lambda_R^1}{\partial t}(\cdot,t)||_{L^\infty} + \frac{\partial \lambda_R^3}{\partial t}(\cdot,t)||_{L^\infty} + \|\frac{\partial \bar{v}_R^2}{\partial t}(\cdot,t)\|_{L^\infty}] \\ & \leq C[\delta(t+1)^{-1} + \delta^2(t+1)^{-\frac{1}{2}} + \delta^3], \qquad (2.42)\end{aligned}$$

where we have used (2.13), (2.14) and (2.17). Therefore, the proof of (2.31) is completed.

The following properties of $\bar{U}_{TW}$ are given in Theorem 2.5 of [9].

Lemma 2.3. Let $\bar{U}_{TW}$ be as constructed above. Then

$$\left|\left|\frac{\partial}{\partial x}\bar{U}_{TW}^i(\cdot,t)\right|\right| \leq C\delta^{3/2} \qquad (2.43)$$

and

$$\left|\left|\bar{U}_{TW}(\cdot,0) - \bar{U}_{TW}(\cdot,-1)\right|\right| \leq C\delta^{3/2}. \qquad (2.44)$$

The following lemma concerning with the difference between $\bar{U}_R$ and $\bar{U}_{TW}$ can be proved in a similar way as Theorem 2.6 of [9].

Lemma 2.4. Let $\bar{U}_R$ and $\bar{U}_{TW}$ be as constructed above. Then

$$\begin{aligned}& ||\bar{U}_R(\cdot,t) - \bar{U}_{TW}(\cdot,t)|| \leq C[\delta^{1/2} + \delta(t+1)^{1/4}]e^{-\delta^2(t+1)/C}, \qquad (2.45)\\ & \int_{-\infty}^0 \left|\int_{-\infty}^x (\bar{U}_R^1 - \bar{U}_{TW}^1)(y,t)dy\right|^2 dx \leq C[\delta^{-1/2} + \delta(t+1)^{1/4}]e^{-\delta^2(t+1)/C}, \qquad (2.46)\\ & \int_0^\infty \left|\int_x^\infty (\bar{U}_R^3 - \bar{U}_{TW}^3)(y,t)dy\right|^2 dx \leq C[\delta^{-1/2} + \delta(t+1)^{1/4}]e^{-\delta^2(t+1)/C}. \qquad (2.47)\end{aligned}$$

We note that from (2.19), it is clear that the approximate solution $\bar{U}_R$ is not conservative, leading to big error. To explicit the main error, we introduce the correction function $\bar{\bar{U}}$ defined by

$$\bar{\bar{U}}_t + F'(U_{m_1})\bar{\bar{U}}_x = \beta\bar{\bar{U}}_{xx} + A_1 + A_3, \quad x \in R,\ t > 0, \tag{2.48}$$

with the initial data

$$\bar{\bar{U}}(x,0) = \bar{U}_R(x,-1) - \bar{U}_R(x,0), \tag{2.49}$$

where $\beta = \frac{1}{\gamma v_{m_1}}$. Thus $\bar{U}_R + \bar{\bar{U}}$ agrees with U at $t = 0$ (see (2.22)), and $U - \bar{U}_R - \bar{\bar{U}}$, satisfies a conservative equation.

Similar to Theorem 3.1 of [9], we have the following lemma on $\bar{\bar{U}}$.

Lemma 2.5. Let $\bar{\bar{U}}$ be constructed above , and assume that $\delta = |U_+ - U_-|$ is sufficiently small. Then

$$\left\|\bar{\bar{U}}(\cdot,t)\right\| \le C \sum_{a-2b\ge 1} \delta^a (t+1)^b, \tag{2.50}$$

$$\int_0^t \int_R |\bar{\bar{U}}|^2 dxdt \le C \sum_{a-2b\ge 0} \delta^a (t+1)^b, \tag{2.51}$$

$$\int_0^t \int_R |\bar{\bar{U}}_x|^2 dxdt \le C \sum_{a-2b\ge 2} \delta^a (t+1)^b, \tag{2.52}$$

$$\left\|\bar{\bar{U}}_x(\cdot,t)\right\| \le C \sum_{a-2b\ge 1} \delta^a (t+1)^b, \qquad t \ge 1, \tag{2.53}$$

$$\int_1^t \int_R |\bar{\bar{U}}_{xx}|^2 dxdt \le C \sum_{a-2b\ge 2} \delta^a (t+1)^b, \qquad t \ge 1, \tag{2.54}$$

$$\int_1^t \int_R \left|\bar{\bar{U}}\right|^4 dxdt \le C \sum_{a-2b\ge 2} \delta^a (t+1)^b, \qquad t \ge 1, \tag{2.55}$$

$$\left\|\bar{\bar{U}}_t(\cdot,t)\right\| \le C\delta, \qquad t \ge 1, \tag{2.56}$$

$$\int_{-\infty}^0 \left|\int_{-\infty}^x \bar{\bar{U}}(y,t)dy\right|^2 dx \le C \sum_{a-2b\ge 0} \delta^a (t+1)^b, \tag{2.57}$$

$$\int_0^\infty \left|\int_x^\infty \bar{\bar{U}}(y,t)dy\right|^2 dx \le C \sum_{a-2b\ge 0} \delta^a (t+1)^b. \tag{2.58}$$

Here $\sum_{a-2b\ge c} \delta^a (t+1)^b$ denotes a finite sum of the terms of the form $\delta^a(t+1)^b$ where a and b are nonnegative and $a - 2b \ge c$.

3. Comparison of U with $\bar{U}_R + \bar{\bar{U}}$

In this section, we derive a priori energy estimates for the difference $U - \bar{U}_R - \bar{\bar{U}}$, and we apply these estimates to show that the solution of (2.1) and (1.5) exists at least up to time $T = O(\delta^{-2-\vartheta})$ for some small positive ϑ.

setting $\Delta U = U - \bar{U}_R - \bar{\bar{U}}$, We obtain from the equations (2.1), (2.19) and (2.48) that

$$\begin{aligned} &\Delta U_t + [F(U) - F(\bar{U}_R) - F'(U_{m_1})\bar{\bar{U}}]_x \\ &\quad = B(U_{m_1})\Delta U_{xx} + [(B(U) - B(U_{m_1}))U_x]_x + (B(U_{m_1}) - \beta_2)\bar{\bar{U}}_{xx} \\ &\qquad - D_x - [(B(U_{m_2}) - B(U_{m_1}))](\bar{U}_R^3)_{xx} + A_2. \end{aligned}$$

From the construction in Section 2, we can define the following anti-derivative variable

$$Y(x,t) = \int_{-\infty}^{x} \Delta U(y,t)dy = (\phi, \varphi, \tilde{z})^t(x,t), \tag{3.1}$$

which satisfies

$$\begin{aligned} &Y_t + [F(U) - F(\bar{U}_R) - F'(U_{m_1})\bar{\bar{U}}] \\ &\quad = B(U_{m_1})Y_{xx} + (B(U) - B(U_{m_1}))U_x + (B(U_{m_1}) - \beta_2)\bar{\bar{U}}_x - D \\ &\qquad - (B(U_{m_2}) - B(U_{m_1}))(\bar{U}_R^3)_x + \tilde{A}_2, \end{aligned} \tag{3.2}$$

with $\tilde{A}_2 = (\bar{v}_R^2)_x(-\frac{1}{\gamma v_{m_1}}, 0, \frac{\bar{p}}{\gamma v_{m_1}})^t$. We recall that the initial data is identically zero for both ΔU and Y by (2.22) and (2.49).

Let

$$z = \frac{\gamma - 1}{R}(\tilde{z} - \bar{u}_R\varphi). \tag{3.3}$$

Inserting (3.3) into (3.2) and linearizing the resulted system, we obtain

$$\begin{cases} \phi_t - \varphi_x = -\beta_2\bar{\bar{v}}_x - \beta_2(\bar{v}_R^2)_x = Q_1, \\ \varphi_t - \frac{\bar{p}_R}{\bar{v}_R}\phi_x + \frac{R}{\bar{v}_R}z_x = \frac{\varphi_{xx}}{v_{m_1}} + (\frac{1}{v} - \frac{1}{v_{m_1}})u_x + (\frac{1}{v_{m_1}} - \beta_2)\bar{\bar{u}}_x \\ \qquad\qquad - (\frac{1}{v_{m_2}} - \frac{1}{v_{m_1}})(\bar{u}_R^3)_x - [\bar{p}_R - \bar{p}_R^1 - \bar{p}_R^3 + \bar{p}_m] \\ \qquad\qquad - [p - \bar{p}_R - \nabla\bar{p}_R \cdot (U - \bar{U}_R) + (\nabla\bar{p}_R - \nabla p(U_{m_1})) \cdot \bar{\bar{U}}] \\ \qquad = \frac{\varphi_{xx}}{v_{m_1}} + Q_2, \\ \frac{R}{\gamma-1}z_t + \bar{p}_R\varphi_x = \frac{1}{v_{m_1}}\frac{R}{\gamma-1}z_{xx} + Q_3, \\ Q_3 = \frac{2}{v_{m-1}}(\bar{u}_R)_x\varphi_x + (\frac{1}{v} - \frac{1}{v_{m_1}})[(e + \frac{u^2}{2})_x - \bar{u}_R u_x] - \frac{\gamma-1}{\gamma v_{m_1}}[(\bar{\bar{e}} + \frac{\bar{\bar{u}}^2}{2})_x - \bar{u}_R\bar{\bar{u}}_x] \\ \qquad - [\bar{p}_R^1(\bar{u}_R - \bar{u}_R^1) + \bar{p}_R^3(\bar{u}_R - \bar{u}_R^3) + \bar{p}_m(\bar{u}_m - \bar{u}_R)] + \frac{\bar{p}_m}{\gamma v_{m_1}}(\bar{v}_R^2)_x + \frac{1}{v_{m_1}}(\bar{u}_R)_{xx}\varphi \\ \qquad - (\frac{1}{v_{m_2}} - \frac{1}{v_{m_1}})[(\bar{e}_R^3 + \frac{(\bar{u}_R^3)^2}{2})_x - \bar{u}_R(\bar{u}_R^3)_x] - (\bar{u}_R)_t\varphi \\ \qquad - [(p - \bar{p}_R)(u - \bar{u}_R) + (\bar{u}_R - \bar{u}_m)\nabla p(U_{m_1}) \cdot \bar{\bar{U}} + (\bar{p}_R - \bar{p}_m)\bar{\bar{u}}]. \end{cases} \tag{3.4}$$

The following lemma gives energy estimates for the solution Y of (3.4) with zero initial data.

Lemma 3.1. For each U_-, there exist positive constants $C = C(U_-)$, η_0, and $\delta^{\frac{1}{4}}/C \leq \eta_1 \leq C\delta^{\frac{1}{4}}$

such that, if the solution (ϕ, φ, z) of (3.4) (and thus U) exists up to time $t > 0$ and satisfies

$$
\begin{aligned}
N_0(t) &= \sup_{0\le\tau\le t} \|(\phi,\varphi,z)(\cdot,\tau)\|_{L^\infty} < \eta_0,\\
N_1(t) &= \sup_{0\le\tau\le t} \|(\phi_x,\varphi_x,z_x)(\cdot,\tau)\| + \|(v_x,u_x,\theta_x)(\cdot,\tau)\| < \eta_1,
\end{aligned}
$$

and if

$$
\delta^3 t, \quad \delta\log(t+1) < \eta_0,
$$

then

$$
\begin{aligned}
&\sup_{0\le\tau\le t} \| (\phi,\varphi,z) \|^2 + \int_0^t \int |f_1|(\varphi^2+z^2)dxd\tau + \int_0^t \|(\phi_x,\varphi_x,z_x)\|^2 d\tau\\
&\quad \le C[\|\varphi_x\|^2(t) + (1+\delta^2 t)\int_0^t \|(\varphi_{xx},z_{xx})\|^2 d\tau + \sum_{a-2b\ge 0} \delta^a(1+t)^b].
\end{aligned}
\tag{3.5}
$$

Proof: Multiplying the first three equations of (3.4) by ϕ, $(\frac{\bar v_R}{\bar p_R}\varphi)$, $\frac{R}{(\bar p_R)^2}z$ respectively, adding then up, and integrating, we have from (2.31) that

$$
\begin{aligned}
&\sup_{0\le\tau\le t} \| (\phi,\varphi,z) \|^2 + \int_0^t \int |f_1|(\varphi^2+z^2)dxd\tau + \int_0^t \|(\phi_x,\varphi_x,z_x)\|^2 d\tau\\
&\quad \le C[\int_0^t \int_R |f_2|(\varphi^2+z^2) + \frac{R}{(\bar p_R)^2}((\bar u_R)_t + (\bar p_R)_x - \frac{(\bar u_R)_{xx}}{v_{m_1}})(\varphi z)\\
&\qquad + |\,(\frac{\bar v_R}{v_{m_1}\bar p_R})_x\,|\,\varphi\varphi_x\,| + |\,[\frac{R^2}{v_{m_1}(\gamma-1)\bar p_R^2}]_x z z_x\,| + |\,Q_1\phi\,| + \frac{\bar v_R\varphi}{\bar p_R}Q_2\\
&\qquad + |\,\frac{Rz}{\bar p_R^2}[Q_3 - (\bar u_R)_t\varphi + (\bar u_R)_{xx}v_{m_1}\varphi]\,|]dxd\tau := \sum_{k=1}^{7} I_k.
\end{aligned}
\tag{3.6}
$$

We apply the estimates (2.26)-(2.35) for $\bar U_R$ and (2.50)-(2.58) for $\bar{\bar U}$ to bound $I_k, (k=1,\cdots 7)$. Firstly, we have from (2.31) that

$$
\begin{aligned}
|I_1| + |I_2| &\le C\int_0^t [\delta^3 + \delta^2(1+\tau)^{-\frac12} + \delta(1+\tau)^{-1}]d\tau(\sup_{0\le\tau\le t}\int_R (\varphi^2+z^2)(x,\tau)dx)\\
&\le C[\delta\log(1+t) + \delta^2(1+t)^{\frac12} + \delta^3 t]\sup_{0\le\tau\le t}\int_R (\varphi^2+z^2)(x,\tau)dx,
\end{aligned}
$$

which can be absorbed into the left-hand side of (3.6) by our assumptions on δ and t.

Similarly, we have

$$
\begin{aligned}
|I_3|+|I_4| &\le \frac{1}{4}\int\int |f_1|\varphi^2 + (C\delta+\frac{1}{4})\int_0^t\int_R \varphi_x^2 \\
&\quad + C[\delta^3 t+\delta^2(1+t)^{\frac{1}{2}}+\delta\log(1+t)]\sup_{0\le\tau\le t}\int_R(\varphi^2+z^2)(x,\tau)\,dx,
\end{aligned}
$$

$$
\begin{aligned}
|I_5| &\le \int_0^t\int_R |\ \phi_x\ |\ |\bar{\bar{v}}|dxd\tau + \int_0^t\int_R |(\bar{v}_R^2)_x\phi|\ dxd\tau \\
&\le \epsilon_1\int_0^t\int_R|\phi_x|^2 + C(\epsilon_1)\int_0^t\int_R(|\bar{\bar{v}}|^2+|(\bar{v}_R^2)_x|) + C\int_0^t\int_R|(\bar{v}_R^2)_x|\phi^2 \\
&\le \epsilon_1\int_0^t\int_R|\phi_x|^2 + C(\epsilon_1)\sum_{a-2b\ge 0}\delta^a(1+t)^b + C\delta^3(1+t)^{\frac{1}{2}}\sup_{0\le\tau\le t}\int_R\phi^2(x,\tau)\,dx,
\end{aligned}
$$

where ϵ_1 is a positive number to be choosen small later.

To estimate the terms I_6 and I_7, we have from the a priori assumptions and Taylor formula that

$$
\begin{aligned}
|Q_2| &= O(1)[(v-\bar{v}_R)^2+(E-\bar{E}_R)^2+(u-\bar{u}_R)^2] + O(1)\ |\ \bar{\bar{U}}\cdot(U_{m_1}-\bar{U}_R)\ | \\
&\quad -\frac{\gamma-1}{\bar{v}_R}\varphi(\bar{u}_R)_x + (\frac{1}{v}-\frac{1}{v_{m_1}})u_x - (\frac{1}{v_{m_1}}-\frac{1}{\gamma v_{m_1}})\bar{\bar{u}}_x \\
&\quad -[\bar{p}_R-\bar{p}_R^1-\bar{p}_R^3+\bar{p}_m] - (\frac{1}{v_{m_2}}-\frac{1}{v_{m_1}})(\bar{u}_R^3)_x
\end{aligned}
$$

and

$$
\begin{aligned}
|Q_3| &= O(1)[(v-\bar{v}_R)^2+(E-\bar{E}_R)^2+(u-\bar{u}_R)^2] + O(1)\ |\ \bar{\bar{U}}\cdot(U_{m_1}-\bar{U}_R)\ | \\
&\quad +\frac{2}{v_{m_1}}(\bar{u}_R)_x\varphi_x + (\frac{1}{v}-\frac{1}{v_{m_1}})[(e+\frac{u^2}{2})_x-\bar{u}_R u_x] - (\frac{1}{v_{m_1}}-\frac{1}{\gamma v_{m_1}})[(\bar{\bar{e}}+\frac{\bar{\bar{u}}^2}{2})_x-\bar{u}_R\bar{\bar{u}}_x] \\
&\quad -[\bar{p}_R^1(\bar{u}_R-\bar{u}_R^1)+\bar{p}_R^3(\bar{u}_R-\bar{u}_R^3)+\bar{p}_m(\bar{u}_m-\bar{u}_R)] + \frac{1}{v_{m_1}}(\bar{u}_R)_{xx}\varphi - (\bar{u}_R)_t\varphi \\
&\quad +\frac{\bar{p}_m}{\gamma v_{m_1}}(\bar{v}_R^2)_x - (\frac{1}{v_{m_2}}-\frac{1}{v_{m_1}})[(\bar{e}_R^3\frac{(\bar{u}_R^3)^2}{2})_x - \bar{u}_R(\bar{u}_R^3)_x].
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
I_6 &\le C\left\{\int_0^t\int_R |\ \varphi\ |\ [(v-\bar{v}_R)^2+(u-\bar{u}_R)^2+(E-\bar{E}_R)^2] - \frac{\gamma-1}{\bar{p}_R}\varphi^2(\bar{u}_R)_x\right. \\
&\quad +|\ \varphi\ ||\ \bar{\bar{U}}\ ||\ U_{m_1}-\bar{U}_R\ | + |\ \varphi\ ||\ v-v_{m_1}\ ||\ u_x\ | + |\ \bar{\bar{u}}\ ||\ \varphi_x\ ||\ \varphi\ | + |\ (\bar{v}_R)_x\varphi\bar{\bar{u}}\ | \qquad (3.7) \\
&\quad +|\ \bar{p}_R-\bar{p}_R^1-\bar{p}_R^3+\bar{p}_m\ ||\ \varphi\ | + |\ v_{m_2}-v_{m_1}\ ||\ (\bar{u}_R^3)_x\ ||\ \varphi\ | := \sum_{i=1}^8 I_6^i,
\end{aligned}
$$

$$
\begin{aligned}
I_7 \ &\leq C\int_0^t \fint \mid z \mid [(v-\bar{v}_R)^2+(u-\bar{u}_R)^2+(E-\bar{E}_R)^2]+\mid z \mid\mid \bar{\bar{U}} \mid\mid U_{m_1}-\bar{U}_R \mid \\
&\quad +\mid z \mid\mid (\bar{u}_R)_x \mid\mid \varphi_x \mid + \mid z \mid\mid v-v_{m_1} \mid\mid E_x-\bar{u}_R u_x \mid + \mid z \mid\mid \bar{\bar{E}}_x-\bar{u}_R\bar{\bar{u}}_x \mid \\
&\quad +\mid z \mid\mid \bar{p}_R^1(\bar{u}_R-\bar{u}_R^1)+\bar{p}_R^3(\bar{u}_R-\bar{u}_R^3)+\bar{p}_m(\bar{u}_m-\bar{u}_R) \mid \\
&\quad +\mid v_{m_1}-v_{m_2} \mid\mid z \mid\mid (\bar{E}_R^3)_x-\bar{u}_R(\bar{u}_R^3)_x \mid + \mid z \mid\mid (\bar{v}_R^2)_x \mid + \mid (\bar{u}_R)_{xx}\varphi z \mid \\
&\quad +\mid (\bar{u}_R)_t\varphi z \mid := \sum_{k=1}^{10} I_7^k.
\end{aligned} \tag{3.8}
$$

Next, we estimate the terms $I_6^i(i=1,2,...,8)$ one by one. By virtue of Sobolev inequality, Cacuhy inequality, (2.29) and (2.51), we have

$$
\begin{aligned}
|I_6^1| \ &\leq CN_0(t)\int_0^t\int_R(\phi_x^2+\varphi_x^2+\tilde{z}_x^2)dxd\tau + C\int_0^t\int_R \mid \bar{\bar{U}} \mid^2 dxd\tau \\
&\leq CN_0(t)\int_0^t\int_R(\phi_x^2+\varphi_x^2)+CN_0(t)\int_0^t\int_R(z_x^2+(\bar{u}_R)_x^2\varphi^2+\varphi_x^2)+C\int_0^t\int_R \mid \bar{\bar{U}} \mid^2 \\
&\leq CN_0(t)\int_0^t\int_R(\phi_x^2+\varphi_x^2+z_x^2)+CN_0(t)\int_0^t\int_R(\bar{u}_R)_x^2\varphi^2+C\sum_{a-2b\geq 0}\delta^a(1+t)^b \\
&\leq CN_0(t)\left(\int_0^t\int_R(\phi_x^2+\varphi_x^2+z_x^2)+(\delta^4 t+\delta^2\log(1+t))\sup_{0\leq\tau\leq t}\int_R\varphi^2\right)+C\sum_{a-2b\geq 0}\delta^a(1+t)^b.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&|I_6^2| \leq C[\delta^2(1+t)^{\frac{1}{2}}+\delta^3 t]\sup_{0\leq\tau\leq t}\int_R\varphi^2, \\
&|I_6^3|+|I_6^6| \leq \frac{1}{8}\sup_{0\leq\tau\leq t}\int_R\varphi^2+C\sum_{a-2b\geq 0}\delta^a(1+t)^b, \\
&I_6^4 \leq CN_0^2(t)\int_0^t\fint\phi_x^2+\frac{1}{8}\sup_{0\leq\tau\leq t}\int_R\varphi^2 \\
&\qquad +C(\delta^2 t+1)\int_0^t\fint\varphi_{xx}^2+C\sum_{a-2b\geq 0}\delta^a(1+t)^b, \\
&I_6^5 \leq \epsilon_1\int_0^t\fint\varphi_x^2+C(\epsilon_1)\sum_{a-2b\geq 0}\delta^a(1+t)^b, \\
&I_6^7 \leq C[\delta+\delta^3(t+1)]+C[\delta^3 t+\delta]\sup_{0\leq\tau\leq t}\int_R\varphi^2, \\
&I_6^8 \leq C\delta^6 t\sup_{0\leq\tau\leq t}\int\varphi^2+C\sum_{a-2b\geq 0}\delta^a(1+t)^b.
\end{aligned}
$$

These imply that

$$
\begin{aligned}
|I_6| &\le C(\delta^2 t+1)\int_0^t \fint \varphi_{xx}^2 + C(N_0(t)+\epsilon_1)\int_0^t \fint (\phi_x^2+\varphi_x^2+z_x^2)\\
&\quad +(\frac{1}{2}+C\delta^3 t+C\delta^2\log(1+t))\sup_{0\le\tau\le t}\int_R \varphi^2 + C(\epsilon_1)\sum_{a-2b\ge 0}\delta^a(1+t)^b.
\end{aligned}
$$

Finally, using (3.8), we can bound the terms $I_7^i, (i=1,2,...,10)$ in a similar way as above. More precisely, we have

$$
\begin{aligned}
&|I_7^1| \le CN_0(T)\int_0^t \fint (\phi_x^2+\varphi_x^2+z_x^2) + \frac{1}{8}\sup_{0\le\tau\le t}\int_R \varphi^2,\\
&|I_7^2| \le \frac{1}{8}\sup_{0\le\tau\le t}\int_R z^2 + C\sum_{a-2b\ge 0}\delta^a(1+t)^b,\\
&|I_7^3| \le \epsilon_1\int_0^t\int_R \varphi_x^2 + C(\epsilon_1)[\delta^2\log(1+t)+\delta^4 t]\sup_{0\le\tau\le t}\int_R z^2,\\
&|I_7^4| \le CN_0^2(t)\int_0^t \fint \phi_x^2 + \frac{1}{4}\sup_{0\le\tau\le t}\int_R z^2\\
&\qquad +C(1+\delta^2 t)\int_0^t \fint \varphi_{xx}^2 + z_{xx}^2 + C\sum_{a-2b\ge 0}\delta^a(1+t)^b,\\
&|I_7^5| \le \frac{1}{8}\sup_{0\le\tau\le t}\int_R z^2 + C\sum_{a-2b\ge 0}\delta^a(1+t)^b,\\
&|I_7^6| \le C[\delta+\delta^3(t+1)] + C[\delta^3 t+\delta]\sup_{0\le\tau\le t}\int_R z^2,\\
&|I_7^7| \le C\delta^6 t\sup_{0\le\tau\le t}\int_R z^2 + C\sum_{a-2b\ge 0}\delta^a(1+t)^b,\\
&|I_7^8| \le C\delta^3(t+1)^{\frac{1}{2}}\sup_{0\le\tau\le t}\int_R z^2 + \delta^3 t,\\
&|I_7^9| + |I_7^{10}| \le C[\delta\log(t+1)+\delta^2(t+1)^{\frac{1}{2}}+\delta^3 t]\sup_{0\le\tau\le t}\int_R(\varphi^2+z^2).
\end{aligned}
$$

These estimates gives

$$
\begin{aligned}
|I_7| &\le [\frac{1}{8}+C(\delta\log(t+1)+\delta^2(t+1)^{\frac{1}{2}}+\delta^3 t)]\sup_{0\le\tau\le t}\int_R(\varphi^2+z^2)\\
&\quad +C(\epsilon_1+N_0(T))\int_0^t \fint (\phi_x^2+\varphi_x^2+z_x^2)\\
&\quad +C(1+\delta^2 t)\int_0^t \fint \varphi_{xx}^2 + z_{xx}^2 + C(\epsilon_1)\sum_{a-2b\ge 0}\delta^a(1+t)^b.
\end{aligned}
$$

Substituting the above estimates for I_1 to I_7 into (3.6), choosing ϵ_1 small, and noting the assumptions listed in Lemma 3.1, we deduce that

$$\sup_{0\le\tau\le t}\|(\phi,\varphi,z)\|(\tau)^2+\int_0^t\int_R|f_1|(\varphi^2+z^2)+\int_0^t\int_R(\varphi_x^2+z_x^2)$$
$$\le C(N_0(T)+\varepsilon_1)\int_0^t\fint\phi_x^2+C(1+\delta^2 t)\int_0^t\fint(\varphi_{xx}^2+z_{xx}^2)+C(\epsilon_1)\sum_{a-2b\ge0}\delta^a(1+t)^b. \quad (3.9)$$

Next, we proceed to estimate the term $\int_0^t\fint\phi_x^2$. To this end, multiplying $(3.4)_2$ by $(-\phi_x)$, and $(3.4)_1$ by (φ_x), respectively, and then adding the resulted equations, we obtain

$$\frac{\bar p_R}{\bar v_R}\phi_x^2-(\phi\varphi_t)_x+(\phi\varphi_x)_t-\varphi_x^2-\frac{R}{\bar v_R}z_x\phi_x=Q_1\varphi_x-\frac{1}{v_{m_1}}\phi_x\varphi_{xx}-Q_2\phi_x.$$

Integrating the above equation, we get

$$\int_0^t\int_R\frac{\bar p_R}{\bar v_R}\phi_x^2\ \le C[\int_R|\phi\varphi_x|(x,t)dx+\int_0^t\fint(\varphi_x^2+z_x^2+\varphi_{xx}^2)dxd\tau$$
$$+\int_0^t\fint(|Q_1\varphi_x|+|Q_2\phi_x|)dxd\tau+\int_0^t|[\phi\varphi_t](0,\tau)|d\tau],$$

which implies

$$\int_0^t\fint\phi_x^2dxd\tau\ \le C[\|\phi\|^2(t)+\|\varphi_x\|^2(t)+\int_0^t\fint(\varphi_x^2+z_x^2+\varphi_{xx}^2)dxd\tau]$$
$$+\int_0^t\fint(|Q_1\varphi_x|+|Q_2\phi_x|)dxd\tau+\int_0^t|[\phi\varphi_t](0,\tau)|d\tau. \quad (3.10)$$

Next, we bound the last three terms on the right-hand side of (3.10) in order. First, we have

$$\int_0^t\fint|Q_1\varphi_x|\ \le C\int_0^t\int_R|\bar{\bar v}_x\varphi_x|+|(\bar v_R^2)_x\varphi_x|$$
$$\le C(\int_0^t\int_R\varphi_x^2+\int_0^t\fint\bar{\bar v}_x^2+(\bar v_R^2)_x^2)$$
$$\le C\int_0^t\int_R\varphi_x^2+C\sum_{a-2b\ge0}\delta^a(1+t)^b, \quad (3.11)$$

$$\int_0^t\fint|Q_2\phi_x|\ \le C\int_0^t\fint|\phi_x|[(v-\bar v_R)^2+(u-\bar u_R)^2+(E-\bar E_R)^2]$$
$$+|\varphi\phi_x(\bar u_R)_x|+|\phi_x||\bar{\bar U}||U_{m_1}-\bar U_R|$$

$$
\begin{aligned}
& | \phi_x || v - v_{m_1} || u_x | + | \bar{\bar{u}}_x \phi_x | + | \bar{p}_R - \bar{p}_R^1 - \bar{p}_R^3 + \bar{p}_m || \phi_x | \\
& + | v_{m_2} - v_{m_1} || (\bar{u}_R^3)_x || \phi_x | := \sum_{i=1}^{7} J_i,
\end{aligned} \tag{3.12}
$$

Similar to those of I_6^i, we have the following estimates for J_i,

$$
\begin{aligned}
|J_1| \quad & \leq C(\delta + N_1(t)) \int_0^t \fint (\phi_x^2 + \varphi_x^2 + z_x^2) \\
& \quad + C(\delta + N_1(t))[\delta_t^4 + \delta^2 \log(1+t)] \sup_{0\leq\tau\leq t} \int_R \varphi^2 + C \sum_{a-2b\geq 0} \delta^a (1+t)^b, \\
|J_2| \quad & \leq C \int_0^t \fint \phi_x^2 \varphi^2 + C \int_0^t \fint (\bar{u}_R)_x^2 \\
& \leq C(N_1^2(t) + \delta) \int_0^t \fint \phi_x^2 + C \sum_{a-2b\geq 0} \delta^a (1+t)^b, \\
|J_3| \quad & \leq C\delta \int_0^t \fint | \phi_x || \bar{\bar{U}} | \\
& \leq C\delta^2 \int_0^t \fint \phi_x^2 + C \sum_{a-2b\geq 0} \delta^a (1+t)^b, \\
|J_4| \quad & \leq C(N_1(t) + \delta) \int_0^t \fint \phi_x^2 + C(1+\delta^2 t) \int_0^t \fint \varphi_{xx}^2 + C \sum_{a-2b\geq 0} \delta^a (1+t)^b, \\
|J_5| + |J_6| \quad & \leq \epsilon_2 \int_0^t \fint \phi_x^2 + C(\epsilon_2) \int_0^t \fint (\bar{\bar{u}}_x^2 + D^2) \\
& \leq \epsilon_2 \int_0^t \fint \phi_x^2 + C(\epsilon_2) (\sum_{a-2b\geq 0} \delta^a (1+t)^b), \\
|J_7| \quad & \leq C\delta^3 \int_0^t \fint | (\bar{u}_R^3)_x || \phi_x | \\
& \leq C\delta^6 \int_0^t \fint \phi_x^2 + C \sum_{a-2b\geq 0} \delta^a (1+t)^b,
\end{aligned}
$$

for any positive ϵ_2. Inserting the above estimates into (3.12), we have

$$
\begin{aligned}
\int_0^t \fint | Q_2 \phi_x | \quad & \leq C(\delta + N_1(t) + 2\epsilon_2) \int_0^t \fint \phi_x^2 + C \int_0^t \fint (\varphi_x^2 + z_x^2) \\
& \quad + C(\delta + N_1(t)) \sup_{0\leq\tau\leq t} \int_R \varphi^2 + C(\epsilon_2) \sum_{a-2b\geq 0} \delta^a (1+t)^b.
\end{aligned} \tag{3.13}
$$

By virtue of Theorem 1.2 in [10], we have

$$\int_0^t \mid [\phi\varphi_t](0,\tau) \mid d\tau = \int_0^t \mid \phi(0,\tau)[\varphi_t](0,\tau) \mid d\tau = \int_0^t \mid \phi(0,\tau)[p - \frac{u_x}{v}](0,\tau) \mid d\tau = 0. \quad (3.14)$$

From (3.10)-(3.11), and (3.13)-(3.14), we obtain by choosing ϵ_2 small that

$$\int_0^t \|\phi_x\|^2 d\tau \le C[\|(\phi,\varphi_x)\|^2(t) + \int_0^t (\|(\varphi_x, z_x)\|^2 + \|\varphi_{xx}\|^2)d\tau + \sum_{a-2b\ge 0} \delta^a (1+t)^b]. \quad (3.15)$$

Therefore, (3.5) follows from (3.9) and (3.15) immediately, if we choose ϵ_1 small enough. The proof of Lemma 3.1 is completed.

We will now derive higher order energy estimates for the solution $Y(t)$. Setting

$$(\Phi,\Psi) = (\phi_x, \varphi_x),\ W = \tilde{z}_x - \frac{1}{2}\Psi^2 - (\bar{u}_R + \bar{\bar{u}})\Psi, \quad (3.16)$$

which is the unique solution of the following problem:

$$\begin{cases} \Phi_t - \Psi_x = -\beta_2 \bar{\bar{v}}_{xx} - \frac{1}{\gamma v_{m_1}}(\bar{v}_R^2)_{xx} = Q_{1x}, \\ \Psi_t - \frac{\bar{p}_R}{\bar{v}_R}\Phi_x + \frac{\gamma-1}{\bar{v}_R} W_x = \frac{\Psi_{xx}}{v_{m_1}} + (\frac{\bar{p}_R}{\bar{v}_R})_x \Phi - (\frac{\gamma-1}{\bar{v}_R})_x [W + \frac{1}{2}\Psi^2 + (\bar{u}_R + \bar{\bar{u}})\Psi - (\bar{u}_R\varphi)_x] \\ \qquad\qquad - \frac{\gamma-1}{\bar{v}_R}[\Psi\Psi_x + ((\bar{u}_R + \bar{\bar{u}})\Psi)_x - (\bar{u}_R\varphi)_{xx}] + Q_{2x}, \\ \qquad = \frac{\Psi_{xx}}{v_{m_1}} + \hat{J}_1 + Q_{2x} \\ W_t + \bar{p}_R \Psi_x = \frac{W_{xx}}{v_{m_1}} - (\bar{p}_R)_x \Psi - [\frac{1}{2}\Psi^2 + (\bar{u}_R + \bar{\bar{u}})\Psi - (\bar{u}_R\varphi)_x]_t \\ \qquad + \frac{1}{v_{m_1}}[\frac{1}{2}\Psi^2 + (\bar{u}_R + \bar{\bar{u}})\Psi - (\bar{u}_R\varphi)_x]_{xx} + Q_{3x} \\ \qquad = \frac{W_{xx}}{v_{m_1}} + \hat{J}_2 + Q_{3x} \\ (\Phi,\Psi,W)(x,0) = 0. \end{cases} \quad (3.17)$$

Multiplying $(3.17)_1$ by Φ, $(3.17)_2$ by $(\frac{\bar{v}_R}{\bar{p}_R}\Psi)$, $(3.17)_3$ by $(\frac{\gamma-1}{(\bar{p}_R)^2}W)$, respectively, and adding, and integrating the results over $\mathbf{R} \times [1,t]$, we obtain

$$\begin{aligned} &\sup_{1\le\tau\le t} \|(\Phi,\Psi,W)\|^2(\tau) + \int_1^t \!\!\!\!\!\!\!\!\int (\Psi_x^2 + W_x^2) \le C\|(\Phi,\Psi,W)\|^2(1) \\ &+ C\int_1^t \!\!\!\!\!\!\!\!\int \{|f_2|(\Psi^2 + W^2) + \mid (\frac{\bar{v}_R}{v_{m_1}\bar{p}_R})_x \Psi\Psi_x \mid + \mid (\frac{\gamma-1}{v_{m_1}(\bar{p}_R)^2})_x W W_x \mid \\ &+ \mid Q_{1x}\Phi \mid + (\hat{J}_1 + Q_{2x})\frac{\bar{v}_R}{\bar{p}_R}\Psi + (\hat{J}_2 + Q_{3x})\frac{\gamma-1}{(\bar{p}_R)^2} W\}\ dxd\tau \\ &+ C\int_1^t [-\Phi\Psi + \frac{\gamma-1}{\bar{p}_R}\Psi W - \frac{\bar{v}_R}{v_{m_1}\bar{p}_R}\Psi\Psi_x - \frac{\gamma-1}{v_{m_1}(\bar{p}_R)^2} W W_x](0,\tau) d\tau \end{aligned}$$

$$:= C\|(\Phi,\Psi,W)\|^2(1) + C\sum_{i=1}^{7} L_i. \tag{3.18}$$

Next, we estimate the terms L_i in order. First, we have the following estimates for the first four terms.

$$\begin{aligned} L_1 &\le C\delta \int_1^t \fint (\Psi^2 + W^2), \\ L_2 + L_3 &\le C\delta \int_1^t \fint (|\ \Psi\Psi_x\ | + |WW_x|) \\ &\le C\delta \int_1^t \fint (\Psi^2 + W^2 + \Psi_x^2 + W_x^2), \\ L_4 &\le C\delta \int_1^t \fint \Phi^2 + \frac{C}{\delta}\int_1^t \fint \{(\bar{\bar{v}}_{xx})^2 + |(\bar{v}_R^2)_{xx}|^2\} \\ &\le C\delta \int_1^t \fint \Phi^2 + \frac{C}{\delta}(\sum_{a-2b\ge 2} \delta^a(1+t)^b + C\delta^6) \\ &\le C\delta \int_1^t \fint \Phi^2 + C\sum_{a-2b\ge 1} \delta^a(1+t)^b, \end{aligned}$$

L_5 is slightly more complicated. We proceed as follows.

$$\begin{aligned} L_5 &\le C\int_1^t \fint \hat{J}_1 \frac{\bar{v}_R}{\bar{p}_R}\Psi + C\int_1^t \fint Q_{2x}\frac{\bar{v}_R}{\bar{p}_R}\Psi \\ &:= L_5^1 + L_5^2, \end{aligned} \tag{3.19}$$

For L_5^1 we have,

$$\begin{aligned} L_5^1 \le C\int_1^t \fint & |\ (\bar{v}_R)_x\Phi\Psi\ | + |\ (\bar{v}_R)_x W\Psi\ | + |\ (\bar{v}_R)_x\Psi^3\ | + |\ (\bar{v}_R)_x\tilde{u}\Psi^2\ | \\ & + |\ (\bar{v}_R)_x\bar{u}_R\Phi\Psi\ | + |\ (\bar{v}_R)_x(\bar{u}_R)_x\varphi\Psi\ | + \frac{\gamma-1}{\bar{p}_R}\Psi^2\Psi_x \\ & + |\ \bar{\bar{u}}\Psi_x\Psi\ | + |\ \bar{\bar{u}}_x\Psi^2\ | + |\ (\bar{u}_R)_{xx}\varphi\Psi\ | + |\ (\bar{u}_R)_x\Psi^2\ | := \sum_{i=1}^{11} L_5^{1,i}, \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} \sum_{i=1}^{5} L_5^{1,i} &\le C\delta \int_1^t \fint (\Phi^2 + \Psi^2 + W^2), \\ L_5^{1,6} &\le C\eta_0^2 \int_1^t \fint (\bar{v}_R)_x^2 + C\int_1^t \fint |(\bar{u}_R)_x|^2\Psi^2 \end{aligned}$$

$$
\begin{aligned}
&\le C\delta^2\int_1^t\fint \Psi^2 + C\sum_{a-2b\ge 1}\delta^a(1+t)^b,\\
|L_5^{1,7}| \quad &= C|\int_1^t\fint \frac{\gamma-1}{3}(\frac{1}{\bar p_R})_x\Psi^3\ dxd\tau| \le C(N_1(t)+\delta)\delta\int_1^t\fint \Psi^2,\\
L_5^{1,8} \quad &+L_5^{1,9}+L_5^{1,11} \le C\delta\int_1^t\fint(\Psi^2+\Psi_x^2),\\
L_5^{1,10} \quad &\le C\delta\int_1^t\fint \Psi^2 + \frac{C}{\delta}\int_1^t\fint \varphi^2(\bar u_R)_{xx}^2\\
&\le C\delta\int_1^t\fint \Psi^2 + \frac{C}{\delta}\int_1^t\fint\{(\delta^2\frac{|x|^2}{\tau}\tau^{-2}+\delta^4\tau^{-1})e^{-\frac{x^2}{2\beta\tau}}+\delta^6 e^{-\frac{\delta|x+\delta\tau|}{2\beta}}\}dxd\tau\\
&\le C\delta\int_1^t\fint \Psi^2 + C[\delta t^{-\frac12}+\delta^3 t^{\frac12}+\delta^4 t]\\
&\le C\delta\int_1^t\fint \Psi^2 + C\sum_{a-2b\ge 1}\delta^a(1+t)^b.
\end{aligned}
$$

Inserting the above estimates into (3.20), we conclude

$$
L_5^1 \le C\delta\int_1^t\fint(\Phi^2+\Psi^2+W^2+\Psi_x^2) + C\sum_{a-2b\ge 1}\delta^a(1+t)^b. \tag{3.21}
$$

Next, we bound the term L_5^2. By integration by parts, we have

$$
\begin{aligned}
L_5^2 \ &= -C\int_1^t\fint_R[Q_2(\frac{\bar v_R}{\bar p_R})_x\Psi + Q_2\frac{\bar v_R}{\bar p_R}\Psi_x]dxdt - \int_1^t[Q_2\frac{\bar v_R}{\bar p_R}\Psi](0,\tau)d\tau\\
&= -C\int_1^t\int_R Q_2(\frac{\bar v_R}{\bar p_R})_x\Psi - C\int_1^t\fint_R Q_2\frac{\bar v_R}{\bar p_R}\Psi_x - \int_1^t\frac{\bar v_R}{\bar p_R}\Psi[Q_2](0,\tau)d\tau\\
&\le C\left(|\int_1^t\int_R Q_2(\frac{\bar v_R}{\bar p_R})_x\Psi| + |\int_1^t\fint_R Q_2\frac{\bar v_R}{\bar p_R}\Psi_x|\right) - \int_1^t\frac{\bar v_R}{\bar p_R}\Psi[\frac{-u_x}{v_{m_1}}+\nabla\bar p_R\cdot U](0,\tau)d\tau\\
&:= L_5^{2,1}+L_5^{2,2}+L_5^{2,3}.
\end{aligned}
$$

It is easy to see from (3.7) that

$$
L_5^{2,1} \le C\delta\int_1^t\fint |\ (\Phi,\Psi,W)\ |^2 + C\sum_{a-2b\ge 1}\delta^a(1+t)^b
$$

and

$$
L_5^{2,2} \ \le C(N_1(t)+\delta+\epsilon_3)\int_1^t\fint \Psi_x^2 + C(\epsilon_3)\int_1^t\fint(\Phi^4+\Psi^4+W^4) + C(\epsilon_3)\sum_{a-2b\ge 1}\delta^a(1+t)^b
$$

$$
\begin{aligned}
&\leq C(N_1(t)+\delta+\epsilon_3)\int_1^t \fint (\Psi_x^2+\Phi_x^2+W_x^2)\\
&\quad +C(\epsilon_3)N_1^4(t)\int_1^t \fint |(\Phi,\Psi,W)|^2 d\tau + C(\epsilon_3)\sum_{a-2b\geq 1}\delta^a(1+t)^b,
\end{aligned}
$$

for every small postive ϵ_3 to be chosen later. Combining the above relations, we obtain the desired estimate for L_5

$$
\begin{aligned}
L_5 \;&\leq C(N_1(t)+\delta+\epsilon_3)\int_1^t \fint (\Psi_x^2+W_x^2+\Phi_x^2)\\
&\quad +C(\epsilon_3)(\delta+N_1^4(t))\int_1^t \fint |(\Phi,\Psi,W)|^2 dx d\tau\\
&\quad +C(\epsilon_3)\sum_{a-2b\geq 1}\delta^a(1+t)^b + \int_1^t \frac{\bar{v}_R}{\bar{P}_R}\Psi[\frac{-u_x}{v_{m_1}}+\nabla\bar{p}_R\cdot U](0,\tau)d\tau. \qquad (3.22)
\end{aligned}
$$

Similarly, we can get the desired estimate for the term L_6

$$
\begin{aligned}
L_6 \;&\leq C(N_1(t)+\delta+\epsilon_3)\int_1^t \fint (\Psi_x^2+W_x^2+\Phi_x^2)\\
&\quad +C(\epsilon_3)(\delta+N_1^4(t))\int_1^t \fint |(\Phi,\Psi,W)|^2 dx d\tau\\
&\quad +C(\epsilon_3)\sum_{a-2b\geq 1}\delta^a(1+t)^b + \int_1^t \frac{\gamma-1}{v_{m_1}(\bar{p}_R)^2}[WW_x](0,\tau)d\tau. \qquad (3.23)
\end{aligned}
$$

Inserting the estimates for $L_i, i=1$ to 6 into (3.18), using the smallnss of $N_1(t)$, δ, and ϵ_3, we obtain

$$
\begin{aligned}
&\sup_{1\leq\tau\leq t}\|\Phi,\Psi,W\|^2 + \int_1^t \|(\Psi_x,W_x)\|^2 d\tau\\
&\leq C\|\Phi,\Psi,W\|^2(1) + C(N_1(t)+\delta+\epsilon_3)\int_1^t \|\Phi_x\|^2 d\tau\\
&\quad +C(\epsilon_3)(\delta+N_1^4(t))\int_1^t \|(\Phi,\Psi,W)\|^2 d\tau + C(\epsilon_3)\sum_{a-2b\geq 1}\delta^a(1+t)^b\\
&\quad +\int_1^t [-\Phi\Psi+\frac{\gamma-1}{\bar{p}_R}\Psi W - \frac{\bar{v}_R}{v_{m_1}\bar{p}_R}\Psi\Psi_x - \frac{\bar{v}_R}{\bar{p}_R}\Psi(\frac{-u_x}{v_{m_1}}+\nabla\bar{p}_R\cdot U)](0,\tau)d\tau. \qquad (3.24)
\end{aligned}
$$

By Theorem 1.2 in [10] and direct computation, one finds that the last term on the right-hand side of (3.24) equals to zero. Therefore, we have

$$
\sup_{1\leq\tau\leq t}\|\Phi,\Psi,W\|^2 + \int_1^t \|(\Psi_x,W_x)\|^2 d\tau
$$

$$\leq C\|\Phi,\Psi,W\|^2(1)+C(N_1(t)+\delta+\epsilon_3)\int_1^t\|\Phi_x\|^2d\tau$$
$$+C(\epsilon_3)(\delta+N_1^4(t))\int_1^t\|(\Phi,\Psi,W)\|^2d\tau+C(\epsilon_3)\sum_{a-2b\geq 1}\delta^a(1+t)^b. \tag{3.25}$$

Applying similar, but simpler, arguments used in obtaining (3.25), we can obtain

$$\sup_{0\leq\tau\leq 1}\|U-\bar{U}\|^2(\tau)+\int_0^1\|(u-\bar{u},\theta-\bar{\theta})_x\|^2d\tau\leq C[\delta+\delta\int_0^1\|U-\bar{U}\|^2d\tau]. \tag{3.26}$$

Applying Gronwall's inequality to (3.26), we get

$$\sup_{0\leq\tau\leq 1}\|U-\bar{U}\|^2(\tau)+\int_0^1\|(u-\bar{u},\theta-\bar{\theta})_x\|^2d\tau\leq C\delta. \tag{3.27}$$

By (3.25), (3.27) and triangle inequality, we conclude that

$$\sup_{0\leq\tau\leq t}\|\Phi,\Psi,W\|^2+\int_0^t\|(\Psi_x,W_x)\|^2d\tau$$
$$\leq C(\delta+N_1^4(t))\int_0^t\|(\Phi,\Psi,W)\|^2d\tau$$
$$+C(N_1(t)+\delta+\epsilon_3)\int_0^t\|\Phi_x\|^2d\tau+C(\epsilon_3)\sum_{a-2b\geq 1}\delta^a(1+t)^b. \tag{3.28}$$

Next, we estimate the term $\int_0^t\|\Phi_x\|^2d\tau$. Setting

$$K(v)=\log v,$$

we have from the first equation in (2.1) that

$$K_{xt}-p_x=u_t.$$

Multiplying the above equation by K_x, we obtain

$$\begin{aligned}(\frac{K_x^2}{2})_t+\frac{R}{v^3}\theta v_x^2 &= \frac{R\theta_x v_x}{v^2}+(u-\bar{u})_tK_x+\bar{u}_tK_x\\ &= \frac{R\theta_x v_x}{v^2}+[(u-\bar{u})_tK_x]_t-(u-\bar{u})K_{xt}+\bar{u}_tK_x.\end{aligned}$$

Integrating the above equation, we can get

$$\sup_{0\le\tau\le t}\|v_x\|^2+\int_0^t\|\Phi_x\|^2d\tau$$
$$\le C\int_0^t\|(\Psi_x,W_x)\|^2+C\delta[\sup_{0\le\tau\le t}\|(\Phi,\Psi,W)\|^2+\sum_{a-2b\ge1}\delta^a(1+t)^b]. \tag{3.29}$$

Combining (3.28) with (3.29), and choosing ϵ_3 sufficiently small, we conclude that

$$\sup_{0\le\tau\le t}[\|(\Phi,\Psi,W)\|^2(\tau)+\|v_x\|^2(\tau)]+\int_0^t\|(\Phi_x,\Psi_x,W_x)\|^2d\tau$$
$$\le C(\delta+N_1^4(t))\int_0^t\|(\Phi,\Psi,W)\|^2d\tau+C\sum_{a-2b\ge1}\delta^a(1+t)^b. \tag{3.30}$$

By virtue of Lemma 3.1 and the crucial estimate (3.30), we are able to prove the following theorem concerning the existence of the solution U up to time $t=O(\delta^{-2-\vartheta})$:

Theorem 3.2. If δ is sufficiently small, the solution of (2.1) and (1.5) exists up to time $T=O(\delta^{-2-\vartheta})$, where $\vartheta>0$ is a global small constant, and satisfies

$$\sup_{0\le\tau\le t}\|(\phi,\varphi,z)\|^2(\tau)+\int_0^t\|(\phi_x,\varphi_x,z_x)\|^2d\tau\le C\sum_{a-2b\ge0}\delta^a(1+t)^b \tag{3.31}$$

$$\sup_{0\le\tau\le t}[\|(\phi_x,\varphi_x,z_x)\|^2+\|(v_x,u_x,\theta_x)\|^2](\tau)$$
$$+\int_0^t[\|(\Phi_x,\Psi_x,W_x)\|^2+\|(u_{xx},\theta_{xx})]\|^2d\tau\le C\sum_{a-2b\ge1}\delta^a(1+t)^b, \tag{3.32}$$

for $t\le T$.

Proof. First of all, under the assumption of Lemma 3.1, (3.5) and (3.30) hold. Choosing a suitably small positive number ϵ_4, adding (3.30) $\times\,\epsilon_4$ and δ times (3.5), and noting the a priori assumption $N_1(t)\le C\delta^{\frac{1}{4}}$, we get (3.31) and

$$\sup_{0\le\tau\le t}[\|(\phi_x,\varphi_x,z_x)\|^2+\|v_x\|^2](\tau)+\int_0^t\|(\Phi_x,\Psi_x,W_x)\|^2d\tau\le C\sum_{a-2b\ge1}\delta^a(1+t)^b. \tag{3.33}$$

Next, we estimate $\|u_x\|^2(\tau)$. Multiplying $(2.1)_2$ by $-u_{xx}$, we obtain

$$\frac{d}{dt}(\frac{1}{2}u_x^2)+\frac{u_{xx}^2}{v}-(u_tu_x)_x=\frac{v_xu_xu_{xx}}{v^2}+\frac{R\theta_x}{v}u_{xx}-\frac{R\theta}{v^2}v_xu_{xx}.$$

Integrating the above equation and using (3.31) and (3.33), we have

$$\|u_x(t)\|^2 + \int_0^t \|u_{xx}\|^2(\tau)d\tau \le C \sum_{a-2b\ge 1} \delta^a(1+t)^b. \tag{3.34}$$

Similarly, we can prove

$$\|\theta_x(t)\|^2 + \int_0^t \|\theta_{xx}\|^2(\tau)d\tau \le C \sum_{a-2b\ge 1} \delta^a(1+t)^b. \tag{3.35}$$

Therefore, (3.32) follows from (3.33)-(3.35).We thus proved (3.31) and (3.32) under the assumption of Lemma 3.1.

We now prove this Theorem with the only assumption of smallness on δ. Noting the zero initial data of Y and ΔU, the smallness of δ, and the local existence result in [10], ensure that there exists a postive time $T_1 > 0$, such that $N_0(t) < \eta_0, N_1(t) < \eta_1$, $\delta^3 t$ and $\delta \log(t+1)$ are sufficiently small for all the time $t \le T_1$. We thus use above argument to get (3.31) and (3.32) as long as the smallness assumption in Lemma 3.1 holds. This enables us to extand our solution beyond T_1. Now, if we assume the solution along with the smallness condition

$$N_0(t) < \eta_0,\ \ N_1(t) < \eta_1$$

holds up to some time $t \le T := \delta^{-2-\vartheta}$, where $\delta^3 t$ and $\delta \log(t+1)$ are arbitrarily small if ϑ and δ are sufficiently small, then the right hand side of (3.31) and (3.32) are bounded respectively by $C\delta^{-\vartheta\mathcal{B}}$ and $C\delta^{1-\vartheta\mathcal{B}}$. A simple Sobolev inequality then shows that, for $t \le T$,

$$\begin{aligned}\|(\phi,\varphi,z)\|_{L^\infty}^4 &\le C\|(\phi_x,\psi_x,z_x)\|^2 \;\|\;(\phi,\varphi,z)\;\|^2 \\ &\le C\delta^{1-\vartheta\mathcal{B}} \ll 1,\end{aligned} \tag{3.36}$$

if ϑ and δ are small. Therefore,

$$N_0(T) < \eta_0.$$

Similarly, we can obtain

$$N_1(T) \le C\delta^{\frac{1}{2}-\frac{\vartheta}{2}\mathcal{B}} < \eta_1.$$

These observations show that we can extend our solution further and keeps the estimates (3.31) and (3.32) to at least a time $T = O(\delta^{-2-\vartheta}$, which complete the proof.

4. Estimates on viscous shock waves and wave interactions

In this section, we first give some fundamental estimates about the viscous shock waves. Then we deal with the wave interactions from the two different characteristic fields. We begin with the following lemma concerning some properties of the viscous shock waves.

Lemma 4.1. Denote $\bar{U}_1 = (V_1, U_1, E_1)^t$ and $\bar{U}_3 = (V_3, U_3, E_3)^t$ by the 1-viscous shock waves (U_-, U_m, s_1) and 3-viscous shock wave (U_m, U_+, s_3) of (2.1), respectively. Here U_m is intermediate constant state, s_1 and s_3 are the shock speeds. Then,

$$
\begin{aligned}
&|(\bar{U}_1 - U_-)(x,t)| \le C\delta_1 e^{-\delta_1|x-s_1t|/C}, \quad x < s_1t, \quad t \ge 0, && (4.1)\\
&|(\bar{U}_1 - U_m)(x,t)| \le C\delta_1 e^{-\delta_1|x-s_1t|/C}, \quad x > s_1t, \quad t \ge 0, && (4.2)\\
&|(\bar{U}_3 - U_m)(x,t)| \le C\delta_3 e^{-\delta_3|x-s_3t|/C}, \quad x < s_3t, \quad t \ge 0, && (4.3)\\
&|(\bar{U}_3 - U_+)(x,t)| \le C\delta_3 e^{-\delta_3|x-s_3t|/C}, \quad x > s_3t, \quad t \ge 0, && (4.4)\\
&U_{i,x}(x,t) < 0,\ |U_{i,x}| \le C\delta_i^2 e^{-\delta_i|x-s_it|/C},\ x \in R,\ t \ge 0, && (4.5)\\
&||(\bar{U}_{\alpha_1,\alpha_3} - \bar{U}_{TW})(\cdot,t)|| \le C[\delta^{\frac12} + \delta^2 t^{\frac12} + \delta^{\frac72} t], && (4.6)\\
&\int_{-\infty}^0 \left|\int_{-\infty}^x (\bar{U}_{1,\alpha_1} - \bar{U}^1_{TW})(y,t)dy\right|^2 dx \le C \sum_{a-2b\ge 0} \delta^a (t+1)^b, && (4.7)\\
&\int_0^\infty \left|\int_x^\infty (\bar{U}_{3,\alpha_3} - \bar{U}^3_{TW})(y,t)dy\right|^2 dx \le C \sum_{a-2b\ge 0} \delta^a (t+1)^b, && (4.8)\\
&\int_0^\infty \left|\int_x^\infty (\bar{U}_{1,\alpha_1} - U_m)(y,t)dy\right|^2 dx \le C[\delta^{-1} e^{-\delta t/C} + \delta^2], && (4.9)\\
&\int_{-\infty}^0 \left|\int_{-\infty}^x (\bar{U}_{3,\alpha_3} - U_m)(y,t)dy\right|^2 dx \le C[\delta^{-1} e^{-\delta t/C} + \delta^2], && (4.10)
\end{aligned}
$$

where $\bar{U}_{i,\alpha_i}(x,t) = \bar{U}_i(x - s_it + \alpha_i)$, $(i = 1,3)$ and $\bar{U}_{\alpha_1,\alpha_3} = \bar{U}_{1,\alpha_1} + \bar{U}_{3,\alpha_3} - U_m$.

Proof: Since the proofs of (4.1)-(4.5) are standard from ODE theory on (1.14) and (1.15), we will focus on the proofs of (4.6)-(4.10).

First, noting that (1.14), (1.15), (2.25), (2.7)-(2.8) and Lemma 2.2 in [21], we can follow the arguments in [9] (see the proof of (5.17) in [9]) step by step to obtain

$$
|\bar{U}_1(x) - \bar{U}^1_{TW}(x)| \le \begin{cases} C\delta^2 e^{-\delta|x|/C}, & x \le 0, \\ C(\delta^3 + \delta^2 e^{-\delta|x|/C}), & x \ge 0, \end{cases} \tag{4.11}
$$

and

$$|\bar{U}_3(x)-\bar{U}_{TW}^3(x)| \le \begin{cases} C(\delta^3+\delta^2 e^{-\delta|x|/C}), & x\le 0,\\ C\delta^2 e^{-\delta|x|/C}, & x\ge 0, \end{cases} \tag{4.12}$$

To prove (4.6), one has

$$\begin{aligned} & \|(\bar{U}_{\alpha_1,\alpha_3}-\bar{U}_{TW})(\cdot,t)\|^2 \\ = & \ ||(\bar{U}_{\alpha_1,\alpha_3}-\bar{U}_{TW})(\cdot,t)||^2_{L^2(\mathbf{R}^-)} + ||(\bar{U}_{\alpha_1,\alpha_3}-\bar{U}_{TW})(\cdot,t)||^2_{L^2(\mathbf{R}^+)} \\ = & \ \bar{J}_1+\bar{J}_2. \end{aligned} \tag{4.13}$$

Using the Cauchy inequality, we have

$$\begin{aligned} \bar{J}_1 \le 4 \Bigg(& \int_{-\infty}^0 \left|\bar{U}_{1,\alpha_1}(x,t)-\bar{U}_{TW}^1(x-\tilde{s}_1 t)\right|^2 dx + \int_{-\infty}^0 \left|\bar{U}_{3,\alpha_3}(x,t)-U_m\ \right|^2 dx \\ & + \int_{-\infty}^0 \left|\bar{U}_{TW}^2(x,t)-U_{m_1}\right|^2 dx + \int_{-\infty}^0 \left|\bar{U}_{TW}^3(x,t)-U_{m_2}\right|^2 dx \Bigg) \\ := & \ \bar{J}_1^1+\bar{J}_1^2+\bar{J}_1^3+\bar{J}_1^4. \end{aligned} \tag{4.14}$$

By virtue of (4.5) and (4.11), we get

$$\begin{aligned} \bar{J}_1^1 & \le C\int_{-\infty}^0 \left(\left|(\bar{U}_1-\bar{U}_{TW}^1)(x+\alpha_1-s_1t)\right|^2 + \left|\bar{U}_{TW}^1(x+\alpha_1-s_1t)-\bar{U}_{TW}^1(x-\tilde{s}_1t)\right|^2 \right) dx \\ & \le C[\int_{-\infty}^{|\alpha_1|-s_1t} \left|(\bar{U}_1-\bar{U}_{TW}^1)(x)\right|^2 dx + (1+|s_1-\tilde{s}_1|^2t^2)\int_{\mathbf{R}} \left|\frac{\partial \bar{U}_{TW}^1}{\partial x}\right|^2 dx] \\ & \le C[\delta^3+\delta^4 t+\delta^7 t^2], \end{aligned} \tag{4.15}$$

where we have used the fact that

$$\begin{aligned} s_1 & = \frac{\lambda_1(U_-)+\lambda_1(U_m)}{2}+O(\delta^2) \\ & = \frac{\lambda_1(U_-)+\lambda_1(\tilde{U}_m)}{2}+O(\delta^2) = \tilde{s}_1+O(\delta^2). \end{aligned}$$

Applying (4.5), we obtain

$$\begin{aligned} \bar{J}_1^2 & = \int_{-\infty}^{\alpha_3-s_3t} \left|\bar{U}_3(x)-U_m\right|^2 dx \\ & \le C[\int_{-\infty}^0 |\bar{U}_3(x)-U_m|^2 dx + \int_0^{|\alpha_3|} |\bar{U}_3(x)-U_m|^2 dx] \\ & \le C\delta. \end{aligned} \tag{4.16}$$

From (2.15) and (2.17), we have

$$\begin{aligned}\bar{J}_1^3 &\le Ct^{-1}|U_{m_1}-U_{m_2}|^2\int_{-\infty}^{0}\left|\int_{0}^{+\infty}e^{-\frac{(x-y)^2}{4\beta_2 t}}dy\right|^2dx\\ &\le C|U_{m_1}-U_{m_2}|^2\int_{-\infty}^{0}\left|\int_{\frac{-x}{\sqrt{4\beta_2 t}}}^{+\infty}e^{-s^2}ds\right|^2dx\\ &\le C|U_{m_1}-U_{m_2}|^2\int_{-\infty}^{0}e^{-\frac{x^2}{2\beta_2 t}}dx\le C\delta^6 t. \end{aligned}\tag{4.17}$$

Similarly, from (2.23) and (2.25), we get

$$\begin{aligned}\bar{J}_1^4 &\le C\tilde{\delta}_3^2\int_{-\infty}^{0}\frac{1}{(1+e^{-\tilde{\delta}_3(x-\tilde{s}_3(t+1))/\beta})^2}dx\\ &\le C\tilde{\delta}_3^2\int_{-\infty}^{-\tilde{s}_3(t+1)}\frac{1}{(1+e^{-\tilde{\delta}_3(x-\tilde{s}_3(t+1))/\beta})^2}dx\\ &\le C\tilde{\delta}_3^2\int_{-\infty}^{0}e^{\tilde{\delta}_3 x/C}dx\le C\delta. \end{aligned}\tag{4.18}$$

Inserting (4.15)-(4.18) into (4.14), we obtain

$$\bar{J}_1\le C[\delta+\delta^4 t+\delta^7 t^2].\tag{4.19}$$

To control $\bar{J}_2$, we have

$$\bar{U}_{\alpha_1,\alpha_3}-\bar{U}_{TW}=(\bar{U}_{3,\alpha_3}-\bar{U}_{TW}^3)+(\bar{U}_{1,\alpha_1}-U_m)+(U_{m_1}-\bar{U}_{TW}^1)+(U_{m_2}-\bar{U}_{TW}^2).\tag{4.20}$$

Then, using the similar arguments just as before, we can obtain

$$\bar{J}_2\le C[\delta+\delta^4 t+\delta^7 t^2].\tag{4.21}$$

Therefore, (4.6) follows from (4.13), (4.19) and (4.21) immediately.

Now, we turn to the proof of (4.7). By virtue of (2.23), (2.25) and (4.11), we have

$$\begin{aligned}&\int_{-\infty}^{0}\left|\int_{-\infty}^{x}(\bar{U}_{1,\alpha_1}-\bar{U}_{TW}^1)(y,t)dy\right|^2dx\\ &\le C\int_{-\infty}^{\alpha_1-s_1 t}\left(\int_{-\infty}^{x}\left|(\bar{U}_1-\bar{U}_{TW}^1)(y)\right|dy\right)^2dx\\ &\quad +C\int_{-\infty}^{0}\left(\int_{-\infty}^{x}\left|\bar{U}_{TW}^1(y+\alpha_1-s_1 t)-\bar{U}_{TW}^1(y-\tilde{s}_1 t)\right|dy\right)^2dx. \end{aligned}\tag{4.22}$$

The first integral on the right hand side of (4.22) can be bounded as follows:

$$
\begin{aligned}
& C\int_{-\infty}^{\alpha_1-s_1t}\left(\int_{-\infty}^{x}\left|(\bar{U}_1-\bar{U}^1_{TW})(y)\right|dy\right)^2 dx \\
& \quad \le C\int_{-\infty}^{0}\left(\int_{-\infty}^{x}\left|(\bar{U}_1-\bar{U}^1_{TW})(y)\right|dy\right)^2 dx \\
& \qquad +C\int_{0}^{|\alpha_1|-s_1t}\left(\int_{-\infty}^{0}\left|(\bar{U}_1-\bar{U}^1_{TW})(y)\right|dy\right)^2 dx \\
& \qquad +C\int_{0}^{|\alpha_1|-s_1t}\left(\int_{0}^{x}\left|(\bar{U}_1-\bar{U}^1_{TW})(y)\right|dy\right)^2 dx \\
& \quad \le C[\delta+\delta^2(t+1)+\delta^6(t+1)^3].
\end{aligned}
$$

The last integral in (4.22) is bounded by

$$
C(1+|s_1-\tilde{s}_1|^2t^2)\int_{-\infty}^{O(1)(t+1)}\left(\int_{-\infty}^{x}|(\bar{U}^1_{TW})'(z)|dz\right)^2 dx.
$$

Using the fact that $|(\bar{U}^1_{TW})'(z)| \le C\delta^2 e^{-\delta|z|/C}$(see (2.23) and (2.25)) and that $|s_1-\tilde{s}_1| \le C\delta^2$, we find that the second term on the right-side of (4.22) can be bounded by $C[\delta^2(t+1)+\delta^6(t+1)^3]$. These estimates prove (4.7). The proof of (4.8) is similar.

Finally, we prove (4.9) and (4.10). It suffices to show (4.9), since the proof of (4.10) is similar. From the fact $s_1<0$, (1.11)-(1.12), and (4.2), we have

$$
\int_0^{\infty}\left|\int_x^{\infty}(\bar{U}_{1,\alpha_1}-U_m)(y,t)dy\right|^2 dx
$$

$$
\le C\Big(\int_0^{|\alpha_1|}+\int_{|\alpha_1|}^{\infty}\Big)\left|\int_x^{\infty}(\bar{U}_{1,\alpha_1}-U_m)(y,t)dy\right|^2 dx \tag{4.23}
$$

$$
\le C[\delta^{-1}e^{-\delta t/C}+\delta^2]. \tag{4.24}
$$

This proves the estimate (4.9). Therefore, the proof of Lemma 4.1 is completed.

To deal with the wave interactions from the two different characteristic fields, we divide $\mathbf{R}\times(0,t)$ into the following two parts

$$
\Omega^- = \left\{(x,t)|x \le \frac{s_1+s_3}{2}t\right\},\ \text{and}\ \Omega^+ = \left\{(x,t)|x > \frac{s_1+s_3}{2}t\right\}.
$$

Then, we have the following lemma concerning the wave interactions estimates:

Lemma 4.2. Let the two viscous shock waves $\bar{U}_{1,\alpha_1}$ and $\bar{U}_{3,\alpha_3}$ be as defined in Lemma 4.1. Then

$$|(\bar{U}_{1,\alpha_1} - U_m)(x,t)| = O(1)\delta_1 e^{-\delta_1(|x|+t)/C}, \qquad \text{in } \Omega^+, \tag{4.25}$$

$$|(\bar{U}_{3,\alpha_3} - U_m)(x,t)| = O(1)\delta_3 e^{-\delta_3(|x|+t)/C}, \qquad \text{in } \Omega^-, \tag{4.26}$$

$$|(\bar{U}_{1,\alpha_1} - U_m)(x,t)||(\bar{U}_{3,\alpha_3})_x(x,t)| = O(1)\delta_3^2\delta_1(e^{-\delta_1(|x|+t)/C} + e^{-\delta_3(|x|+t)/C}), \tag{4.27}$$

$$|(\bar{U}_{3,\alpha_3} - U_m)(x,t)||(\bar{U}_{1,\alpha_1})_x(x,t)| = O(1)\delta_1^2\delta_3(e^{-\delta_1(|x|+t)/C} + e^{-\delta_3(|x|+t)/C}). \tag{4.28}$$

Proof: Since the inviscid system (1.6) is strictly hyperbolic, then $s_1 < 0 < s_3$. When t is large, the major parts of two viscous shock waves will decouple. With this in mind, (4.24)-(4.27) can be proved easily. Indeed, set $t_0 = 4\max_{i=1,3}\{|\alpha_i|\}/s_3$. When $t \le t_0$, the estimates (4.24)-(4.27) are obvious. For $t > t_0$, we have, in Ω^+,

$$x + \alpha_1 - s_1 t > (s_1 + s_3)t/2 + \alpha_1 - s_1 t \ge (s_3 - s_1)t/4 > 0. \tag{4.29}$$

By Lemma 4.1,

$$|(\bar{U}_{1,\alpha_1} - U_m)(x,t)| = O(1)\delta_1 e^{-\delta_1|x+\alpha_1-s_1t|/C}. \tag{4.30}$$

It is clear that $|x + \alpha_1 - s_1 t| \ge Ct$, by (4.28). When $x \le 0$, we have

$$|x + \alpha_1 - s_1 t| = x + \alpha_1 - s_1 t \ge x + (s_1 - s_3)t/4 - s_1 t \ge x - 3x/2 = |x|/2. \tag{4.31}$$

When $x > 0$, if $s_1 + s_3 \le 0$,

$$|x + \alpha_1 - s_1 t| \ge x + \alpha_1 + s_3 t \ge x + 3|\alpha_1| \ge x = |x|; \tag{4.32}$$

if $s_1 + s_3 > 0$,

$$|x + \alpha_1 - s_1 t| \ge x - (s_1 + s_3)t/4 \ge x - x/2 = |x|/2. \tag{4.33}$$

Therefore $|x + \alpha_1 - s_1 t| \ge C|x|$. Now, we have proved the estimate (4.24). The other estimates in Lemma 4.2 can be treated similarly. We omit the details.

Therefore, the proof of lemma 4.2 is completed.

5. Intermediate-Time estimate for $U - \tilde{U}$

We see from previous sections that $\bar{U}_R$ provided good approximation to U in short time and intermediate time. However, visous shock waves would be the main part of U in large time. Therefore, $\bar{U}_R$ is not a good comparison background to U, and we will prepare the transition estimates for the large time regime in this section. In particular, we will first construct the sharp large time ansatz $\tilde{U}$, which will contain viscous shocks and a linear diffusion wave. We then make estimates on $U - \tilde{U}$ in the intermediate time regime. These estimates will lay a solid base for the final proof of Theorem 1.1 in next section.

Recalling $\bar{U} = \bar{U}_1 + \bar{U}_3 - U_m$, the composite viscous shock waves, we note that the Riemann data (1.5) is a nontrivial discountious perturbation near $\bar{U}$. In general, the integral $\int_{-\infty}^{\infty} (U - \bar{U})(x,0)dx$ is not zero. We note that the three vectors $\tilde{r}_1 = (v_m - v_-, u_m - u_-, E_m - E_-)^t$, $\tilde{r}_2 = r_2(U_m)$ and $\tilde{r_3} = (v_+ - v_m, u_+ - u_m, E_+ - E_m)^t$ are linearly independent in $\mathbf{R}^3$ if $\delta_1 + \delta_3$ is small. Therefore, it holds that

$$\int_{-\infty}^{\infty} (U(x,0) - \bar{U}(x,0))dx = \sum_{i=1}^{3} \alpha_i \tilde{r}_i, \tag{5.1}$$

where $\alpha_i (i = 1,2,3)$ are constants uniquely determined by the initial data. The excessive mass $\alpha_1 \tilde{r}_1$ in the first characteristic field can be eliminated by the translation in 1-viscous shock wave with a shift α_1, i.e., $\bar{U}_1(x - s_1 t + \alpha_1)$. Similarly, we can eliminate $\alpha_3 \tilde{r}_3$ by shifting $\bar{U}_3(x - s_3 t)$ to $\bar{U}_3(x - s_3 t + \alpha_3)$. So the remaining problem is how to remove the excessive mass in the second characteristic field, i.e., $\alpha_2 \tilde{r}_2$, which in general is not zero. For this, we apply the technique developed in [12] to look for our ansatz $\tilde{U} = (\tilde{v}, \tilde{u}, \tilde{E})^t$ in the form

$$\begin{aligned}
\tilde{v} &= V_1(x - s_1 t + \alpha_1) + V_3(x - s_3 t + \alpha_3) - v_m + \Theta(x,t), \\
\tilde{u} &= U_1(x - s_1 t + \alpha_1) + U_3(x - s_3 t + \alpha_3) - u_m + f(x,t), \\
\tilde{E} &= E_1(x - s_1 t + \alpha_1) + E_3(x - s_3 t + \alpha_3) - E_m + \frac{p_m}{\gamma - 1}\Theta + g(x,t).
\end{aligned} \tag{5.2}$$

Here, we expect $(\Theta, 0, \frac{p_m}{\gamma-1}\Theta)^t = \Theta r_2$ to be a basic approximation of the diffusion wave associated with the second characteristic field which tends to zero as $t \to +\infty$, and carries excessive mass $\alpha_2 \tilde{r}_2$, i.e.

$$|\Theta(x,t)| \le \frac{C|\alpha_2|}{\sqrt{1+t}} e^{-\frac{cx^2}{1+t}}, \quad \int_{-\infty}^{\infty} \Theta(x,t)dx = \alpha_2. \tag{5.3}$$

We also expect that $f(x,t)$ and $g(x,t)$ are higher order correction terms compared with $\Theta(x,t)$ which behave like derivatives of the diffusion wave, i.e.,

$$|f(x,t)| + |g(x,t)| \le \frac{C|\alpha_2|}{1+t} e^{-\frac{cx^2}{1+t}}, \tag{5.4}$$

and carry zero mass, i.e.,

$$\int_{-\infty}^{\infty} f(x,t)dx = \int_{-\infty}^{\infty} g(x,t)dx = 0. \tag{5.5}$$

Finally, we should expect that $\tilde{U} = (\tilde{v}, \tilde{u}, \tilde{E})^t$ gives a good approximate solution of (2.1) in the sense that

$$\begin{cases} \tilde{v}_t - \tilde{u}_x = 0, \\ \tilde{u}_t + \tilde{p}_x = (\frac{\tilde{u}_x}{\tilde{v}})_x + \tilde{R}_{1,x}, \\ (\tilde{e} + \frac{\tilde{u}^2}{2})_t + (\tilde{p}\tilde{u})_x = \mu(\frac{\tilde{\theta}_x}{\tilde{v}})_x + (\frac{\tilde{u}\tilde{u}_x}{\tilde{v}})_x + \tilde{R}_{2,x}, \end{cases} \tag{5.6}$$

where the error terms $\tilde{R}_i$(i=1, 2) should decay fast enough for a priori estimates we will establish (see (5.20)-(5.21)). From the estimates for wave interactions (4.24)-(4.27), we expect $\tilde{R}_i (i = 1, 2)$ satisfy

$$|\tilde{R}_i|,\ |\tilde{R}_{i,x}| \leq C(\delta^2 + |\alpha_2|\delta^{\frac{3}{2}})e^{-c\delta(|x|+t)} + C\frac{|\alpha_2|}{(1+t)^{\frac{3}{2}}}e^{-\frac{cx^2}{1+t}} + C(\delta + |\alpha_2|)e^{-c\delta(|x|+t)}, \tag{5.7}$$

where and in what follows the constant c and C are uniformly positive constants with respect to small δ and α_2. We will also denote $A \approx B$ if

$$|A - B| \leq C(\delta^2 + |\alpha_2|\delta^{\frac{3}{2}})e^{-c\delta(|x|+t)} + C\frac{|\alpha_2|}{(1+t)^{\frac{3}{2}}}e^{-\frac{cx^2}{1+t}} + C(\delta + |\alpha_2|)e^{-c\delta(|x|+t)}. \tag{5.8}$$

In order to achieve (5.6) and (5.7), we now try to choose the functions (Θ, f, g) satisfying (5.3)-(5.5). Substituting (5.2) into $(2.1)_1$ and using the definitions of $\bar{U}_1$ and $\bar{U}_2$, we obtain

$$\Theta_t - f_x = 0. \tag{5.9}$$

By virtue of (5.3), (5.4) and Lemma 4.2, we get

$$\tilde{\theta} = \frac{\gamma - 1}{R}(\tilde{E} - \frac{\tilde{u}^2}{2}) \approx \Theta_1 + \Theta_3 - \theta_m + \frac{p_m}{R}\Theta + \frac{\gamma - 1}{R}g - \frac{\gamma - 1}{R}u_m f. \tag{5.10}$$

Hence we choose $g = u_m f$ so that

$$\tilde{\theta} = \frac{\gamma - 1}{R}(\tilde{E} - \frac{\tilde{u}^2}{2}) \approx \Theta_1 + \Theta_3 - \theta_m + \frac{p_m}{R}\Theta. \tag{5.11}$$

In the same way, we have

$$\begin{cases} \tilde{p} = \frac{R\tilde{\theta}}{\tilde{v}} \approx P_1 + P_3 - p_m, \\ \tilde{p}\tilde{u} \approx P_1 U_1 + P_3 U_3 - p_m u_m + p_m f, \\ \dfrac{\tilde{u}_x}{\tilde{v}} \approx \dfrac{U_{1,x}}{V_1} + \dfrac{U_{3,x}}{V_3},\ \dfrac{\tilde{\theta}_x}{\tilde{v}} \approx \dfrac{\Theta_{1,x}}{V_1} + \dfrac{\Theta_{3,x}}{V_3} + \dfrac{p_m}{Rv_m}\Theta_x, \\ \dfrac{\tilde{u}\tilde{u}_x}{\tilde{v}} \approx \dfrac{U_1 U_{1,x}}{V_1} + \dfrac{U_3 U_{3,x}}{V_3}. \end{cases} \tag{5.12}$$

Substituting (5.2), (5.11)-(5.12) into $(2.1)_3$ and using (1.12), we have

$$\frac{p_m}{\gamma-1}\Theta_t + p_m f_x = \frac{\mu p_m}{Rv_m}\Theta_{xx} + \varepsilon_x,\ \varepsilon \approx 0. \tag{5.13}$$

Combining (5.9) and (5.13), we obtain

$$\Theta_t = \frac{(\gamma-1)\mu}{\gamma R v_m}\Theta_{xx} + \varepsilon_x,\ \varepsilon \approx 0. \tag{5.14}$$

Therefore we choose the function Θ

$$\Theta(x,t) = \frac{\alpha_2}{\sqrt{4\pi\tilde{a}(1+t)}}e^{-\frac{x^2}{4\tilde{a}(1+t)}},\ \tilde{a} = \frac{(\gamma-1)\mu}{\gamma R v_m} > 0 \tag{5.15}$$

as the unique solution of

$$\Theta_t = \tilde{a}\Theta_{xx},\ \Theta|_{t=-1} = \alpha_2\delta(x),\ \int_{-\infty}^{\infty}\Theta(x,t)dx = \alpha_2, \tag{5.16}$$

and also define f and g by

$$f(x,t) = \tilde{a}\Theta_x,\ g(x,t) = \tilde{a}u_m\Theta_x. \tag{5.17}$$

It is clear that (Θ, f, g) defined above satisfy (5.3)-(5.5), and so (5.11), (5.12), (5.6) and (5.7) are justified.

Now we return to (5.2). Substituting (Θ, f, g) defined as in (5.15)-(5.17) into (5.2), we finally reach the defintion of $\tilde{U}$,

$$\begin{aligned}
\tilde{v} &= V_1(x - s_1t + \alpha_1) + V_3(x - s_3t + \alpha_3) - v_m + \Theta,\\
\tilde{u} &= U_1(x - s_1t + \alpha_1) + U_3(x - s_3t + \alpha_3) - u_m + \tilde{a}\Theta_x,\\
\tilde{E} &= E_1(x - s_1t + \alpha_1) + E_3(x - s_3t + \alpha_3) - E_m + \frac{p_m}{\gamma-1}\Theta + \tilde{a}u_m\Theta_x.
\end{aligned} \tag{5.18}$$

It follows from (5.1) that

$$\begin{aligned}
&\int_{-\infty}^{\infty}(U(x,0) - \tilde{U}(x,0))dx\\
&= \int_{-\infty}^{\infty}(U(x,0) - \bar{U}(x,0))dx + \int_{-\infty}^{\infty}(\bar{U}(x,0) - \tilde{U}(x,0))dx\\
&= \sum_{i=1}^{3}\alpha_i\tilde{r}_i + \sum_{i\neq 2}\int_{-\infty}^{\infty}(V_i(x) - V_i(x+\alpha_i), U_i(x) - U_i(x+\alpha_i),\\
&\qquad E_i(x) - E_i(x+\alpha_i))^t - \alpha_2\tilde{r}_2 = 0,
\end{aligned} \tag{5.19}$$

where we have used the fact that $\int_{-\infty}^{\infty} \Theta dx = \alpha_2$. Thus $\tilde{U}$ is the desired ansatz.

The following theorem is about L^2 bounds for $U - \tilde{U}$ and its piecewise H^1-norm for $U - \tilde{U}$ at time $t \leq \delta^{-2-\vartheta}$. These bounds give the control of the "initial" data for the energy estimates of Section 6, where we finally conclude the proof of Theorem 1.1.

Theorem 5.1. Under the condition of Theorem 1.1, if δ, and ϑ are sufficiently small, then for $t \leq \delta^{-2-\vartheta}$, it holds that

$$||U - \tilde{U}\|\!|^2 + \|(U - \tilde{U})_x\|\!|^2 \leq C \sum_{a-2b\geq 1} \delta^a (t+1)^b, \tag{5.20}$$

$$\left|\left|\int_{-\infty}^{x} (U - \tilde{U})(y,t)dy\right|\right|^2 \leq C[\sum_{a-2b\geq 0} \delta^a (t+1)^b \\ +(\delta^{-1} + (t+1)^{1/2})e^{-\delta^2 t/C} + \delta^2 (t+1)^{3/2} e^{-t/C}]. \tag{5.21}$$

In addition, there is a positive number M, depending only on U_- and U_+, such that, if

$$M\delta^{-2} \log \delta^{-1} \leq t \leq \delta^{-2-\vartheta}$$

(which is possible for small δ), then at time t,

$$||U - \tilde{U}\|\!|^2 + \|(U - \tilde{U})_x\|\!|^2 \leq C\delta^{1-\vartheta\mathcal{B}} \tag{5.22}$$

and

$$\left|\left|\int_{-\infty}^{x} (U - \tilde{U})(y,t)dy\right|\right|^2 \leq C\delta^{-\vartheta\mathcal{B}}, \tag{5.23}$$

where $\mathcal{B}$ is a positive constant depending only on U_-.

Proof: It is clear that the bounds (5.22) and (5.23) follow from (5.20) and (5.21), respectively. Therefore, it is sufficient to prove (5.20) and (5.21).

To prove (5.20), we triangulate as follows:

$$U - \tilde{U} = (U - \bar{U}_R - \bar{\bar{U}}) + (\bar{U}_R - \bar{U}_{TW}) + \bar{\bar{U}} \\ +(\bar{U}_{TW} - \bar{U}_{\alpha_1,\alpha_3}) - (\Theta, \tilde{a}\Theta_x, \frac{p_m}{\gamma - 1}\Theta + \tilde{a}u_m\Theta_x)^t. \tag{5.24}$$

By virtue of (3.32), (2.45), (2.50), (4.6), (5.15) and (5.24), we have

$$||U - \tilde{U}\|\!|^2 \leq C \sum_{a-2b\geq 1} \delta^a (t+1)^b. \tag{5.25}$$

From (3.32) and (5.15), we obtain

$$\|(U-\tilde{U})_x\|^2 \le C\sum_{a-2b\ge 1}\delta^a(t+1)^b. \tag{5.26}$$

Thus, (5.20) follows from (5.25) and (5.26) immediately. To prove (5.21), we treat the case $x\le 0$ and $x\ge 0$ separately. First, for $x\le 0$,

$$\begin{aligned} U-\tilde{U} &= (U-\bar{U}_R-\bar{\bar{U}}) + (\bar{U}_R^1-\bar{U}_{TW}^1) + (\bar{U}_R^2-U_{m_1}) + (\bar{U}_R^3-U_{m_2}) + \bar{\bar{U}} \\ &\quad + (\bar{U}_{TW}^1-\bar{U}_{1,\alpha_1}) + (U_m-\bar{U}_{3,\alpha_3}) - (\Theta, \tilde{a}\Theta_x, \frac{p_m}{\gamma-1}\Theta+\tilde{a}u_m\Theta_x)^t. \end{aligned} \tag{5.27}$$

From the definition of $\bar{U}_R^2$ in (2.17), we have

$$\begin{aligned} \int_{-\infty}^0\left|\int_{-\infty}^x(\bar{U}_R^2-U_{m_1})(y,t)dy\right|^2dx &\le C|U_{m_1}-U_{m_2}|^2\int_{-\infty}^0\left|\int_{-\infty}^x e^{-\frac{y^2}{4\beta_2 t}}dy\right|^2dx \\ &\le C|U_{m_1}-U_{m_2}|^2 t\int_{-\infty}^0 e^{-\frac{x^2}{2\beta_2 t}}dx \le C\delta^6 t^{\frac{3}{2}}. \end{aligned} \tag{5.28}$$

Again from (5.15), we also have

$$\int_{-\infty}^0\left|\int_{-\infty}^x(\Theta, \tilde{a}\Theta_x, \frac{p_m}{\gamma-1}\Theta+\tilde{a}u_m\Theta_x)^t(y,t)dy\right|^2dx \le C. \tag{5.29}$$

So, from (3.31), (2.46), (5.28), (2.33), (2.57), (4.7), (4.10), (5.29) and (5.27), we can obtain the desired estimate for

$$\int_{-\infty}^0\left|\int_{-\infty}^x(U-\tilde{U})(y,t)dy\right|^2dx.$$

For $x\ge 0$ we triangulate differently:

$$\begin{aligned} U-\tilde{U} &= (U-\bar{U}_R-\bar{\bar{U}}) + (\bar{U}_R^3-\bar{U}_{TW}^3) + (\bar{U}_R^1-U_{m_1}) + (\bar{U}_R^2-U_{m_2}) + \bar{\bar{U}} \\ &\quad + (\bar{U}_{TW}^3-\bar{U}_{3,\alpha_3}) + (U_m-\bar{U}_{1,\alpha_1}) - (\Theta, \tilde{a}\Theta_x, \frac{p_m}{\gamma-1}\Theta+\tilde{a}u_m\Theta_x)^t. \end{aligned} \tag{5.30}$$

Similarly, by combining (2.1), (5.6) and (5.19), and noting that

$$\int_{-\infty}^x(U-\tilde{U})(y,t)dy = -\int_x^\infty(U-\tilde{U})(y,t)dy,$$

we finally get (5.21).

The proof of Theorem 5.1 is completed.

6. Proof of Theorem 1.1

In this section, we combine the results of the previous sections to complete the proof of Theorem 1.1. Setting

$$(\bar{\Phi}, \bar{\Psi}, \bar{Z}) = \int_{-\infty}^{x} (U - \tilde{U})(y,t)dy, \tag{6.1}$$

then,

$$v - \tilde{v} = \bar{\Phi}_x,\ u - \tilde{u} = \bar{\Psi}_x,\ \frac{R}{\gamma - 1}(\theta - \tilde{\theta}) + \frac{1}{2}|\bar{\Psi}_x|^2 + \tilde{u}\bar{\Psi}_x = \bar{Z}_x. \tag{6.2}$$

Instead of the variable $\bar{Z}$, which is the anti-derivative of the total energy, it is more convenient to introduce another variable related to the absolute temperature

$$Z = \frac{\gamma - 1}{R}(\bar{Z} - \tilde{u}\bar{\Psi}),$$

which turns (6.2) into

$$v - \tilde{v} = \bar{\Phi}_x,\ u - \tilde{u} = \bar{\Psi}_x,\ \theta - \tilde{\theta} = Z_x - \frac{\gamma - 1}{R}(\frac{1}{2}|\bar{\Psi}_x|^2 + \tilde{u}_x\bar{\Psi}). \tag{6.3}$$

Subtracting (5.6) from the system (2.1) and integrating the resulting system, we have the following system for $(\bar{\Phi}, \bar{\Psi}, Z)$:

$$\begin{cases} \bar{\Phi}_t - \bar{\Psi}_x = 0, \\ \bar{\Psi}_t - (\frac{\tilde{p}}{\tilde{v}} - \frac{\tilde{u}_x}{\tilde{v}^2})\bar{\Phi}_x + \frac{R}{\tilde{v}}Z_x + \frac{\gamma - 1}{\tilde{v}}\tilde{u}_x\bar{\Psi} = \frac{1}{\tilde{v}}\bar{\Psi}_{xx} + \tilde{J}_1 - \tilde{R}_1, \\ \frac{R}{\gamma - 1}Z_t + (\tilde{p} - \frac{\tilde{u}_x}{\tilde{v}})\bar{\Psi}_x + \tilde{u}_t\bar{\Psi} - \frac{(\gamma - 1)\mu}{\tilde{v}R}(\tilde{u}_x\bar{\Psi})_x + \frac{\mu\tilde{\theta}_x}{v\tilde{v}}\bar{\Phi}_x \\ = \frac{\mu}{\tilde{v}}Z_{xx} + \tilde{J}_2 - \tilde{R}_3, \end{cases} \tag{6.4}$$

where $\tilde{R}_3 = \tilde{R}_2 - \tilde{u}\tilde{R}_1$ and

$$\begin{cases} \tilde{J}_1 = \frac{\gamma - 1}{2\tilde{v}}\bar{\Psi}_x^2 + \frac{1}{v\tilde{v}}\tilde{u}_x\bar{\Phi}_x^2 - \frac{1}{v\tilde{v}}\bar{\Psi}_{xx}\Phi_x - (p - \tilde{p} + \frac{\tilde{p}}{\tilde{v}}\bar{\Phi}_x - \frac{R}{\tilde{v}}(\theta - \tilde{\theta})), \\ \tilde{J}_2 = (\tilde{p} - p)\Psi_x + (\frac{u_x}{v} - \frac{\tilde{u}_x}{\tilde{v}})\bar{\Psi}_x - \frac{(\gamma - 1)\mu}{R\tilde{v}}\bar{\Psi}_x\bar{\Psi}_{xx} - \frac{\mu\bar{\Phi}_x(\theta - \tilde{\theta})_x}{v\tilde{v}}. \end{cases} \tag{6.5}$$

In what follows, we use the notations

$$\bar{\phi} = \bar{\Phi}_x, \quad \bar{\psi} = \bar{\Psi}_x, \quad \zeta = Z_x - \frac{\gamma - 1}{R}(\frac{1}{2}\bar{\Psi}_x^2 - \tilde{u}_x\bar{\Psi}). \tag{6.6}$$

which exactly correspond to $v - \tilde{v}$, $u - \tilde{u}$ and $\theta - \tilde{\theta}$ by (6.3). Following the proof of Theorem 1 in [12] and using the results of the previous sections, we first show the following a priori estimates:

Lemma 6.1. For each U_-, there exist positive constants $C = C(U_-)$, η_0 and $\delta^{\frac{1}{4}}/C \le \eta_1 \le C\delta^{\frac{1}{4}}$ such that, if $\delta = |U_- - U_+| < \eta_0$, and if a solution U of (2.1)-(1.5) exists for $M\delta^{-2}\log\delta^{-1} = t_0 \le t \le t_1$ and satisfies

$$N_0(t_0,t_1) = \sup_{t_0\le\tau\le t_1} \|(\bar{\Phi},\bar{\Psi},Z)(\cdot,\tau)\|_{L^\infty} \le \eta_0,$$
$$N_1(t_0,t_1) = \sup_{t_0\le\tau\le t_1} \|(\bar{\phi},\bar{\psi},\zeta)(\cdot,\tau)\|\!\!| + \|(\bar{\phi}_x,\bar{\psi}_x,\zeta_x)(\cdot,\tau)\|\!\!| \le \eta_1,$$

then

$$\begin{aligned}&\sup_{t_0\le\tau\le t_1} \|(\bar{\Phi},\bar{\Psi},Z)(\tau)\|^2 + \int_{t_0}^{t_1} \|(\bar{\Phi}_x,\bar{\Psi}_x,Z_x)\|\!\!|^2 d\tau \\ &\quad + \int_{t_0}^{t_1}\int_R (|(U_{1,\alpha_1})_x| + |(U_{2,\alpha_2})_x|)(\bar{\Psi}^2 + Z^2)dxd\tau \\ &\le C\left[\|(\bar{\Phi},\bar{\Psi},Z)(t_0)\|^2 + \|\bar{\Phi}_x(t_0)\|\!\!|^2 + \delta^{\frac{1}{2}} + \int_{t_0}^{t_1}(\|\zeta_x\|\!\!|^2 + \|\Psi_{xx}\|\!\!|^2)d\tau\right],\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}&\sup_{t_0\le\tau\le t_1} [\|(\bar{\phi},\bar{\psi},\zeta)(\tau)\|\!\!|^2 + \|(\bar{\phi}_x,\bar{\psi}_x,\zeta_x)(\tau)\|\!\!|^2] \\ &\quad + \int_{t_0}^{t_1} [\|(\bar{\phi}_x,\bar{\psi}_x,\zeta_x)\|\!\!|^2 + \|(\bar{\psi}_{xx},\zeta_{xx})\|\!\!|^2]d\tau \\ &\le C\left[\|(\bar{\phi},\bar{\psi},\zeta)(t_0)\|^2 + \|(\bar{\phi}_x,\bar{\psi}_x,\zeta_x)(t_0)\|\!\!|^2 + \delta\int_{t_0}^{t_1} \|(\bar{\phi},\bar{\psi},\zeta)\|\!\!|^2 dt + \delta\right].\end{aligned} \tag{6.8}$$

Remark 6.2. In contrast to [12], our energy estimates are new and different here. Indeed, the proof of main theorem in [12] depends crucially on the smallness of the H^2–norm of the spatial antiderivative of the initial perturbation (see the assumptions of Theorem 1 on page 850 of [12]). However, as mentioned before, the H^2–norm of $\int_{-\infty}^{x}(U(y,t) - \bar{U}_{\alpha_1,\alpha_3}(y,t))dy$ is arbitrarily large here. Therefore, the energy methods in [12] do not work here (see the proofs of (6.7) and (6.8)). Furthermore, we need to deal with the boundary integral terms arising from the non–smooth initial perturbations carefully (see (6.41)).

Proof: It is clear that $\inf(\tilde{p} - \frac{\tilde{u}_x}{\tilde{v}}) > 0$ due to Lemma 4.1 and the smallness of δ. Let $L = \inf(\tilde{p} - \frac{\tilde{u}_x}{\tilde{v}})^{-1}$. Then, as in [12], multiplying $(6.4)_1$ by $\bar{\Phi}$, $(6.4)_2$ by $\tilde{v}L\bar{\Psi}$, and $(6.4)_3$ by RL^2Z respectively, adding all the resultant equations, we obtain

$$E_{1,t} + E_2 + E_3 + E_4 = -\tilde{v}L\tilde{R}_1\bar{\Psi} + \tilde{v}L\tilde{J}_1\bar{\Psi} - \tilde{R}_3RL^2Z + RL^2\tilde{J}_2Z + E_{5,x}, \tag{6.9}$$

with

$$
\begin{aligned}
&E_1 = \frac{1}{2}(\bar{\Phi}^2 + \frac{R^2}{\gamma-1}L^2Z^2 + \tilde{v}L\bar{\Psi}^2),\\
&E_2 = A\bar{\Psi}^2 + L_x\bar{\Psi}_x\bar{\Psi} + L\bar{\Psi}_x^2,\ A = -\frac{1}{2}(\tilde{v}L)_t + (\gamma-1)L\tilde{u}_x,\\
&E_3 = -\frac{R^2}{\gamma-1}LL_tZ^2 + (\frac{R\mu}{\tilde{v}}L^2)_xZZ_x + \frac{R\mu}{\tilde{v}}L^2Z_x^2,\\
&E_4 = R(L^2\tilde{u}_t - L_x)\bar{\Psi}Z + \frac{(\gamma-1)\mu}{\tilde{v}}\tilde{u}_xL^2Z_x\bar{\Psi}\\
&\qquad +(\gamma-1)\mu\tilde{u}_x(\frac{L^2}{\tilde{v}})_x\bar{\Psi}Z + \frac{\mu}{v\tilde{v}}RL^2\tilde{\theta}_x\bar{\Phi}_xZ,\\
&E_5 = L\bar{\Psi}\bar{\Psi}_x + \frac{R\mu}{\tilde{v}}L^2ZZ_x - RL\bar{\Psi}Z + \frac{(\gamma-1)\mu}{\tilde{v}}\tilde{u}_xL^2\bar{\Psi}Z + \bar{\Phi}\bar{\Psi}.
\end{aligned}
\tag{6.10}
$$

We estimate $E_i (i = 2, 3, 4)$ one by one. From (5.11) and (5.12), we have

$$
\begin{aligned}
L_t &\approx -\frac{(P_1 - \frac{U_{1,x}}{V_1})_t + (P_3 - \frac{U_{3,x}}{V_3})}{P_1 + P_3 - p_m - \frac{U_{1,x}}{V_1} - \frac{U_{3,x}}{V_3}} \approx -\frac{(P_1 - \frac{U_{1,x}}{V_1})_t}{(P_1 - \frac{U_{1,x}}{V_1})^2} - \frac{(P_3 - \frac{U_{3,x}}{V_3})_t}{(P_3 - \frac{U_{3,x}}{V_3})^2},\\
&= L_{1,t} + L_{3,t},\\
L_x &\approx L_{1,x} + L_{3,x},
\end{aligned}
\tag{6.11}
$$

where $L_i = (P_i - \dfrac{U_{i,x}}{V_i})^{-1},\ i = 1, 3$, which, together with (5.6), implies

$$
A \approx (-\frac{1}{2}(V_1L_1)_t + (\gamma-1)L_1U_{1,x}) + (-\frac{1}{2}(V_3L_3)_t + (\gamma-1)L_3U_{3,x}) =: A_1 + A_3. \tag{6.12}
$$

We will treat all the errors arising from the relation " $\approx$ " later. Furthermore, we can obtain from Lemma 4.1, that for $i = 1, 3$,

$$
A_i = \frac{1}{2}s_iV_{i,x}L_i + \frac{1}{2}s_iV_iL_i^2s_i^2V_{i,x} + (\gamma-1)L_iU_{i,x} \geq C|U_{i,x}|((3-\gamma)p_m - C\delta_i). \tag{6.13}
$$

Therefore, if δ is small, we get the following inequality for E_2 with the help of the Cauchy inequality and Lemma 4.1,

$$
E_2 \geq C\{(|U_{1,x}| + |U_{3,x}|)\bar{\Psi}^2 + \bar{\Psi}_x^2\}. \tag{6.14}
$$

For E_3, we observe that

$$
-L_{1,t} - L_{3,t} = s_1L_1^2s_1^2V_{1,x} + s_3L_3^2V_{3,x} = |U_{1,x}|s_1^2L_1^2 + |U_{3,x}|s_3^2L_3^2, \tag{6.15}
$$

$$
(\frac{R\mu}{\tilde{v}}L^2)_xZZ_x \leq \frac{R\mu}{2\tilde{v}}L^2Z_x^2 + C(|U_{1,x}|^2 + |U_{3,x}|^2 + \Theta_x^2)Z^2,
$$

and the fact $\Theta_x^2 \approx 0$, we have

$$E_3 \geq C\{(|U_{1,x}| + |U_{3,x}|)Z^2 + Z_x^2\}. \tag{6.16}$$

On the other hand, we note that

$$L^2\tilde{u}_t - L_x = L^2(\tilde{u}_t + \tilde{p}_x - (\frac{\tilde{u}_x}{\tilde{v}})_x) = -L^2\tilde{R}_{1,x} \approx 0. \tag{6.17}$$

Thus the Cauchy inequality and Lemma 4.1 give

$$E_4 \leq C\delta(|U_{1,x}| + |U_{3,x}|)(\bar{\Psi}^2 + Z^2) + C\delta(Z_x^2 + \bar{\Phi}_x^2). \tag{6.18}$$

Next, we estimate the terms $\tilde{J}_1\bar{\Psi}$ and $\tilde{J}_2 Z$. From (6.5), (6.6) and the a priori assumptions , we can get

$$\begin{aligned} |\tilde{J}_1\bar{\Psi}| &\leq CN_0(t_0,t_1)(\bar{\Phi}_x^2 + \bar{\Psi}_x^2 + Z_x^2 + \bar{\Psi}_{xx}^2) + C\delta^2(|U_{1,x}| + |U_{3,x}|)\bar{\Psi}^2, \\ |\tilde{J}_2 Z| &\leq CN_0(t_0,t_1)(\bar{\Phi}_x^2 + \bar{\Psi}_x^2 + Z_x^2 + \zeta^2 + \bar{\Psi}_{xx}^2) + C\delta^2(|U_{1,x}| + |U_{3,x}|)\bar{\Psi}^2. \end{aligned} \tag{6.19}$$

Finally, we estimate all the error terms arising from the relation " $\approx$ " and also the terms $\tilde{R}_1\bar{\Psi}$ and $\tilde{R}_3 Z$. It is easy to see that all the integrals of these terms on $(t_0,t_1)\times\mathbf{R}$ can be written as follows:

$$\begin{aligned} &\int_{t_0}^{t_1}\!\!\!\!\!\!\!\!\!\!\quad\;\; \int |\tilde{R}|[(\bar{\Psi}^2 + |Z\bar{\Psi}| + Z^2) + (|\bar{\Psi}\bar{\Psi}_x| + |ZZ_x| + |Z_x\bar{\Psi}| + |\bar{\Phi}_x Z|) \\ &\quad +(|\bar{\Psi}| + |Z|)] =: M_1 + M_2 + M_3. \end{aligned} \tag{6.20}$$

We estimate the terms $M_i(\ i = 1,2,3)$ one by one. Firstly, as in [12], we have

$$\begin{aligned} M_1 &\leq C\int_0^t\!\!\int |[(\delta^2 + |\alpha_2|\delta^{\frac{3}{2}})e^{-c\delta(|x|+\tau)} + \frac{|\alpha_2|}{(1+\tau)^{\frac{3}{2}}}e^{-\frac{cx^2}{1+\tau}} + (\delta + |\alpha_2|)e^{-c(|x|+\tau)}](\bar{\Psi}^2 + Z^2) \\ &\leq C(\delta^{\frac{1}{2}} + \delta^{-\frac{1}{2}}e^{-ct_0} + t_0^{-\frac{1}{2}})\sup_{t_0\leq\tau\leq t_1}\|(\bar{\Psi},Z)\|^2 \leq C\delta^{\frac{1}{2}}\sup_{t_0\leq\tau\leq t_1}\|(\bar{\Psi},Z)\|^2. \end{aligned} \tag{6.21}$$

Similarly, we have

$$\begin{aligned} M_2 &\leq C\int_{t_0}^{t_1}\!\!\int |\tilde{R}|(\bar{\Psi}^2 + Z^2 + \bar{\Phi}_x^2 + \bar{\Psi}_x^2 + Z_x^2) \\ &\leq C\delta^{\frac{1}{2}}\sup_{t_0\leq\tau\leq t_1}\|(\bar{\Psi},Z)\|^2 + C\delta\int_{t_0}^{t_1}\|(\bar{\Phi}_x,\bar{\Psi}_x,Z_x)\|^2, \end{aligned} \tag{6.22}$$

and

$$M_3 \leq C\int_{t_0}^{t_1}\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\! \qquad \int |\tilde{R}|^{\frac{5}{6}}(\bar{\Psi}^2+Z^2)+C\int_{t_0}^{t_1}\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\qquad \int |\tilde{R}|^{\frac{7}{6}} \leq C\delta^{\frac{1}{2}}\sup_{t_0\leq\tau\leq t_1}\|(\bar{\Psi},Z)\|^2+C\delta^{\frac{1}{2}}. \tag{6.23}$$

Combining the relations (6.20)-(6.23), we conclude

$$\begin{aligned}&\int_{t_0}^{t_1}\!\!\int |\tilde{R}|[(\bar{\Psi}^2+|Z\bar{\Psi}|+Z^2)+(|\bar{\Psi}\bar{\Psi}_x|+|ZZ_x|+|Z_x\bar{\Psi}|+|\bar{\Phi}_xZ|)\\ &\quad+(|\bar{\Psi}|+|Z|)]\leq C\delta^{\frac{1}{2}}\sup_{t_0\leq\tau\leq t_1}\|(\bar{\Psi},Z)\|^2+C\delta\int_{t_0}^{t_1}\|(\bar{\Phi}_x,\bar{\Psi}_x,Z_x)\|^2+C\delta^{\frac{1}{2}}.\end{aligned} \tag{6.24}$$

Integrating (6.9) over $(t_0,t_1)\times\mathbf{R}$ and using (6.10), (6.14), (6.16), (6.18), (6.19), (6.24) and the a priori assumptions, we obtain

$$\begin{aligned}&\sup_{t_0\leq\tau\leq t_1}\|(\bar{\Phi},\bar{\Psi},Z)(\tau)\|^2+\int_{t_0}^{t_1}\|(\bar{\Psi}_x,Z_x)\|^2d\tau\\ &\quad+\int_{t_0}^{t_1}\int_R(|U_{1,x}|+|U_{3,x}|)(\bar{\Psi}^2+Z^2)dxd\tau\\ &\leq C\|(\bar{\Phi},\bar{\Psi},Z)(t_0)\|^2+C(N_0(t_0,t_1)+\delta)\int_{t_0}^{t_1}(\|\bar{\Phi}_x\|^2+\|\zeta_x\|^2+\|\bar{\Psi}_{xx}\|^2)d\tau+C\delta^{\frac{1}{2}}.\end{aligned} \tag{6.25}$$

Next, we estimate the term $\int_{t_0}^{t_1}\|\bar{\Phi}_x\|^2d\tau$. From $(6.4)_1$ and $(6.4)_2$, we have

$$\frac{1}{\tilde{v}}\bar{\Phi}_{xt}-\bar{\Psi}_t+\frac{1}{\tilde{v}L}\bar{\Phi}_x=\frac{R}{\tilde{v}}Z_x+\frac{\gamma-1}{\tilde{v}}\tilde{u}_x\bar{\Psi}+\tilde{R}_1-\tilde{J}_1. \tag{6.26}$$

Multiplying (6.26) by $\bar{\Phi}_x$ yields

$$(\frac{1}{2\tilde{v}}\bar{\Phi}_x^2)_t-(\frac{1}{2\tilde{v}})_t\bar{\Phi}_x^2-\bar{\Phi}_x\bar{\Psi}_t+\frac{1}{\tilde{v}L}\bar{\Phi}_x^2=(\frac{R}{\tilde{v}}Z_x+\frac{\gamma-1}{\tilde{v}}\tilde{u}_x\bar{\Psi}+\tilde{R}_1-\tilde{J}_1)\bar{\Phi}_x. \tag{6.27}$$

Since

$$\bar{\Phi}_x\bar{\Psi}_t=(\bar{\Phi}_x\bar{\Psi})_t-(\bar{\Phi}_t\bar{\Psi})_x+\bar{\Psi}_x^2, \tag{6.28}$$

we get,

$$\begin{aligned}&\sup_{t_0\leq\tau\leq t_1}\|\bar{\Phi}_x\|^2+\int_{t_0}^{t_1}\|\bar{\Phi}_x\|^2d\tau\leq C[\sup_{t_0\leq\tau\leq t_1}\|\bar{\Psi}\|^2+\|\bar{\Phi}_x(t_0)\|^2+\int_{t_0}^{t_1}\|(\bar{\Psi}_x,Z_x\|^2d\tau\\ &\quad+\delta\int_{t_0}^{t_1}\int_R(|U_{1,x}|+|U_{3,x}|)\bar{\Psi}^2dxd\tau+\int_{t_0}^{t_1}\int_R(\tilde{J}_1^2+\tilde{R}_1^2)dxd\tau.\end{aligned} \tag{6.29}$$

By virtue of (6.5), Cauchy inequality and Sobolev inequality, we have

$$\begin{aligned}\int_{t_0}^{t_1}\int_R \tilde{J}_1^2 dxd\tau \quad &\le CN_1^2(t_0,t_1)\int_{t_0}^{t_1}\|(\bar{\Phi}_x,\bar{\Psi}_x,Z_x,)\|^2 d\tau \\ &+CN_1^2(t_0,t_1)\int_{t_0}^{t_1}\|\bar{\Psi}_{xx}\|^2 + C\delta\int_{t_0}^{t_1}\int_R(|U_{1,x}|+|U_{3,x}|)\bar{\Psi}^2 dxd\tau. \end{aligned} \tag{6.30}$$

From (5.7), it is easy to see

$$\int_{t_0}^{t_1}\int_R \tilde{R}_1^2 dxd\tau \le C\delta^2. \tag{6.31}$$

Then, (6.7) follows from (6.25) and (6.29)-(6.31) immediately.

We now prove (6.8). Applying ∂_x to the system (6.4) and using the notations (6.6), we obtain

$$\begin{cases}\bar{\phi}_t-\bar{\psi}_x=0,\\ \bar{\psi}_t-\frac{\tilde{p}}{\tilde{v}}\bar{\phi}_x+\frac{R}{\tilde{v}}\zeta_x-(\frac{\tilde{p}}{\tilde{v}})_x\bar{\phi}+(\frac{R}{\tilde{v}})_x\zeta-(\frac{1}{\tilde{v}})_x\bar{\psi}_x+(\frac{\tilde{u}_x\bar{\phi}}{\tilde{v}^2})_x=\frac{1}{\tilde{v}}\bar{\psi}_{xx}-\tilde{R}_{1,x}+\tilde{Q}_{1,x},\\ \frac{R}{\gamma-1}\zeta_t+\tilde{p}\bar{\psi}_x+(p-\tilde{p})\tilde{u}_x+\frac{1}{v\tilde{v}}\tilde{u}_x^2\bar{\phi}-\frac{2}{v}\tilde{u}_x\bar{\psi}_x-(\frac{\mu}{\tilde{v}})_x\zeta_x+(\frac{\mu\tilde{\theta}_x\bar{\phi}}{\tilde{v}^2})_x\\ =\frac{\mu}{\tilde{v}}\zeta_{xx}+\tilde{u}\tilde{R}_{1,x}-\tilde{R}_{2,x}+(p-\tilde{p})\bar{\psi}_x+\frac{\bar{\psi}_x}{v}+\tilde{Q}_{2,x},\end{cases} \tag{6.32}$$

where

$$\tilde{Q}_1=\left\{\frac{1}{v\tilde{v}^2}\tilde{u}_x\bar{\phi}^2-\frac{1}{v\tilde{v}^2}\tilde{\psi}_x\bar{\phi}-\left(p-\tilde{p}+\frac{p}{v}\bar{\phi}-\frac{R}{\tilde{v}}(\theta-\tilde{\theta})\right)\right\}, \tag{6.33}$$

$$\tilde{Q}_2=\left(\frac{\mu\tilde{\theta}_x}{v\tilde{v}^2}\bar{\phi}^2\right)-\left(\frac{\mu\bar{\phi}\zeta_x}{v\tilde{v}}\right). \tag{6.34}$$

Multiplying $(6.32)_1$ by $\bar{\phi}$, $(6.32)_2$ by $\frac{\tilde{v}}{\tilde{p}}\bar{\psi}$, and $(6.32)_3$ by $\frac{R}{\tilde{p}^2}\zeta$, and adding and integrating, we find

$$\begin{aligned}&\sup_{t_0\le\tau\le t_1}\|(\bar{\phi},\bar{\psi},\zeta)(\tau)\|^2+\int_{t_0}^{t_1}\|(\bar{\psi}_x,\zeta_x)\|^2 d\tau\\ &\le C\|(\bar{\phi},\bar{\psi},\zeta)(t_0)\|^2+C\int_{t_0}^{t_1}\int\Big\{|(\frac{\tilde{v}}{2\tilde{p}})_t|\bar{\psi}^2+|(\frac{R^2\tilde{v}}{2(\gamma-1)\tilde{p}^2})_t|\zeta^2+|(\frac{1}{\tilde{p}})_x\bar{\psi}\bar{\psi}_x|\\ &\quad-|(\frac{\mu R}{\tilde{v}\tilde{p}^2})_x\zeta\zeta_x|+|(\frac{\mu R}{\tilde{p}^2}\zeta)_x\frac{\tilde{\theta}_x\bar{\phi}}{\tilde{v}}|+|\frac{\tilde{u}_x\bar{\phi}}{\tilde{v}^2}(\frac{\tilde{v}}{\tilde{p}}\bar{\psi})_x|+|(\frac{\tilde{p}}{\tilde{v}})_x\frac{\tilde{v}}{\tilde{p}}\bar{\phi}\bar{\psi}+(\frac{R}{\tilde{p}})_x\bar{\psi}\zeta-(\frac{R}{\tilde{v}})_x\frac{\tilde{v}}{\tilde{p}}\bar{\psi}\zeta|\\ &\quad+|[(p-\tilde{p})\tilde{u}_x+\frac{1}{v\tilde{v}}\tilde{u}_x\bar{\phi}-\frac{2}{v}\tilde{u}_x\bar{\psi}_x-(\frac{\mu}{\tilde{v}})_x\zeta_x]\frac{R\zeta}{\tilde{p}^2}|+[-\tilde{R}_{1,x}+\tilde{Q}_{1,x}]\frac{\tilde{v}}{\tilde{p}}\bar{\psi}\\ &\quad+[\tilde{u}\tilde{R}_{1,x}-\tilde{R}_{2,x}+\tilde{Q}_{2,x}]\frac{R\zeta}{\tilde{p}^2}+|[(p-\tilde{p})\bar{\psi}_x+\frac{\bar{\psi}_x^2}{v}]\frac{R}{\tilde{p}^2}\zeta|\end{aligned}$$

$$-(-\bar{\phi}\bar{\psi}+\frac{R}{\tilde{p}}\bar{\psi}\zeta-\frac{1}{\tilde{p}}\bar{\psi}\bar{\psi}_x-\frac{\mu R}{\tilde{p}^2\tilde{v}}\zeta\zeta_x+\frac{\mu R\tilde{\theta}_x}{\tilde{v}^2\tilde{p}^2}\bar{\phi}\zeta+\frac{\tilde{u}_x}{\tilde{v}\tilde{p}}\bar{\phi}\bar{\psi})_x$$

$$=: C\|(\bar{\phi},\bar{\psi},\zeta)(t_0)\|^2+\sum_{i=1}^{12}\tilde{A}_i. \tag{6.35}$$

Next, we estimate the terms $\tilde{A}_i$, $i=1$ to 12. It is easy to see

$$\sum_{i=1}^{8}\tilde{A}_i\le C\delta\int_{t_0}^{t_1}[\|(\bar{\phi},\bar{\psi},\zeta)\|^2+\|(\bar{\psi}_x,\zeta_x)\|^2], \tag{6.36}$$

Using integration by parts, we have

$$\tilde{A}_9\le C\int_{t_0}^{t_1}\fint|\tilde{R}_{1,x}||\bar{\psi}|+|\tilde{Q}_1(\frac{\tilde{v}}{\tilde{p}}\bar{\psi})_x|-\int_{t_0}^{t_1}[\tilde{Q}_1\bar{\psi}](0,\tau)d\tau=:\tilde{A}_9^1+\tilde{A}_9^2+\tilde{A}_9^3, \tag{6.37}$$

It is clear that

$$\tilde{A}_9^1\le C\delta+C\delta\int_{t_0}^{t_1}\|\bar{\psi}\|^2d\tau.$$

By virtue of Cauchy inequality, Hőlder inequality and Sobolev inequality, we have

$$\begin{aligned}
\tilde{A}_9^2 &\le C\int_{t_0}^{t_1}\fint(\bar{\phi}^2+\zeta^2+|\bar{\psi}\bar{\phi}|)|\bar{\psi}|dxd\tau\\
&\le CN_1(t_0,t_1)\int_{t_0}^{t_1}\|\bar{\psi}_x\|^2d\tau+C\int_{t_0}^{t_1}(\|\bar{\phi}\|^{\frac{3}{2}}\|\bar{\phi}_x\|^{\frac{1}{2}}\|\bar{\psi}_x\|+\|\zeta\|^{\frac{3}{2}}\|\zeta_x\|^{\frac{1}{2}}\|\bar{\psi}_x\|)d\tau\\
&\le CN_1(t_0,t_1)\int_{t_0}^{t_1}\|\bar{\psi}_x\|^2d\tau+C\alpha\int_{t_0}^{t_1}\|\bar{\phi}_x\|^2d\tau\\
&\quad+\frac{1}{8}\int_{t_0}^{t_1}\|(\bar{\psi}_x,\zeta_x)\|^2d\tau+C\int_{t_0}^{t_1}\|(\bar{\phi},\zeta)\|^6d\tau\\
&\le CN_1(t_0,t_1)\int_{t_0}^{t_1}\|\bar{\psi}_x\|^2d\tau+C\alpha\int_{t_0}^{t_1}\|\bar{\phi}_x\|^2d\tau\\
&\quad+\frac{1}{8}\int_{t_0}^{t_1}\|(\bar{\psi}_x,\zeta_x)\|^2d\tau+CN_1^4(t_0,t_1)\int_{t_0}^{t_1}\|(\bar{\phi},\zeta)\|^2d\tau.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\tilde{A}_9 &\le C(\delta+N_1^4(t_0,t_1))\int_{t_0}^{t_1}\|(\bar{\phi},\zeta)\|^2d\tau+C\alpha\int_{t_0}^{t_1}\|\bar{\phi}_x\|^2d\tau\\
&\quad+(CN_1(t_0,t_1)+\frac{1}{8})\int_{t_0}^{t_1}\|(\bar{\psi}_x,\zeta_x)\|^2d\tau+\tilde{A}_9^3.
\end{aligned} \tag{6.38}$$

Similarly, we have

$$
\begin{aligned}
\tilde{A}_{10} &\leq C(\delta+N_1^4(t_0,t_1))\int_{t_0}^{t_1}\|(\bar{\phi},\zeta)\|^2d\tau+C\alpha\int_{t_0}^{t_1}\|\bar{\phi}_x\|^2d\tau\\
&\quad+(CN_1(t_0,t_1)+C\delta+\frac{1}{8})\int_{t_0}^{t_1}\|\zeta_x\|^2d\tau-C\int_{t_0}^{t_1}[\tilde{Q}_2\frac{R\zeta}{\tilde{p}^2}](0,\tau)d\tau.
\end{aligned}\tag{6.39}
$$

Using Cauchy inequality, we obtain

$$
\tilde{A}_{11}\leq CN_1^4(t_0,t_1)\int_{t_0}^{t_1}\|(\bar{\phi},\zeta)\|^2d\tau+(CN_1(t_0,t_1)+\frac{1}{8})\int_{t_0}^{t_1}\|\bar{\psi}_x\|^2d\tau.\tag{6.40}
$$

Finally, we deal with the boundary integral term arising from the jump discontinuous. Using Theorem 1.2 in [10], (6.33) and (6.34) and making direct computation, we obtain

$$
\int_{t_0}^{t_1}[-\bar{\phi}\bar{\psi}+\frac{R}{\tilde{p}}\bar{\psi}\zeta-\frac{1}{\tilde{p}}\bar{\psi}\bar{\psi}_x-\frac{\mu R}{\tilde{p}^2\tilde{v}}\zeta\zeta_x+\frac{\mu R\tilde{\theta}_x}{\tilde{v}^2\tilde{p}^2}\bar{\phi}\zeta+\frac{\tilde{u}_x}{\tilde{v}\tilde{p}}\bar{\phi}\bar{\psi}-\tilde{Q}_1\bar{\psi}-\tilde{Q}_2\frac{R\zeta}{\tilde{p}^2}](0,\tau)=0.\tag{6.41}
$$

Combining the above relations (6.35)-(6.41), we conclude

$$
\begin{aligned}
&\sup_{t_0\leq\tau\leq t_1}\|(\bar{\phi},\bar{\psi},\zeta)(\tau)\|^2+\int_{t_0}^{t_1}\|(\bar{\psi}_x,\zeta_x)\|^2d\tau\\
&\leq C\|(\bar{\phi},\bar{\psi},\zeta)(t_0)\|^2+C(\delta+N_1^4(t_0,t_1))\int_{t_0}^{t_1}\|(\bar{\phi},\bar{\psi},\zeta)\|^2d\tau\\
&\quad+C\alpha\int_{t_0}^{t_1}\|\bar{\phi}_x\|^2d\tau+C\delta.
\end{aligned}\tag{6.42}
$$

Next, we estimate the term $\int_{t_0}^{t_1}\|\bar{\phi}_x\|^2d\tau$. Using $(6.32)_1$, we can rewrite $(6.32)_2$ as

$$
\frac{1}{\tilde{v}}\bar{\phi}_{xt}-\bar{\psi}_t+\frac{\tilde{p}}{\tilde{v}}\bar{\phi}_x=\frac{R}{\tilde{v}}\zeta_x-(\frac{\tilde{p}}{\tilde{v}})_x\bar{\phi}+(\frac{R}{\tilde{v}})_x\zeta-(\frac{1}{\tilde{v}})_x\bar{\psi}_x+(\frac{\tilde{u}_x\bar{\phi}}{\tilde{v}^2})_x+\tilde{R}_{1,x}-\tilde{Q}_{1,x}.\tag{6.43}
$$

Multiplying (6.43) by $\bar{\phi}_x$, we have

$$
\frac{1}{\tilde{v}}\bar{\phi}_{xt}\bar{\phi}_x-\bar{\phi}_x\bar{\psi}_t+\frac{\tilde{p}}{\tilde{v}}\bar{\phi}_x^2=[\frac{R}{\tilde{v}}\zeta_x-(\frac{\tilde{p}}{\tilde{v}})_x\bar{\phi}+(\frac{R}{\tilde{v}})_x\zeta-(\frac{1}{\tilde{v}})_x\bar{\psi}_x+(\frac{\tilde{u}_x\bar{\phi}}{\tilde{v}^2})_x]\bar{\phi}_x+(\tilde{R}_{1,x}-\tilde{Q}_{1,x})\bar{\phi}_x.\tag{6.44}
$$

Since

$$
\bar{\phi}_x\bar{\psi}_t=(\bar{\phi}_x\bar{\psi})_t-(\bar{\phi}_t\bar{\psi})_x+\bar{\psi}_x^2,\tag{6.45}
$$

we obtain

$$
\begin{aligned}
&\sup_{t_0\le\tau\le t_1}\|\bar\phi_x\|^2+\int_{t_0}^{t_1}\|\bar\phi_x\|^2d\tau\\
&\quad\le C[\sup_{t_0\le\tau\le t_1}\|\bar\psi\|^2+\|\bar\phi_x(t_0)\|^2+\int_{t_0}^{t_1}\|(\bar\psi_x,\zeta_x)\|^2d\tau\\
&\qquad+\delta\int_{t_0}^{t_1}\|(\bar\phi,\zeta)\|^2d\tau+\int_{t_0}^{t_1}\|\tilde R_{1,x}\|^2+\|\tilde Q_{1,x}\|^2d\tau+\int_{t_0}^{t_1}|[\bar\phi_t\bar\psi](0,\tau)|d\tau. \qquad (6.46)
\end{aligned}
$$

From (5.7), it is easy to see

$$
\int_{t_0}^{t_1}\|\tilde R_{1,x}\|^2d\tau\le C\delta. \qquad (6.47)
$$

By virtue of (6.33) and the a priori assumptions, we have

$$
\tilde Q_{1,x}=O(1)(\bar\phi^2+\zeta^2+|\bar\phi_x||\bar\phi|+|\bar\psi_x||\bar\phi_x|+\zeta_x^2+|\bar\psi_{xx}||\bar\phi|). \qquad (6.48)
$$

Applying the similar arguments used in obtaining (6.38) and (6.48), we can get

$$
\begin{aligned}
\int_{t_0}^{t_1}\|\tilde Q_{1,x}\|^2d\tau &\le CN_1^4(t_0,t_1)\int_{t_0}^{t_1}\|(\bar\phi,\zeta)\|^2d\tau+CN_1(t_0,t_1)\int_{t_0}^{t_1}\|(\bar\psi_{xx},\zeta_{xx})\|^2d\tau\\
&\quad+(CN_1(t_0,t_1)+\frac{1}{4})\int_{t_0}^{t_1}\|(\bar\phi_x,\bar\psi_x,\zeta_x)\|^2d\tau. \qquad (6.49)
\end{aligned}
$$

From Theorem 1.2 in [10], we have

$$
\begin{aligned}
\int_{t_0}^{t_1}|[\bar\phi_t\bar\psi](0,\tau)|d\tau &=\int_{t_0}^{t_1}|\bar\psi[\bar\psi_x](0,\tau)|d\tau\le C\int_{t_0}^{t_1}\|\bar\psi\|^{\frac12}\|\bar\psi_x\|^{\frac12}|[u_x(0,t_0)]|e^{-c(\tau-t_0)}d\tau\\
&\le C\int_{t_0}^{t_1}\|\bar\psi_x\|^2d\tau+C\sup_{t_0\le\tau\le t_1}\|\bar\psi\|^{\frac23}|[u_x(0,t_0)]|^{\frac43}\int_{t_0}^{t_1}e^{-c(\tau-t_0)}d\tau\\
&\le C\int_{t_0}^{t_1}\|\bar\psi_x\|^2d\tau+C\delta. \qquad (6.50)
\end{aligned}
$$

Inserting the estimates (6.47), (6.49) and (6.50) into (6.46), we obtain

$$
\begin{aligned}
&\sup_{t_0\le\tau\le t_1}\|\bar\phi_x\|^2+\int_{t_0}^{t_1}\|\bar\phi_x\|^2d\tau\\
&\quad\le C[\sup_{t_0\le\tau\le t_1}\|\bar\psi\|^2+\|\bar\phi_x(t_0)\|^2+\int_{t_0}^{t_1}\|(\bar\psi_x,\zeta_x)\|^2d\tau\\
&\qquad+(\delta+N_1^4(t_0,t_1))\int_{t_0}^{t_1}\|(\bar\phi,\zeta)\|^2d\tau+N_1(t_0,t_1)\int_{t_0}^{t_1}\|(\bar\psi_{xx},\zeta_{xx}\|^2d\tau+\delta]. \qquad (6.51)
\end{aligned}
$$

Combining (6.42) with (6.51), we have

$$\sup_{t_0\le\tau\le t_1}(\|(\bar\phi,\bar\psi,\zeta)(\tau)\|^2+\|\bar\phi_x(\tau)\|^2)+\int_{t_0}^{t_1}\|(\bar\phi_x,\bar\psi_x,\zeta_x)\|^2d\tau$$
$$\le C[\|(\bar\phi,\bar\psi,\zeta)(t_0)\|^2+\|\bar\phi_x(t_0)\|^2+(\delta+N_1^4(t_0,t_1))\int_{t_0}^{t_1}\|(\bar\phi,\bar\psi,\zeta)\|^2d\tau$$
$$+N_1(t_0,t_1)\int_{t_0}^{t_1}\|(\bar\psi_{xx},\zeta_{xx})\|^2d\tau+\delta]. \tag{6.52}$$

Finally, we estimate the term $\|(\bar\psi_x,\zeta_x)\|^2$. Multiplying $(6.32)_2$ by $-\bar\psi_{xx}$, $(6.32)_3$ by $-\zeta_{xx}$, respectively and adding and integrating, we have

$$\sup_{t_0\le\tau\le t_1}\|(\bar\psi_x,\zeta_x)\|^2+\int_{t_0}^{t_1}\|(\bar\psi_{xx},\zeta_{xx})\|^2d\tau$$
$$\le C\|(\bar\psi_x,\zeta_x)(t_0)\|^2+C\int_{t_0}^{t_1}\!\!\!\!\!\!\!\!\!\!\int\Big\{[\,|\frac{\tilde p}{\tilde v}\bar\phi_x|+|\frac{R}{\tilde v}\zeta_x|+|(\frac{\tilde p}{\tilde v})_x\bar\phi|+|(\frac{R}{\tilde v})_x\zeta|$$
$$+|(\frac{1}{\tilde v})_x\zeta|+|(\frac{R}{\tilde v})_x\bar\psi_x|+|(\frac{\tilde u_x\bar\phi}{\tilde v^2})_x|]|\bar\psi_{xx}|+[|\tilde p\bar\psi_x|+|p-\tilde p|\tilde u_x|+|\frac{1}{v\tilde v}\tilde u_x^2\bar\phi|$$
$$+|\frac{2}{v}\tilde u_x\bar\psi_x|+|(\frac{\mu}{\tilde v})_x\zeta_x|+|(\frac{\mu\tilde\theta_x\bar\phi}{\tilde v^2})_x|]|\zeta_{xx}|+(|\tilde R_{1,x}|+|\tilde Q_{1,x}|)|\bar\psi_{xx}|$$
$$+[|\tilde u\tilde R_{1,x}|+|\tilde R_{2,x}|+|\tilde Q_{2,x}+(p-\tilde p)\bar\psi+\frac{\bar\psi_x^2}{v}|]|\zeta_{xx}|\Big\}$$
$$+C\int_{t_0}^{t_1}(|[\bar\psi_x\bar\psi_t]|+|\zeta_x\zeta_t|)(0,\tau)d\tau=:C\|(\bar\psi_x,\zeta_x)(t_0)\|^2+\sum_{i=1}^5\tilde B_i. \tag{6.53}$$

Using Cauchy inequality, we have

$$\tilde B_1+\tilde B_2\le C\delta\int_{t_0}^{t_1}\|(\bar\phi,\zeta)\|^2d\tau+C\int_{t_0}^{t_1}\|(\bar\phi_x,\bar\psi_x,\zeta_x)\|^2d\tau+\frac18\int_{t_0}^{t_1}\|(\bar\psi_{xx},\zeta_{xx})\|^2d\tau. \tag{6.54}$$

Applying the similar arguments used in getting the estimates (6.38)-(6.40), we have

$$\tilde B_3+\tilde B_4\le CN_1^4(t_0,t_1)\int_{t_0}^{t_1}\|(\bar\phi,\zeta)\|^2d\tau+C\int_{t_0}^{t_1}\|(\bar\phi_x,\zeta_x)\|^2d\tau$$
$$+(\frac14+CN_1(t_0,t_1))\int_{t_0}^{t_1}\|(\bar\psi_{xx},\zeta_{xx})\|^2d\tau+C\delta. \tag{6.55}$$

The boundary term $\tilde B_5$ can be estimated in the similar way as (6.50). In fact, we have

$$\tilde B_5\le C\delta. \tag{6.56}$$

Combining the above relations (6.53)-(6.56), we conclude

$$\begin{aligned}
&\sup_{t_0\le\tau\le t_1}\|(\bar\psi_x,\zeta_x)\|^2+\int_{t_0}^{t_1}\|(\bar\psi_{xx},\zeta_{xx})\|^2d\tau\\
&\le C\|(\bar\psi_x,\zeta_x)(t_0)\|^2+C(\delta+N_1^4(t_0,t_1))\int_{t_0}^{t_1}\|(\bar\phi,\zeta)\|^2d\tau\\
&\quad+C\int_{t_0}^{t_1}\|(\bar\phi_x,\bar\psi_x,\zeta_x)\|^2d\tau+C\delta. \qquad (6.57)
\end{aligned}$$

Therefore, (6.8) follows from (6.52), (6.57) and $\eta_1=C\delta^{\frac14}$ immediately.

This conculdes the proof of Lemma 6.1.

In the following theorem, we apply the a priori estimates above to obtain global existence:

Theorem 6.3. Given $U_-=[\begin{array}{c}v_-\\ v_+\end{array}]$, there is a small constant ε_0 depending only on U_- such that , if $\delta=|U_+-U_-|\le\varepsilon_0$, then the Cauchy problem (2.1)-(1.5) has a unique global solution U defined for all positive time and satisfying

$$\begin{aligned}
&\sup_{t\le\tau}\|(\bar\Phi,\bar\Psi,Z)(\tau)\|^2+\int_t^\infty\|(\bar\Phi_x,\bar\Psi_x,Z_x)\|^2d\tau\\
&\le\begin{cases} C\delta^{-\vartheta\mathcal{B}}[1+\delta t+\delta^{-1}e^{(-\delta^2t)/C}], & t\le t_0,\\ C\delta^{-\vartheta\mathcal{B}}, & t\ge t_0,\end{cases} \qquad (6.58)
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{0\le\tau}[\|(\bar\phi,\bar\psi,\zeta)(\tau)\|^2+\|(\bar\phi_x,\bar\psi_x,\zeta_x)(\tau)\|^2]\\
&\quad+\int_0^\infty[\|(\bar\phi_x,\bar\psi_x,\zeta_x)\|^2+\|(\bar\psi_{xx},\zeta_{xx})\|^2]d\tau\le C\delta^{1-\vartheta\mathcal{B}}, \qquad (6.59)
\end{aligned}$$

where $t_0=M\delta^{-2}\log\delta^{-1}$, M and $\mathcal{B}$ are positive constants depending only on U_-.

Proof. We take t_0 as indicated above, so that the estimates (5.22) and (5.23) hold. Adding a small multiple of (6.8) to δ times (6.7), we see that, for $t\ge t_0$,

$$\begin{aligned}
&\sup_{t_0\le\tau\le t_1}\delta\|(\bar\Phi,\bar\Psi,Z)(\tau)\|^2+\delta\int_{t_0}^{t_1}\|(\bar\Phi_x,\bar\Psi_x,Z_x)\|^2d\tau\\
&\quad+\sup_{t_0\le\tau\le t_1}[\|(\bar\phi,\bar\psi,\zeta)(\tau)\|^2+\|(\bar\phi_x,\bar\psi_x,\zeta_x)(\tau)\|^2]\\
&\quad+\int_{t_0}^{t_1}[\|(\bar\phi_x,\bar\psi_x,\zeta_x)\|^2+\|(\bar\psi_{xx},\zeta_{xx})\|^2]d\tau\le C\delta^{1-\vartheta\mathcal{B}}. \qquad (6.60)
\end{aligned}$$

Separating terms out, we then conclude that (6.58) and (6.59) hold for all $t\ge t_0$ as a priori bounds, that is, provided that $N_0(t_0,t) < \eta_0$ and $N_1(t_0,t) < \eta_1$. On the other hand, (6.58), (6.59) and the relation (6.6) imply that

$$\begin{aligned}\|(\bar{\Phi},\bar{\Psi},Z)\|^2_{L^\infty} &\le C\|(\bar{\Phi},\bar{\Psi},Z)\|^2 \|(\bar{\phi},\bar{\psi},\zeta)(\tau)\|\!|^2\\ &\le C\delta^{1-2\vartheta\mathcal{B}} \ll 1,\end{aligned}$$

if ϑ and δ are small. Therefore,

$$N_0(t_0,t) < \eta_0.$$

Again from (6.59), we have

$$\|(\bar{\phi},\bar{\psi},\zeta)\|\!| + \|(\bar{\phi}_x,\bar{\psi}_x,\zeta_x)\|\!| \le C\delta^{\frac{1}{2}-\frac{\vartheta\mathcal{B}}{2}} < \eta_1,$$

if ϑ and δ are small. Therefore, we have

$$N_1(t_0,t) < \eta_1.$$

These observations, together with the local existence result, the Theorem 1.2 in [10], and the intermediate-time result, Theorem 3.2, prove the global existence of U, and show that (6.58) and (6.59) hold for all $t \ge t_0$. From (5.20), (5.21) and Triangle inequality, it is easy to see that (6.58) and (6.59) hold for all $t \le t_0$.

Therefore, (6.58) and (6.59) hold for all $t \ge 0$, and the proof of Theorem 6.3 is completed.

Proof of Theorem 1.1. First, it is easy to see that, if U^ϵ and U are the solutions of (1.1)-(1.3) and (2.1)-(1.3), respectively, then

$$U^\epsilon(x,t) = U(\frac{x}{\epsilon},\frac{t}{\epsilon}). \tag{6.61}$$

The global existence of U^ϵ, its regularity and the information (1.20) concerning the jump discontinuities in U^ϵ then follow directly from Theorem 1.2 in [10] and Theorem 6.3. To prove Theorem 1.1, it is sufficient to prove the convergence results (1.21)– (1.23). Setting $G(t) = \|(U-\bar{U}_{\alpha_1,\alpha_3})(\cdot,t)\|\!|^2$ and using (6.58)-(6.59), (5.18), the relation (6.6) and the equations (1.1), we have

$$\int_{t_0}^\infty \left\{G(t) + \left|\frac{d}{dt}G(t)\right|\right\} dt < \infty.$$

This yields

$$\lim_{t\to\infty} \|(U-\bar{U}_{\alpha_1,\alpha_3})(\cdot,t)\|\!|^2 = 0,$$

from which and Sobolev's inequality it follows

$$\lim_{t\to\infty}\sup_{x\neq 0}|U(x,t)-\bar{U}_{\alpha_1,\alpha_3}(x,t)|=0. \tag{6.62}$$

This together with the estimate (1.20) gives

$$\lim_{t\to\infty}\sup_{x\in\mathbf{R}}|U(x,t)-\bar{U}_{\alpha_1,\alpha_3}(x,t)|=0. \tag{6.63}$$

Noting (1.14)-(1.15), (1.19) and the definitions of $\bar{U}_1$, $\bar{U}_3$ and $\bar{U}_{\alpha_1,\alpha_3}$(see Lemma 4.1), we have

$$\begin{aligned}\bar{U}^{\epsilon}_{\alpha_1^{\epsilon},\alpha_3^{\epsilon}}(x,t) &= \bar{U}^{\epsilon}_1(x-s_1t+\alpha_1\epsilon)+\bar{U}^{\epsilon}_3(x-s_3t+\alpha_3\epsilon)-U_m\\ &= \bar{U}_1(\frac{x-s_1t+\alpha_1\epsilon}{\epsilon})+\bar{U}_3(\frac{x-s_3t+\alpha_3\epsilon}{\epsilon})-U_m\\ &= \bar{U}_1(\frac{x-s_1t}{\epsilon}+\alpha_1)+\bar{U}^{\epsilon}_3(\frac{x-s_3t}{\epsilon}+\alpha_3)-U_m\\ &= \bar{U}_{\alpha_1,\alpha_3}(\frac{x}{\epsilon},\frac{t}{\epsilon}).\end{aligned} \tag{6.64}$$

Then, (1.21) follows from (6.61), (6.63) and (6.64) immediately. By using (6.61) and (6.64), one can write

$$\begin{aligned}U^{\epsilon}(x,t)-U^0(x,t) &= U^{\epsilon}(x,t)-\bar{U}^{\epsilon}_{\alpha_1^{\epsilon},\alpha_3^{\epsilon}}(x,t)+\bar{U}^{\epsilon}_{\alpha_1^{\epsilon},\alpha_3^{\epsilon}}(x,t)-U^0(x,t)\\ &= \left(U(\frac{x}{\epsilon},\frac{t}{\epsilon})-\bar{U}_{\alpha_1,\alpha_3}(\frac{x}{\epsilon},\frac{t}{\epsilon})\right)+\left(\bar{U}_{\alpha_1,\alpha_3}(\frac{x}{\epsilon},\frac{t}{\epsilon})-U^0(x,t)\right).\end{aligned} \tag{6.65}$$

From (6.63), it is easy to see that

$$\lim_{\epsilon\to 0}\sup_{|x-s_it|\geq h, i=1,3}\left|U(\frac{x}{\epsilon},\frac{t}{\epsilon})-\bar{U}_{\alpha_1,\alpha_3}(\frac{x}{\epsilon},\frac{t}{\epsilon})\right|=0. \tag{6.66}$$

By virtue of (1.21) and Lemma 4.1, we have

$$\lim_{\epsilon\to 0}\sup_{|x-s_it|\geq h, i=1,3}\left|\bar{U}_{\alpha_1,\alpha_3}(\frac{x}{\epsilon},\frac{t}{\epsilon})-U^0(x,t)\right|=0. \tag{6.67}$$

Then, (1.22) follows from (6.65)-(6.67) immediately.

From (1.9), it is clear that

$$U^0(x,t)=U^0(\frac{x}{\epsilon},\frac{t}{\epsilon}). \tag{6.68}$$

By (6.65) and (6.68), (1.23) follows directly from (6.59), Lemma 4.1 and Sobolev inequality.

Therefore, the proof of Theorem 1.1 is completed.

Appendix

$\bar{U}_1^\epsilon$,	(1.14)
$\bar{U}_3^\epsilon$,	(1.15)
$m^\epsilon,\ \bar{m}^\epsilon$	(1.16)
$\tilde{m}^\epsilon$	(1.17)
$\bar{U}^\epsilon_{\alpha_1^\epsilon,\alpha_3^\epsilon}$	(1.19)
$\bar{U}_R$	(2.18)
$\bar{U}_{TW}$	(2.25)
$\bar{\bar{U}}$	(2.48)
(ϕ,φ,z)	(3.1) and (3.3)
(Φ,Ψ,W)	(3.16)
$\bar{U}_1,\ \bar{U}_3,\ \bar{U}_{\alpha_1,\alpha_3}$	Lemma 4.1
$\tilde{U}$	(5.2)
$(\bar{\Phi},\bar{\Psi},Z)$	(6.1) and the equality below (6.2)
$(\bar{\phi},\bar{\psi},\zeta)$	(6.6)

Acknowledgement. The authors would like to thank the referee for the valuable comments for improving the presentation of the paper. This work is part of the first author's PhD thesis, and is partially supported by National Science Foundation under grant DMS-0807406, NSFC Grant No. 10801128 and No. 10976026, and the Research Foundation of Education Bureau of Hunan Province Grant No. 11C0628. Part of this work was done while the first author was visiting School of Mathematics at Georgia Institute of Technology. The first author expresses his gratitude to Prof. Ronghua Pan for his hospitality and supervision. The research of R. Pan is partly supported by National Science Foundation under grant DMS-0807406 and DMS-1108994.

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