Communications in Mathematical Analysis
Special Volume in Honor of Prof. Peter Lax
Volume 8, Number 3, pp. 1–22 (2010)
ISSN 1938-9787
www.commun-math-anal.org

INITIAL BOUNDARY VALUE PROBLEM FOR COMPRESSIBLE EULER EQUATIONS WITH RELAXATION

QIANGCHANG JU*
Institue of Applied Physics and Computational Mathematics
P.O. Box 8009-28
Beijing, 100088, P. R. China

YONG LI†
College of Science
Beijing University of Technology
Beijing, 100022, P. R. China

RONGHUA PAN‡
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332, USA

(Communicated by Toka Diagana)

Abstract

In this paper, we study the global exisitence of L^∞ weak entropy solution to the initial boundary value problem for compressible Euler equations with relaxtion and the large time asymptotic behavior of the solution. Motivated by the sub-characterisitic conditions, we proposed some structural conditions on the relaxation term comparing with the pressure function. These conditions are proved to be sufficient to construct the global L^∞ entropy weak solution and to prove the equilibrium state is the global attactor of all physical weak solutions. Furthermore, the convergence rate is proved to be exponential in time. The proof is based on the entropy dissipation principle.

AMS Subject Classification: 35Q35, 35L04, 35L65.

Keywords: Compressible Euler Equations, Relaxation, Entropy weak solutions, Large time behavior, Compensated compactness.

*E-mail address: qiangchang_ju@yahoo.com
†E-mail address: liy@bjut.edu.cn, Li's research is supported by NSFC grant 10541001.
‡E-mail address: panrh@math.gatech.edu,Pan's research is supported by NSF under grant DMS-0807406.

1 Introduction

The relaxation phenomena occurs in many physical applications, including gas dynamics away from thermo-equilibrium, chromatography, river flow, traffic flow, reacting flow and etc; see for instance [33]. In this paper, we consider the compressible Euler equations with relaxation

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = \dfrac{Q(\rho) - \rho u}{\tau}. \end{cases} \tag{1.1}$$

Here ρ, u and P denote the density, velocity and pressure. Assuming the flow is a polytropic perfect gas with the adiabatic exponent γ, then $P(\rho) = P_0\rho^\gamma$, $\gamma > 1$, where P_0 is a positive constant. τ denotes the relaxation time. Without loss of generality, we take $P_0 = \frac{1}{\gamma}$, $\tau = 1$ throughout this paper. Since the particle velocity is not well-defined at vacuum, the momentum $m = \rho u$ is often used to rewrite (1.1) as follows:

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + \left(\dfrac{m^2}{\rho} + P(\rho)\right)_x = Q(\rho) - m. \end{cases} \tag{1.2}$$

The system (1.2) is supplemented by the following initial value and boundary conditions:

$$\begin{cases} \rho(x,0) = \rho_0(x),\ m(x,0) = m_0(x),\ 0 < x < 1, \\ m|_{x=0} = 0,\ m|_{x=1} = 0,\ t \geq 0, \\ m_0(0) = m_0(1) = 0, \\ \displaystyle\int_0^1 \rho_0(x)\,dx = \rho^* > 0. \end{cases} \tag{1.3}$$

The last condition in (1.3) is imposed to avoid the trivial case: $\rho \equiv 0$.

It is clear that the system (1.2) is hyperbolic with characteristic speeds

$$\lambda_1 = \frac{m}{\rho} - \rho^\theta,\ \lambda_2 = \frac{m}{\rho} + \rho^\theta, \tag{1.4}$$

with $\theta = \dfrac{\gamma - 1}{2}$. The corresponding Riemann invariants are

$$w = \frac{m}{p} + \frac{\rho^\theta}{\theta}, z = \frac{m}{p} - \frac{\rho^\theta}{\theta}. \tag{1.5}$$

The nonlinearity of the problem, along with the degeneracy at vacuum where $\rho = 0$ and relaxation effect in the source term, gives the system (1.2) rich phenomena to study. The vanishing relaxation limit of (1.1) as $\tau \to 0$ toward the equilibrium system

$$\rho_t + Q(\rho)_x = 0 \tag{1.6}$$

and the related wave stability problem in large time are two significant problems. In the past two decades, the mathematical theories on hyperbolic conservation laws with relaxation have been developed extensively since the pioneer work of Liu [22]. Both the zero relaxation limit and the nonlinear stability of elementary waves were studied in a great deal, see [2], [3], [17], [8],[9], [10], [11], [12], [16], [17], [18], [19], [20], [21], [25], [26], [27], [28], [29], [34], [35], [37] and [38] for a limited partial list of references. The purpose of current paper is to study the fundamental problems on the global existence of weak solution and the large time behavior of the the solutions to (1.2)–(1.3).

It has been a general belief that the nonlinearity and hyperbolicity lead to the blow up of smooth solutions in finite time when the dissipation in the source term can not win the compitition for some initial data. For the system (1.2), the study of [34] and [19] clearly shows that when initial data is not restricted in certain range, the solution will break down in finite time. Therefore, as in current paper, when the intial data could be large and rough, the weak solution is the choice. We now give the defintion of the weak solution to our problem.

Definition 1.1. For every $T > 0$, we define a weak solution of (1.2)-(1.3) to be a pair of bounded measurable functions $v(x,t) = (\rho(x,t), m(x,t))$ satisfying the following pair of integral identities:

$$\int_0^T\int_0^1 (\rho\phi_t + m\phi_x)\,dxdt + \int_{t=0}\rho_0\phi\,dx = 0, \tag{1.7}$$

$$\begin{aligned}\int_0^T\int_0^1 \Big(m\phi_t + \Big(\frac{m^2}{\rho} + P(\rho)\Big)\phi_x\Big)\,dxdt \\ + \int_0^T\int_0^1 (Q(\rho) - m)\phi\,dxdt + \int_{t=0} m_0\phi\,dx = 0,\end{aligned} \tag{1.8}$$

for all $\phi \in C_0^\infty(I_T)$ satisfying $\phi(x,T) = 0$ for $0 \le x \le 1$ and $\phi(0,t) = \phi(1,t) = 0$ for $t \ge 0$, where $I_T = (0,1)\times(0,T)$. Moreover, (ρ, m) satisfy the initial boundary conditions (1.3) in the sense of trace.

In order to identify the physical relevant weak solutions, the entropy addmissible condition is often imposed, motivated by the second law of thermodynamics. For system (1.2), the entropy and entropy flux pairs are defined as follows.

Definition 1.2. A pair of functions $\eta(\rho,m)$ and $q(\rho,m)$ is called an entropy-entropy flux pair if it satisfies the following equations

$$\nabla q = \nabla\eta\nabla f,$$

where

$$f = (m, \frac{m^2}{\rho} + P(\rho)).$$

Among all entropies, the most natural entropy is the mechanical energy

$$\eta_e(\rho,m) = \frac{m^2}{2\rho} + \frac{\rho^\gamma}{\gamma(\gamma-1)},$$

which plays a very important role in estimates for entropy dissipation measures.

Definition 1.3. The weak solution $v(x,t) = (\rho(x,t), m(x,t))$ defined in Definition 1.1 is said to be entropy admissible if for any convex entropy η and associated entropy flux q, the following entropy inequality holds

$$\eta_t + q_x + \eta_m(m - Q(\rho)) \leq 0, \tag{1.9}$$

in the sense of distribution.

Regarding the study of the L^∞ weak entropy solution of hyperbolic system with source term, very few references are available. Concerning to our system (1.2), a closely related system is the compressible Euler equations with linear damping where $Q(\rho) = 0$. For the latter system, the existence of weak entropy solutions and large time behavior has been studied in [4, 13, 14, 32] for Cauchy problem, and in [30] for initial boundary value problem. Motivated by the method in [13, 14, 30], we shall constructs the global L^∞ entropy weak solutions to (1.2)-(1.3) by means of Godunov scheme and the compensation compactness frameworks see [5, 6, 23, 24]. Based on the entropy dissipation and energy method, the exponential decay of any L^∞ weak entropy solutions to the equilibrium state is shown under some assumptions on relaxation term $Q(\rho)$.

On the way to our goal, one interesting issue is to solicite the appropriate conditions to ensure the uniform L^∞ estimate for the solutions and the stability of the equilibrium $(\rho^*, 0)$ under any perturbation with finite amplitude. In the traditional setting, the sub-characteristic condition

$$\lambda_1 < Q'(\rho) < \lambda_2 \tag{1.10}$$

is often proposed to ensure at least the linear stability of the equilibrium with small perturbation; see for instance [3] and [33]. However, in the context of stability with large amplitude perturbation, the sub-characteristic condition (1.10) seems not enough to ensure the stability. Indeed, since one expects $Q(\rho)$ and m approch to each other, some deeper relations are waiting for further investigations. We shall propose some in this paper as an attemp in this direction.

For this purpose, we define the following quantity

$$\alpha_0 = \max\{\sup_x w_0(x), -\inf_x z_0(x)\}, \tag{1.11}$$

which will measure the L^∞ bounds of the solution. Here, w_0 and z_0 are initial Riemann invariants. For the convenience of presentation, we also introduce the following notations

$$\begin{aligned} f_1(\rho,\rho^*) &= P(\rho) - P(\rho^*) - P'(\rho^*)(\rho-\rho^*) \equiv f_2(\rho,\rho^*)(\rho-\rho^*)^2, \\ f_3(\rho,\rho^*) &= [P(\rho) - P(\rho^*)](\rho-\rho^*) \equiv f_4(\rho,\rho^*)(\rho-\rho^*)^2, \\ f_5(\rho,\rho^*) &= \frac{Q(\rho) - Q(\rho^*)}{\rho-\rho^*}. \end{aligned} \tag{1.12}$$

Clearly, the above functions f_i are well defined when $\rho \neq \rho^*$, the difference quotients will be replaced by the corresponding derivatives when $\rho = \rho^*$ for the defintion of f_2, f_4 and f_5.

In section 3, the following assumption on $Q(\rho)$ plays an important role in the proof of uniform L^∞ bound for solutions to (1.2)-(1.3).

(A1) *For* $0 \le \rho \le (\theta\alpha_0)^{1/\theta}$, $Q(\rho)$ *is* C^2 *and it holds that*

$$|Q(\rho)| \le \rho\Big(\alpha_0 - \frac{\rho^\theta}{\theta}\Big), \tag{1.13}$$

where α_0 is defined in (1.11).

Remark 1.4. This assumption simply askes the bounds on $Q(\rho)$ near vacuum and the large ρ. One observes that from (1.13) that $Q(0) = 0$ which is necessary to confirm that $(0,0)$ is a solution to (1.2)-(1.3). Since $\frac{Q(\rho)}{\rho}$ is the difference quotient of $Q(\rho)$ over the interval $[0,\rho]$, the inequality (1.13) is closely related to the sub-characteristic condition (1.10). The exisitence of such $Q(\rho)$ is obivious. For example, we can take $Q(\rho) = \varepsilon\rho(\alpha_0 - \frac{\rho^\theta}{\theta})$, or $Q(\rho) = \varepsilon\rho(\frac{\rho^\theta}{\theta} - \alpha_0)$, where $0 \le \varepsilon \le 1$. It is also worthy to remark that the right hand side of (1.13) with $\theta = 1$ appears in the traffic flow models, see [33].

To investigate the large time behavior of weak entropy solutions to the initial boundary value problem (1.2)-(1.3), we will need the following strong **sub-slope condition** on $Q(\rho)$.

(A2) *There are* $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ *and* $1/a_1 + 1/a_2 + 1/a_3 = 1$ *such that, for* $0 < \rho \le M$, *it holds that*

$$f_4(\rho,\rho^*) > \frac{a_1 M}{\rho}[f_5(\rho,\rho^*)]^2, \tag{1.14}$$

and

$$\frac{8C^*}{a_2 a_3} f_4(\rho,\rho^*) \ge [f_5(\rho,\rho^*)]^2, \tag{1.15}$$

where $C^* = \frac{1}{\gamma}(\rho^*)^{\gamma-1}$.

Remark 1.5. This condition states some relation on the slopes of P and Q. We thus call it the **strong sub-slope condition**. The assumption (1.14) implies that

$$\frac{P(\rho) - P(\rho^*)}{\rho - \rho^*} \ge \Big(\frac{Q(\rho) - Q(\rho^*)}{\rho - \rho^*}\Big)^2,$$

which implies the sub-characteristic condition (1.10) when $\frac{m}{\rho} = 0$. We remark that sub-characteristic condition involves the velocity $\frac{m}{\rho}$, which does not appear in the limiting equation (1.6). However, at the equilibrium where $m = Q$, $(A2)$ is very close to the sub-characteristic condition. Condition $(A2)$ gives a global picture between P and Q without involving the velocity, which will give us some advantage in the proof of large time asymptotic behavior of weak solutions in section 4.

The plan of the rest of this paper is organized as follows. In section 2 we give some elementary notations and basic facts. The main results will also be stated there. In section 3, the global existence of L^∞ weak entropy solutions will be proved. Finally, we will investigate the large time behavior of any L^∞ weak solutions in section 4.

2 Preliminaries and main results

In this section, we will present some preliminaries for the foundation of our studies in next sections.

The homogeneous system corresponding to system of (1.2) reads

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + \Big(\dfrac{m^2}{\rho} + P(\rho)\Big)_x = 0. \end{cases} \tag{2.1}$$

For a smooth solution, (2.1) can be rewritten as

$$v_t + \nabla f(v) v_x = 0, \tag{2.2}$$

where $v = (\rho, m)^T, f(v) = (m, m^2/\rho + \rho^\gamma/\gamma)^T$, and

$$\nabla f = \begin{pmatrix} 0 & 1 \\ -\dfrac{m^2}{\rho^2} + \rho^{\gamma-1} & \dfrac{2m}{\rho} \end{pmatrix}. \tag{2.3}$$

The eigenvalues of (2.3) are

$$\lambda_1 = \frac{m}{\rho} - \rho^\theta, \;\; \lambda_2 = \frac{m}{\rho} + \rho^\theta, \tag{2.4}$$

and the Riemann invariants are

$$w = \frac{m}{\rho} + \frac{\rho^\theta}{\theta}, \;\; z = \frac{m}{\rho} - \frac{\rho^\theta}{\theta},$$

for $\theta = (\gamma - 1)/2$. We note that (w, z) satisfies

$$\begin{cases} z_t + \lambda_1 z_x = \dfrac{Q(\rho)}{\rho} - \dfrac{m}{\rho} \\ w_t + \lambda_2 w_x = \dfrac{Q(\rho)}{\rho} - \dfrac{m}{\rho} \end{cases} \tag{2.5}$$

For the Riemann problem

$$\begin{cases} (2.2), \quad t > 0, \quad x \in \mathbb{R}, \\ (\rho, m)|_{t=0} = \begin{cases} (\rho_l, m_l), & x < 0, \\ (\rho_r, m_r), & x > 0, \end{cases} \end{cases} \tag{2.6}$$

where ρ_l, ρ_r, m_l and m_r are constants satisfying $0 \le \rho_l, \rho_r, |m_l/\rho_l|, |m_r/\rho_r| < \infty$, there are two distinct types of rarefaction waves and shock waves, called elementary waves, which are labelled 1-rarefaction or 2-rarefaction waves and 1-shock or 2-shock waves, respectively.

Lemma 2.1. *There exists a global weak entropy solution of* (2.6) *which is piecewise smooth function satisfying*

$$\begin{aligned} &w(x,t) = w(\frac{x}{t}) \le \max\{w(\rho_l, m_l), w(\rho_r, m_r)\}, \\ &z(x,t) = z(\frac{x}{t}) \ge \min\{z(\rho_l, m_l), z(\rho_r, m_r)\}, \\ &w(x,t) - z(x,t) \ge 0. \end{aligned}$$

It follows that the region $\Lambda = \{(\rho, m) : w \le w_0, z \ge z_0, w - z \ge 0\}$ is an invariant region for the Riemann problem (2.6).

Lemma 2.2. *If* $\{(\rho, m) : a \le x \le b\} \subset \Lambda$*, then*

$$\Big(\frac{1}{b-a}\int_a^b \rho\, dx, \quad \frac{1}{b-a}\int_a^b m\, dx\Big) \in \Lambda. \tag{2.7}$$

Lemma 2.3. *For the mixed problem*

$$\begin{cases} (2.2), \qquad t > 0, \quad x > 0, \\ (\rho, m)|_{t=0} = (\rho_0, m_0), \qquad x > 0, \\ m|_{x=0} = 0, \qquad t \ge 0, \end{cases} \tag{2.8}$$

where (ρ_0, m_0) *are constants, there exists a weak entropy solution in the region* $\{(x,t) : x \ge 0, t \ge 0\}$ *satisfying the following estimates*

$$w(x,t) \le \max\{w(\rho_0, m_0), -z(\rho_0, m_0)\},$$
$$z(x,t) \ge z(\rho_0, m_0) \quad \text{and} \quad w(x,t) - z(x,t) \ge 0.$$

Similarly, we can solve the following mixed problem in the region $\{(x,t) : x \le 1, t \ge 0\}$

$$\begin{cases} (2.2), \qquad t > 0, \quad x < 1, \\ (\rho, m)|_{t=0} = (\rho_0, m_0), \qquad x < 1, \\ m|_{x=1} = 0, \qquad t \ge 0. \end{cases} \tag{2.9}$$

The weak solution for (2.9) satisfies the following estimates

$$z(x,t) \ge \min\{z(\rho_0, m_0), -w(\rho_0, m_0)\},$$
$$w(x,t) \le w(\rho_0, m_0) \quad \text{and} \quad w(x,t) - z(x,t) \ge 0.$$

Lemma 2.4. *Suppose that* $(\rho(x,t), m(x,t))$ *is a solution to* (2.6) *or* (2.8) *or* (2.9). *Then the jump strength of* $m(x,t)$ *across an elementary wave can be dominated by that of* $\rho(x,t)$ *across the same elementary wave, i.e.,*

across a shock wave: $\quad |m_r - m_l| \le C|\rho_r - \rho_l|,$
across a rarefaction wave: $\quad |m - m_l| \le C|\rho - \rho_l| \le C|\rho_r - \rho_l|,$

where C *depends only on the bounds of* ρ *and* $|m|$.

Lemma 2.5. *For any* $\varepsilon > 0$*, there exist constants* $h > 0$ *and* $k > 0$ *such that the solution of* (2.6) *in the region* $\{(x,t) : |x| < h, 0 \le t < k\}$ *satisfies*

$$\int_{-h}^{h} |\rho(x,t) - \rho(x,0)|\, dx \le Ch\varepsilon, \quad 0 \le t \le k, \tag{2.10}$$

where C *depends only on the bounds of* ρ *and* $|m|$*, and the mesh lengths* h *and* k *satisfy* $\max_{i=1,2} \sup |\lambda_i(\rho, m)| < \dfrac{h}{2k}$.

Theorem 2.6. *Assume that the initial data* (ρ_0, m_0) *satisfy the following conditions*

$$0 \le \rho_0(x) \le M_1,\ \rho_0 \not\equiv 0,\ |m_0(x)| \le M_2\rho_0(x),$$

for some positive constants $M_i(i=1,2)$. *And assume* $Q(\rho)$ *satisfies the assumption (A1). Then, for* $\gamma > 1$, *there exists a positive constant M, the initial boundary value problem* (1.2)-(1.3) *has a global weak solution* $(\rho(x,t), m(x,t))$ *satisfying the following estimates and entropy condition*

$$0 \le \rho(x,t) \le M,\ \ |m(x,t)| \le M\rho \quad \text{a.e.},$$
$$\int_0^T\int_0^1 (\eta(\rho,m)\psi_t + q(\rho,m)\psi_x)\,dxdt + \int_0^T\int_0^1 \eta_m(\rho,m)(Q(\rho)-m)\psi\,dxdt \ge 0,$$

for all weak and convex entropy pairs (η, q) *for* (1.2)-(1.3) *and for all nonnegative smooth functions* $\psi \in C_0^1(I_T)$.

Theorem 2.7. *Let* (ρ, m) *be any* L^∞ *entropy weak solution of the initial boundary problem* (1.2)-(1.3), *satisfying* $\int_0^1 \rho_0(x)\,dx = \rho^*$, $Q(\rho^*) = 0$ *and*

$$0 \le \rho(x,t) \le M < \infty,\ |m(x,t)| \le M_1\rho(x,t),$$

where M and M_1 *are positive constants. And assume* $Q(\rho)$ *satisfies the assumption A(2). Then, there exist constants* $C > 0$ *and* $\delta > 0$ *depending on* γ, ρ^*, M *and initial data such that*

$$\|(\rho-\rho^*, m)(\cdot,t)\|^2_{L^2([0,1])} \le Ce^{-\delta t}.$$

3 Global existence of weak entropy solutions

We begin with the construction of approximate solution by modified Godunov Scheme in the spirit of operator splitting. Let us take the space mesh length $h = 1/N$, where N is a positive integer. The time mesh length $k = k(h)$ will be determined later so that the Courant-Friedrich-Lewy condition

$$\max_{i=1,2}(\sup|\lambda_i(v)|) < \frac{h}{2k} \tag{3.1}$$

holds for a given $T > 0$. We partition the interval $[0,1]$ into cells, with the $j-$th cell centered at $x_j = jh, j = 1,\cdots,N-1$, and denote t_i by ik. Set $x_0 = 0$ and $x_N = 1$. Now we consider the solution $\underline{v}_h = (\underline{\rho}_h, \underline{m}_h)^T$ of the Riemann problems (2.6) in the region $R_j^1 \equiv \{(x,t) : x_{j-\frac{1}{2}} \le x < x_{j+\frac{1}{2}}, 0 \le t < k\}$:

$$\begin{cases} \dfrac{\partial}{\partial t}\underline{v}_h + \dfrac{\partial}{\partial x} f(\underline{v}_h) = 0, \\ \underline{v}_h|_{t=0} = \begin{cases} (\rho_j^0, m_j^0), & x < x_j, \\ (\rho_{j+1}^0, m_{j+1}^0), & x > x_j, \quad j = 1,\cdots,N-1, \end{cases} \end{cases} \tag{3.2}$$

where

$$\rho_j^0 = \frac{1}{h}\int_{x_{j-1}}^{x_j} \rho_0(x)\,dx, \quad m_j^0 = \frac{1}{h}\int_{x_{j-1}}^{x_j} m_0(x)\,dx, \quad \text{for } j = 1,\cdots,N. \tag{3.3}$$

At the same time we also solve the mixed problem (2.8) and (2.9) with (ρ_1^0, m_1^0) and (ρ_N^0, m_N^0), in regions $\{(x,t) : 0 \le x < x_{1/2}, 0 \le t < k\}$ and $\{(x,t) : x_{N-1/2} \le x < 1, 0 \le t < k\}$, respectively. Then we set, for $0 \le x \le 1,\ 0 \le t < k$,

$$\begin{cases} v_h(x,t) = (\rho_h(x,t), m_h(x,t))^T, \\ \rho_h(x,t) = \underline{\rho}_h(x,t), \\ m_h(x,t) = \underline{m}_h(x,t) - \underline{m}_h(x,t)t + Q(\underline{\rho}_h(x,t))t \end{cases} \tag{3.4}$$

and

$$v_j^1 = \frac{1}{h}\int_{x_{j-1}}^{x_j} v_h(x, t_1 - 0)\,dx, \quad j = 1, \cdots, N. \tag{3.5}$$

Next we will define approximate solutions v_h for $t_i \le t < t_{i+1}$ through using approximate solutions defined in $0 \le t < t_i$. Suppose that we have defined approximate solutions $v_h(x,t)$ for $0 \le t < t_i$. we then define approximate solutions for $0 \le x \le 1,\ t_i \le t < t_{i+1}$ as follows

$$\begin{cases} v_h(x,t) = (\rho_h(x,t), m_h(x,t))^T, \\ \rho_h(x,t) = \underline{\rho}_h(x,t), \\ m_h(x,t) = \underline{m}_h(x,t) - \underline{m}_h(x,t)(t-t_i) + Q(\underline{\rho}_h(x,t))(t-t_i) \end{cases} \tag{3.6}$$

where $\underline{v}_h(x,t) = (\underline{\rho}_h(x,t), \underline{m}_h(x,t))$ are piecewise-smooth functions defined as solutions of Riemann problems in the region $R_j^{i+1} \equiv \{(x,t) : x_{j-\frac{1}{2}} \le x < x_{j+\frac{1}{2}}, t_i \le t < t_{i+1}\}$:

$$\begin{cases} (2.2), \\ \underline{v}_h(x,t)|_{t=t_i} = \begin{cases} v_j^i, & x < x_j, \\ v_{j+1}^i, & x > x_j, \quad j = 1, \cdots, N-1, \end{cases} \end{cases} \tag{3.7}$$

and as solutions of mixed problems in the two side regions $R_0^{i+1} = \{(x,t) : 0 \le x < x_{1/2}, t_i \le t < t_{i+1}\}$ and $R_N^{i+1} = \{(x,t) : x_{N-1/2} \le x < 1, t_i \le t < t_{i+1}\}$:

$$\begin{cases} (2.2), & t > t_i, \quad x > 0, \\ \underline{v}_h|_{t=t_i} = v_1^i, & x > 0, \\ \underline{m}_h|_{x=0} = 0, \end{cases} \tag{3.8}$$

and

$$\begin{cases} (2.2), & t > t_i, \quad x < 1, \\ \underline{v}_h|_{t=t_i} = v_N^i, & x < 1, \\ \underline{m}_h|_{x=1} = 0. \end{cases} \tag{3.9}$$

And we set

$$v_j^{i+1} = \frac{1}{h}\int_{x_{j-1}}^{x_j} v_h(x, t_{i+1} - 0)\,dx, \quad 1 \le j \le N. \tag{3.10}$$

Therefore the approximate solutions $v_h = (\rho_h, m_h)$ are well defined in the region $\{0 \le x \le 1, 0 \le t \le T\}$ for any $T > 0$ since $\underline{\rho}_h \ge 0$.

For $t_i \le t < t_{i+1}$, due to (2.5), we can obtain the expression of $(w_h(x,t), z_h(x,t))$ as follows:

$$\begin{aligned} w_h(x,t) &= \underline{w}_h(x,t) - \frac{\underline{w}_h + \underline{z}_h}{2}(t-t_i) + \frac{Q(\underline{\rho}_h)}{\underline{\rho}_h}(t-t_i), \\ z_h(x,t) &= \underline{z}_h(x,t) - \frac{\underline{w}_h + \underline{z}_h}{2}(t-t_i) + \frac{Q(\underline{\rho}_h)}{\underline{\rho}_h}(t-t_i), \end{aligned} \tag{3.11}$$

where $\underline{w}_h$ and $\underline{z}_h$ are Riemann invariants corresponding to the Riemann solution $\underline{v}_h$.

We prove the following uniform bound for the approximate solutions.

Lemma 3.1. *Suppose that the initial data (ρ_0, m_0) satisfy the following conditions:*

$$0 \le \rho_0(x) \le M_1,\ \rho_0 \not\equiv 0,\ |m_0(x)| \le M_2 \rho_0(x). \tag{3.12}$$

And assume $Q(\rho)$ satisfies the assumption (A1). Then the approximate solutions (ρ_h, m_h) derived by the Godunov scheme are uniformly bounded in the region $\bar{I}_T \equiv \{(x,t) : 0 \le x \le 1, 0 \le t \le T\}$ for any $T > 0$; that is, there is a constant C independent of t such that

$$0 \le \rho_h(x,t) \le C,\ \ |m_h(x,t)| \le C\rho_h(x,t). \tag{3.13}$$

Proof. Assume that $0 < k < 1$. Firstly, for $0 \le t < t_1$, the Riemann invariant properties imply that

$$\underline{w}_h(x,t) \le \alpha_0,\ \ \underline{z}_h(x,t) \ge -\alpha_0,\ \text{ and }\ \underline{w}_h(x,t) - \underline{z}_h(x,t) \ge 0,$$

where

$$\alpha_0 = \max\{\sup_x w_0(x), -\inf_x z_0(x)\}. \tag{3.14}$$

Then it holds that

$$0 \le \underline{\rho}_h(x,t) \le (\theta\alpha_0)^{1/\theta},\ \ |\underline{m}_h(x,t)| \le \alpha_0 \underline{\rho}_h(x,t).$$

From (A1), we have

$$\begin{aligned} w_h(x,t) &= \underline{w}_h(x,t)(1-t) + \Big[\frac{\underline{w}_h - \underline{z}_h}{2} + \frac{Q(\underline{\rho}_h)}{\underline{\rho}_h}\Big]t \\ &\le \underline{w}_h(x,t)(1-t) + \Big(\frac{\underline{\rho}_h^\theta}{\theta} + \frac{Q(\underline{\rho}_h)}{\underline{\rho}_h}\Big)t \\ &\le \underline{w}_h(x,t)(1-t) + \alpha_0 t \\ &\le \alpha_0, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} z_h(x,t) &= \underline{z}_h(x,t)(1-t) + \Big[\frac{Q(\underline{\rho}_h)}{\underline{\rho}_h} - \frac{\underline{w}_h + \underline{z}_h}{2}\Big]t \\ &= \underline{z}_h(x,t)(1-t) + \Big(\frac{Q(\underline{\rho}_h)}{\underline{\rho}_h} - \frac{\underline{\rho}_h^\theta}{\theta}\Big)t \\ &\ge \underline{z}_h(x,t)(1-t) - \alpha_0 t \\ &\ge -\alpha_0. \end{aligned} \tag{3.16}$$

Inductively, we can prove that for $t_i \le t < t_{i+1}$,

$$w_h(x,t) \le \alpha_0,\ \ z_h(x,t) \ge -\alpha_0,\ \ w_h(x,t) - z_h(x,t) = \underline{w}_h(x,t) - \underline{z}_h(x,t) \ge 0.$$

Then there is a constant $C > 0$ independent of h, k and t such that

$$0 \le \rho_h(x,t) \le C,\ \ |m_h(x,t)| \le C\rho_h(x,t).$$

This completes the proof of Lemma 3.1.

□

Finally, we can choose the time mesh length $k = k(h)$. Let

$$\lambda = \max_{i=1,2} \Big\{ \sup_{0\le\rho\le C, |m|\le C\rho} |\lambda_i(\rho, m)| \Big\},$$

then we take

$$k = \frac{T}{n}, \qquad \text{where} \quad n = \max\Big\{ \Big[\frac{4\lambda T}{h}\Big] + 1, \Big[\frac{T}{2}\Big] + 1 \Big\}.$$

For this k, both the CFL condition and $0 < k < 1$ hold.

Set $g(v) = (0, Q(\rho) - m)^T$. Then (1.2)-(1.3) can be rewritten as

$$\begin{cases} v_t + f(v)_x = g(v), \\ v(x,0) = v_0(x), \quad x \in (0,1), \\ m(0,t) = m(1,t) = 0. \end{cases} \tag{3.17}$$

In the following, we will show that the approximate solutions constructed above admit a convergent subsequence whose limit is a weak entropy solution of problem (1.2)-(1.3). The convergence is achieved by the compensated compactness, the boundary conditions are verified in the sense of trace.

In view of the uniform L^∞ estimates given in Lemma 3.1, and the specific structure of system (1.2), then it is standard to apply the compensated compactness framework to the approximate solution $\{v_h\}$, to conclude that there exists a convergent subsequence, still labeled $\{v_h\}$, such that

$$(\rho_h(x,t), m_h(x,t)) \to (\rho(x,t), m(x,t)) \qquad \text{a.e.} \tag{3.18}$$

Clearly, there is a positive constant C such that

$$0 \le \rho(x,t) \le C, \qquad |m(x,t)| \le C\rho(x,t) \qquad \text{a.e..} \tag{3.19}$$

For any $\phi \in C^\infty(\bar{I}_T)$ satisfying $\phi(x,T) = 0$, $\phi(0,t) = \phi(1,t) = 0$, we consider the following integral identity

$$\int_0^T \int_0^1 (\rho_h \phi_t + m_h \phi_x)\, dxdt + \int_{t=0} \rho_h \phi\, dx = A(\phi) + R(\phi), \tag{3.20}$$

where

$$A(\phi) = \sum_{i,j} \int_{x_{j-1}}^{x_j} (\rho_h^i - \rho_j^i)\phi^i\, dx,$$
$$R(\phi) = \sum_{i,j} \int_{t_i}^{t_{i+1}} \int_{x_{j-1}}^{x_j} (m_h - \underline{m}_h)\phi_x\, dxdt,$$

with $\rho_h^i = \rho(x, t_i - 0)$. Similar to [31], with the help of Hölder inequality and the uniform bound of v_h, we have

$$A(\phi) \le Ch^{1/2} \|\phi\|_{C^1} \to 0, \qquad \text{as} \quad h \to 0. \tag{3.21}$$

$$\begin{aligned} R(\phi) &\le \sum_{i,j} \int_{t_i}^{t_{i+1}} \int_{x_{j-1}}^{x_j} |(Q(\underline{\rho}_h) - \underline{m}_h)(t - t_i)||\phi_x|\, dxdt \\ &\le Ch\|\phi\|_{C^1} \to 0, \qquad \text{as} \quad h \to 0. \end{aligned} \tag{3.22}$$

Then from (3.21)-(3.22), it gives

$$\lim_{h\to 0}\int_0^T\int_0^1 (\rho_h\phi_t + m_h\phi_x)\,dxdt + \lim_{h\to 0}\int_{t=0}\rho_h\phi\,dx = 0. \tag{3.23}$$

By virtue of the dominated convergence theorem to (3.23), we get

$$\int_0^T\int_0^1 (\rho\phi_t + m\phi_x)\,dxdt + \int_{t=0}\rho_0(x)\phi\,dx = 0. \tag{3.24}$$

For every function $\phi\in C^1(\bar{I}_T)$ satisfying $\phi(x,T)=0$ for $0\le x\le 1$ and $\phi(0,t)=\phi(1,t)=0$ for $t\ge 0$, we consider the integral identity

$$\int_0^T\int_0^1 (m_h\phi_t + f_1(v_h)\phi_x + V(v_h)\phi)\,dxdt + \int_{t=0} m_h\phi\,dx = B(\phi)+S(\phi), \tag{3.25}$$

with $f_1(v)=m^2/\rho+\rho^\gamma/\gamma$, $V(v)=Q(\rho)-m$, and

$$\begin{aligned} B(\phi) &= \sum_{i,j}\int_{x_{j-1}}^{x_j}(\underline{m}_h^i - m_j^i)\phi^i\,dx + \sum_{i,j}\int_{t_i}^{t_{i+1}}\int_{x_{j-1}}^{x_j} V(\underline{v}_h)\phi\,dxdt,\\ S(\phi) &= \sum_{i,j}\int_{t_i}^{t_{i+1}}\int_{x_{j-1}}^{x_j}[(m_h-\underline{m}_h)\phi_t\\ &\qquad\qquad + (f_1(v_h)-f_1(\underline{v}_h)\phi_x + (V(v_h)-V(\underline{v}_h))\phi]\,dxdt. \end{aligned}$$

Using the uniform bound of v_h and $|m_h-\underline{m}_h|\le k(|\underline{m}_h|+C'|\underline{\rho}_h|)$, we have

$$S(\phi)\le Ch\|\phi\|_{C^1}\to 0, \qquad \text{as} \quad h\to 0. \tag{3.26}$$

Due to $m_h^i = \underline{m}_h^i + \int_{t_i}^{t_{i+1}} V(\underline{v}_h^i)\,dt$, then $B(\phi)$ can be bounded by

$$\begin{aligned} B(\phi) &= \sum_{i,j}\int_{x_{j-1}}^{x_j}(\underline{m}_h^i - m_j^i)(\phi^i-\phi_j^i)\,dx\\ &\quad + \sum_{i,j}\int_{t_i}^{t_{i+1}}\int_{x_{j-1}}^{x_j} V(\underline{v}_h)(\phi-\phi_j^i)\,dxdt\\ &\quad + \sum_{i,j}\int_{t_i}^{t_{i+1}}\int_{x_{j-1}}^{x_j}[V(\underline{v}_h)-V(\underline{v}_h^i)]\phi_j^i\,dxdt\\ &\le Ch^{1/2}\|\phi\|_{C^1} + Ch\|\phi\|_{C^1} + C\varepsilon\|\phi\|_\infty, \end{aligned}$$

where ε is an arbitrarily small constant. Then, it implies

$$B(\phi)\to 0, \qquad \text{as} \quad h\to 0, \quad \varepsilon\to 0. \tag{3.27}$$

Then due to (3.26)-(3.27) and the dominated convergence theorem, we obtain

$$\begin{aligned} &\int_0^T\int_0^1 \Big(m\phi_t + \Big(\frac{m^2}{\rho}+P(\rho)\Big)\phi_x\Big)\,dxdt\\ &\qquad + \int_0^T\int_0^1 (Q(\rho)-m)\phi\,dxdt + \int_{t=0} m_0\phi\,dx = 0. \end{aligned} \tag{3.28}$$

For every weak and convex entropy pair (η, q) and every nonnegative smooth function ψ which has a compact support in I_T, we study the integral identity

$$\int_0^T\int_0^1 (\eta(v_h)\psi_t + q(v_h)\psi_x)\ dxdt = A(\psi_h) + R(\psi_h) + \Sigma(\psi_h) + S(\psi_h), \tag{3.29}$$

where

$$\begin{aligned}
A(\psi_h) &= \sum_{i,j}\int_{x_{j-1}}^{x_j} [\eta(v_h^i) - \eta(v_j^i)]\psi(x,t_i)\,dx,\\
R(\psi_h) &= \sum_{i,j}\int_{x_{j-1}}^{x_j} [\eta(\underline{v}_h^i) - \eta(v_h^i)]\psi(x,t_i)\,dx,\\
\Sigma(\psi_h) &= \int_0^T \Sigma\{\sigma[\eta] - [q]\}\psi(x(t),t)\,dt,\\
S(\psi_h) &= \int_0^T\int_0^1 [\eta(v_h) - \eta(\underline{v}_h)]\psi_t + [q(v_h) - q(\underline{v}_h)]\psi_x\,dxdt.
\end{aligned}$$

Since (η, q) is a convex entropy pair and $\psi \geq 0$, similar to [31], we have

$$\begin{aligned}
A(\psi_h) &\geq \sum_{i,j}\int_{x_{j-1}}^{x_j} [\eta(v_h^i) - \eta(v_j^i)](\psi^i - \psi_j^i)\,dx\\
&\geq -Ch^{\alpha-1/2}\|\psi\|_{C_0^\alpha}, \qquad 1/2 \leq \alpha \leq 1, &(3.30)\\
\Sigma(\psi_h) &\geq 0, &(3.31)\\
S(\psi_h) &\geq -Ch\|\psi\|_{H_0^1}, &(3.32)\\
R(\psi_h) &= \sum_{i,j}\int_{x_{j-1}}^{x_j}\int_0^1 \nabla\eta(v_h^i + \theta(\underline{v}_h^i - v_h^i))(\underline{v}_h^i - v_h^i)d\theta\psi^i\,dx\\
&\geq -\sum_i \int_0^1 \Big(\int_0^1 \eta_m(v_h^i + \theta(\underline{v}_h^i - v_h^i))d\theta\cdot[Q(\underline{\rho}_h^i) - m_h^i](t-t_i)\psi^i\Big)\,dx\\
&\quad -Ch. &(3.33)
\end{aligned}$$

With the help of these above inequalities and the fact that $v_h \to v$ a.e., letting $h \to 0$, then we have the following entropy condition

$$\int_0^T\int_0^1 (\eta(v)\psi_t + q(v)\psi_x)\ dxdt + \int_0^T\int_0^1 \eta_m(v)V(v)\psi_t\ dxdt \geq 0. \tag{3.34}$$

Now we turn to the boundary conditions of weak solutions. The exact meaning of traces for weak solutions is given below. Let $v(x,t) = (\rho(x,t), m(x,t))$ be a weak solution of (1.2) obtain in (3.18). We introduce the generalized function $\mathcal{A} : C_0^1(\mathbb{R}^2) \to \mathbb{R}^2$ as follows: for $\phi \in C_0^1(\mathbb{R}^2)$,

$$\mathcal{A}(\phi) = -\int_0^T\int_0^1 [v\phi_t + f(v)\phi_x + g(v)\phi]\,dxdt, \tag{3.35}$$

with $f(v) = (m, m^2/\rho + \rho^\gamma/\gamma)^T$, $g(v) = (0, Q(\rho - m))^T$.

We take smooth $\zeta_0(t), \zeta_T(t), \xi_0(x), \xi_1(x)$ with

$$\begin{aligned}
&\zeta_0(0) = 1, \quad \zeta_0(T) = 1; \quad \zeta_T(0) = 0, \quad \zeta_T(T) = 1;\\
&\xi_0(0) = 1, \quad \xi_0(T) = 1; \quad \xi_T(0) = 0, \quad \xi_T(T) = 1.
\end{aligned} \tag{3.36}$$

For any $\chi(x)$, we define the generalized functions:

$$
\begin{aligned}
&v^*(\cdot,0)(\chi) = \mathcal{A}(\chi\cdot\zeta_0) - \chi(0)\mathcal{A}(\xi_0\cdot\zeta_0) - \chi(1)\mathcal{A}(\xi_0\cdot\zeta_0);\\
&v^*(\cdot,T)(\chi) = -\mathcal{A}(\chi\cdot\zeta_T) + \chi(0)\mathcal{A}(\xi_0\cdot\zeta_T) + \chi(1)\mathcal{A}(\xi_0\cdot\zeta_T);\\
&f^*(v)(0,\cdot)(\chi) = \mathcal{A}(\xi_0\cdot\chi);\\
&f^*(v)(1,\cdot)(\chi) = -\mathcal{A}(\xi_1\cdot\chi),
\end{aligned}
$$

where $\chi\cdot\zeta_0(x,t) = \chi(x)\zeta_0(t)$ and so mean the tensor product.

Then we define the trace of v along the segments $(0,1)\times\{0\}$ and $(0,1)\times\{T\}$, and the trace of $f(v)$ along the segments $\{0\}\times(0,T)$ and $\{1\}\times(0,T)$ respectively as $v^*(\cdot,0), v^*(\cdot,T), f^*(v)(0,\cdot)$ and $f^*(v)(1,\cdot)$. Similarly, for any $t\in(0,T)$, we can also define $v^*(\cdot,t)$ as the trace of v along the segment $(0,1)\times\{t\}$. For any $x\in(0,1)$, we can also define $f^*(v)(x,\cdot)$ as the trace of $f(v)$ along the segment $\{x\}\times(0,1)$.

Similar to [7], we have the following lemma.

Lemma 3.2. *Let v satisfy* (1.2) *in distributional sense, then*

$$
\begin{aligned}
&v^*(\cdot,0)|_{(0,1)}, \quad v^*(\cdot,T)|_{(0,1)} \in L^\infty_{loc}(0,1)\\
&f^*(v)(0,\cdot)|_{(0,T)}, \quad f^*(v)(1,\cdot)|_{(0,T)} \in L^\infty_{loc}(0,T),
\end{aligned}
$$

and $\forall\ \phi\in C^1_0(\mathbb{R}^2)$,

$$
\begin{aligned}
&\int_0^T\int_0^1 [v\phi_t + f(v)\phi_x + g(v)\phi]\,dxdt\\
&\quad = \int_0^1 v^*(x,T)\phi(x,T);dx - \int_0^1 v^*(x,0)\phi(x,T)\,dx\\
&\quad\quad + \int_0^T f^*(v)(1,t)\phi(1,t)\,dt - \int_0^T f^*(v)(0,t)\phi(0,t)\,dt.
\end{aligned}
\tag{3.37}
$$

Theorem 3.3. *Let $v_h(x,t) = (\rho_h(x,t), m_h(x,t))$ be the approximate solutions of* (1.2)-(1.3) *constructed in this section and $v(x,t) = (\rho(x,t), m(x,t))$ be the limit function obtained in* (3.18). *Then $v = (\rho, m)$ satisfy the initial-boundary conditions*

$$
m^*(0,t) = m^*(1,t) = 0, \qquad t\in(0,T) \tag{3.38}
$$

$$
v^*(x,0) = v_0(x), \qquad x\in(0,1). \tag{3.39}
$$

Proof. From (1.7)- (1.8), it gives, for any $\phi\in C^1_0(\mathbb{R}^2)$, that

$$
\lim_{h\to 0}\Big[\int_0^T\int_0^1 [v_h\phi_t + f(v_h)\phi_x + g(v_h)\phi]\,dxdt + \int_{t=0} v_h\phi\,dx - \int_{t=T} v_h\phi\,dx\Big] = 0,
$$

which implies

$$
\int_0^T\int_0^1 [v\phi_t + f(v)\phi_x + g(v)\phi]\,dxdt + \lim_{h\to 0}\Big[\int_{t=0} v_h\phi\,dx - \int_{t=T} v_h\phi\,dx\Big] = 0, \tag{3.40}
$$

Inserting (3.37) into (3.40), we get

$$\begin{aligned}\lim_{h\to 0}\Big[\int_{t=T} v_h\phi\,dx-\int_{t=0} v_h\phi\,dx\Big]&\\ =\int_0^1 v^*(x,T)\phi(x,T)\,dx-\int_0^1 v^*(x,0)\phi(x,0)\,dx&\\ +\int_0^T f^*(v)(1,t)\phi(1,t)\,dt-\int_0^T f^*(v)(0,t)\phi(0,t)\,dt.&\end{aligned}\tag{3.41}$$

So the first component of above equality reads

$$\begin{aligned}\int_0^1\rho^*(x,T)\phi(x,T)\,dx-\int_0^1\rho^*(x,0)\phi(x,0)\,dx+\int_0^T m^*(v)(1,t)\phi(1,t)\,dt&\\ -\int_0^T m^*(v)(0,t)\phi(0,t)\,dt+\int_{t=0}\rho_0(x)\phi\,dx-\int_{t=T}\rho\phi\,dx=0.&\end{aligned}\tag{3.42}$$

Taking $\phi(x,t)=\zeta(x)\chi(t)\in C_0^1(\mathbb{R}^2)$ with $\zeta,\chi\in C_0^1(\mathbb{R})$, and $\chi(0)=1,\chi(T)=0,\zeta(0)=\zeta(1)=0$ in (3.41), we have

$$\int_0^1\rho^*(x,0)\zeta(x)\,dx=\int_0^1\rho_0(x)\zeta(x)\,dx,$$

which implies $\rho^*(x,0)=\rho_0(x)$ on $(0,1)$.

Similarly, using the second component of (3.41), it is easy to show that $m^*(x,0)=m_0(x)$ on $(0,1)$. Taking $\phi(x,t)=\zeta(x)\chi(t)\in C_0^1(\mathbb{R}^2)$ with $\zeta,\chi\in C_0^1(\mathbb{R})$, and $\chi(0)=\chi(T)=0,\zeta(0)=1,\zeta(1)=0$ in (3.41), one can get

$$\int_0^1 m^*(0,t)\xi(t)\,dx=0.$$

Therefore $m^*(0,t)=0$ on $(0,T)$. It is similar to obtain that $m^*(1,t)=0$ on $(0,T)$. This completes the proof Theorem 3.3 □

4 Large time behavior of weak solution

In this section, we will prove the asymptotic behavior of the weak solution, namely, Theorem 2.7. For this purpose, we assume that (ρ,m) is an entorpy weak solution in L^∞ such that

$$0\le\rho\le M,\ |m(x,t)|\le C_0\rho(x,t)$$

for some positive constants M and C_0.

Due to conservation law of total mass, we have

$$\int_0^1\rho(x,t)\,dx=\int_0^1\rho_0(x)\,dx=\rho^*>0.\tag{4.1}$$

Without the loss of generality, we assume $\rho^*<M$, otherwise one has $\rho\equiv M$ and $m=Q(M)=0$, the trivial constant solution, or inconsistance in the case $Q(M)\neq 0$. For the same reason, we require in Theorem 2.7 the condition

$$Q(\rho^*)=0,$$

to ensure $(\rho^*,0)$ a equilibrium solution to (2)-(3).

In order to control the sigularity near vacuum, the following lemma proved in [30] plays an important role.

Lemma 4.1. *Let* $0\le\rho\le M<\infty$. *There are positive constants* C_1, C_2 *and* C_3 *such that*

$$C_1\le f_2(\rho,\rho^*),\ f_4(\rho,\rho^*)\le C_2,\quad f_1(\rho,\rho^*)\le C_3 f_3(\rho,\rho^*). \tag{4.2}$$

This lemma is a direct consequence of the mean value theorem for $P(\rho)$.

Remark 4.2. We remark that, due to the convexity of $P(\rho)=\frac{1}{\gamma}\rho^\gamma$, it is clear that

$$f_4(\rho,\rho^*)\ge f_4(0,\rho^*)=\frac{1}{\gamma}(\rho^*)^{\gamma-1}\equiv C^*.$$

Set

$$y=-\int_0^x(\rho-\rho^*)dr, \tag{4.3}$$

then

$$y_x=-(\rho-\rho^*),\quad y_t=m. \tag{4.4}$$

Due to the conservation of mass we have

$$y(0)=y(1)=0. \tag{4.5}$$

The momentum equation becomes

$$y_{tt}+y_t+\left(\frac{m^2}{\rho}\right)_x+(P(\rho)-P(\rho^*))_x=Q(\rho). \tag{4.6}$$

Multiplying y with (4.6) and integrating the resulting equation over $[0,1]$, we have

$$\begin{aligned}&\frac{d}{dt}\int_0^1(y_ty+\frac{1}{2}y^2)\,dx+\int_0^1(f_3(\rho,\rho^*)-y_t^2-\frac{m^2}{\rho}y_x)\,dx\\ =\ &\int_0^1(Q(\rho)-Q(\rho^*))y\,dx.\end{aligned} \tag{4.7}$$

Let

$$\eta_e=\frac{m^2}{2\rho}+\frac{P(\rho)}{\gamma-1},\ \ q_e=\frac{m^3}{2\rho^2}+\frac{\rho^{\gamma-1}m}{\gamma-1}$$

be the mechanical energy and related flux, respectively. We define

$$\eta_*=\eta_e-\frac{1}{\gamma-1}P'(\rho^*)(\rho-\rho^*)-\frac{1}{\gamma-1}P(\rho^*). \tag{4.8}$$

Thus, by the definition of weak entropy solution, the following entropy inequality holds in the sense of distribution:

$$\eta_{*t}+\frac{1}{\gamma-1}[P'(\rho^*)(\rho-\rho^*)]_t+q_{ex}+\frac{m^2}{\rho}-\frac{m}{\rho}Q(\rho)\le 0. \tag{4.9}$$

By the conservation of mass and theory of divergence-measure fields [1], we have

$$\frac{d}{dt}\int_0^1 \eta_*\,dx+\int_0^1\Big(\frac{y_t^2}{\rho}-\frac{m}{\rho}Q(\rho)\Big)\,dx\leq 0. \tag{4.10}$$

Now we add (4.7) to (4.10)$\times\lambda_0$, for some $\lambda_0>2M$ chosen later,

$$\frac{d}{dt}E(t)+D(t)\leq 0, \tag{4.11}$$

where

$$\begin{aligned} E(t)&=\int_0^1(\lambda_0\eta_*+yy_t+\frac{1}{2}y^2)\,dx,\\ D(t)&=\int_0^1\Big(\frac{\lambda_0-\rho^*}{\rho}y_t^2-\lambda_0\frac{m}{\rho}Q(\rho)+f_3(\rho,\rho^*)-(Q(\rho)-Q(\rho^*))y\Big)\,dx. \end{aligned} \tag{4.12}$$

We note, from (4.4), that

$$\begin{aligned} D(t)&=\int_0^1(\frac{\lambda_0-\rho^*}{\rho}y_t^2+\lambda_0\frac{1}{\rho}f_5(\rho,\rho^*)y_ty_x+f_4(\rho,\rho^*)y_x^2+f_5(\rho,\rho^*)y_xy)\,dx\\ &\equiv\int_0^t I_1+I_2\,dx, \end{aligned}$$

where

$$\begin{aligned} I_1&=\frac{\lambda_0-\rho^*}{\rho}y_t^2+\lambda_0\frac{1}{\rho}f_5(\rho,\rho^*)y_ty_x+\frac{1}{a_1}f_4(\rho,\rho^*)y_x^2\\ I_2&=(\frac{1}{a_2}+\frac{1}{a_3})f_4(\rho,\rho^*)y_x^2+f_5(\rho,\rho^*)y_xy. \end{aligned} \tag{4.13}$$

Due to $\lambda_0>2M$, it holds that

$$\frac{\lambda_0-\rho^*}{\rho}\geq 1. \tag{4.14}$$

Now we claim that there is a positive constant $\delta_0>0$, such that

$$\lambda_0=2M+\delta_0, \tag{4.15}$$

satisfying

$$2\sqrt{\frac{\lambda_0-\rho^*}{\rho}}\sqrt{\frac{f_4(\rho,\rho^*)}{a_1}}>\frac{\lambda_0}{\rho}f_5(\rho,\rho^*), \tag{4.16}$$

or equivalently

$$\frac{(f_5(\rho,\rho^*))^2}{\rho^2}\lambda_0^2-\frac{4}{a_1\rho}f_4(\rho,\rho^*)\lambda_0+\frac{4\rho^*}{a_1\rho}f_4(\rho,\rho^*)<0,$$

for all $\rho\in[0,M]$. In fact, one can define the polynomial in λ,

$$F(\lambda)=\frac{(f_5(\rho,\rho^*))^2}{\rho^2}\lambda^2-\frac{4}{a_1\rho}f_4(\rho,\rho^*)\lambda+\frac{4\rho^*}{a_1\rho}f_4(\rho,\rho^*).$$

It is easy to verify that under the assumption of (1.14),

$$F(2M) < -\frac{4(M-\rho^*)}{a_1\rho} f_4(\rho,\rho^*)$$
$$\leq -\frac{4(M-\rho^*)}{a_1 M} C^* < 0,$$

Therefore, by the continuity of $F(\lambda)$, there is a $\delta_0 > 0$ such that

$$F(2M+\delta_0) < -\frac{2(M-\rho^*)}{a_1 M} C^*. \tag{4.17}$$

This verifies our claim (4.15). We now fix this λ_0. With the help of the estimate (4.17), we conclude that there is a $\delta_1 > 0$ and $C_4 > 0$ such that

$$\delta_1 (\frac{y_t^2}{\rho} + y_x^2) \leq I_1 \leq C_4 (\frac{y_t^2}{\rho} + y_x^2). \tag{4.18}$$

We now take care of I_2. Using the Poincaré's inequality for y,

$$\int_0^1 y^2 \, dx \leq \frac{1}{2} \int_0^1 y_x^2 \, dx,$$

one has

$$\frac{1}{a_3} \int_0^1 f_4(\rho,\rho^*) y_x^2 \, dx \geq \frac{C^*}{a_3} \int_0^1 y_x^2 \, dx \geq \frac{2C^*}{a_3} \int_0^1 y^2 \, dx. \tag{4.19}$$

Thus we have

$$\int_0^1 I_2 \, dx \geq \int_0^1 \left\{ \frac{1}{a_2} f_4(\rho-\rho^*) y_x^2 + f_5(\rho,\rho^*) y_x y + \frac{2C^*}{a_3} y^2 \right\} dx.$$

The assumption (1.15) implies

$$0 \leq \int_0^1 I_2 \, dx \leq C_5 \int_0^t y_x^2 \, dx, \tag{4.20}$$

for some positive constant C_5. This estimate, together with (4.18) leads to

$$\delta_1 \int_0^1 (\frac{y_t^2}{\rho} + y_x^2) \, dx \leq D(t) \leq (C_4 + C_5) \int_0^1 (\frac{y_t^2}{\rho} + y_x^2) \, dx. \tag{4.21}$$

We now turn to $E(t)$. Using the expression of η_*, we have

$$E(t) = \int_0^1 (\frac{\lambda_0}{2\rho} y_t^2 + y y_t + \frac{1}{2} y^2 + \frac{\lambda_0}{\gamma-1} f_1(\rho,\rho^*)) \, dx. \tag{4.22}$$

In view of Lemma 4.1, one has

$$C_6 y_x^2 \leq \frac{\lambda_0}{\gamma-1} f_1(\rho,\rho^*) \leq C_7 y_x^2, \tag{4.23}$$

for two positive constants C_6 and C_7.

By the fact that $\lambda_0 = 2M + \delta_0$ and $0 \le \rho \le M < \infty$, we have

$$\begin{aligned}\int_0^1 \Big(\frac{\lambda_0}{2\rho}y_t^2 + yy_t + \frac{1}{2}y^2\Big)\,dx &\ge C_8\Big(\int_0^1 y_t^2\,dx + \int_0^1 y^2\,dx\Big)\\ &\ge C_8\int_0^1 y_t^2\,dx,\end{aligned} \tag{4.24}$$

for some postive constant C_8. On the other hand, by Poincaré's inequality and Cauchy-Schwartz inequality, there is a positive constant C_9 such that

$$\begin{aligned}\int_0^1 \Big(\frac{\lambda_0}{2\rho}y_t^2 + yy_t + \frac{1}{2}y^2\Big)\,dx &\le C\Big(\int_0^1 \frac{y_t^2}{\rho}\,dx + \int_0^1 y^2\,dx\Big)\\ &\le C_9\int_0^1 \Big(\frac{y_t^2}{\rho} + y_x^2\Big)\,dx.\end{aligned} \tag{4.25}$$

Therefore, we have the following estimate for $E(t)$,

$$\alpha\int_0^1 (y_t^2 + y_x^2)\,dx \le E(t) \le \beta D(t), \tag{4.26}$$

for some postive constant α and β. One thus concludes from (4.26) and (4.11) that

$$\frac{d}{dt}E(t) + \frac{1}{\beta}E(t) \le 0,$$

which leads to

$$E(t) \le E(0)e^{-\frac{1}{\beta}t}. \tag{4.27}$$

Finally, we have from (4.26) that

$$\int_0^1 y_t^2 + (\rho - \rho^*)^2\,dx \le \frac{1}{\alpha}E(0)e^{-\frac{1}{\beta}t}. \tag{4.28}$$

This completes the proof of Theorem 2.7.

Acknowledgments

Part of this research was carried out when R. Pan was visiting Beijing Institute of Applied Physics and Computational Mathematics as an Intellegent Scholar during the summer of 2009. R. Pan is grateful for the generous support of the Beijing IAPCM and the warm hospitality of his hosts, Prof. Jiang and Prof. Ju.

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