

# Zero Relaxation Limit to Centered Rarefaction Waves for a Rate-Type Viscoelastic System

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Received January 12, 1997; revised November 12, 1998

We study a rate-type viscoelastic system proposed by I. Suliciu (1990, *Internat. J. Engrg. Sci.* **28**, 827–841), which is a  $3 \times 3$  hyperbolic system with relaxation. As the relaxation time tends to zero, this system convergences to the well-known  $p$ -system formally. In the case where the initial data are the Riemann data such that the corresponding solutions of the  $p$ -system are centered rarefaction waves, we show that if the wave strength is suitably small, then the solution for the relaxation system exists globally in time and converges to the solution of the corresponding rarefaction waves uniformly as the relaxation time goes to zero, except for an initial layer. The jump discontinuities in the solutions are decaying exponentially fast as time tends to infinity. © 1999 Academic Press

*Key Words:* zero relaxation limit; centered rarefaction waves; piecewise energy estimates; characteristic analysis; viscoelasticity.

## 1. INTRODUCTION

We study the asymptotic behavior of the following rate-type viscoelastic system,

$$\begin{cases} v_t - u_x = 0 \\ u_t + p_x = 0, & x \in \mathbf{R}^1, \quad t > 0, \\ (p + Ev)_t = \frac{p_R(v) - p}{\varepsilon}, \end{cases} \quad (1.1)$$

as the relaxation time  $\varepsilon$  goes to zero. Here  $v$  and  $-p$  denote strain and stress, respectively,  $u$  is related to the particle velocity, and  $E$  is a positive constant, called the dynamic Young's modulus.

This system was proposed in [18] to introduce a relaxation approximation to the following system

$$\begin{cases} v_t - u_x = 0 \\ u_t + p_R(v)_x = 0. \end{cases} \quad (1.2)$$

\* To be supported partially by NSF of China.

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Since the system (1.2) can be obtained from (1.1) by an expansion procedure as the first order, it is natural to expect that (1.2) governs the evolution of the solutions to (1.1) as  $\varepsilon \rightarrow 0$ . For smooth flow, this statement can be easily verified by Hilbert expansion and a standard energy estimate argument. When shocks occur in the solutions of (1.2), [9] proved this fact by use of the matched asymptotic analysis. In this paper, we study the case where the discontinuous initial data are chosen so that the corresponding solutions of (1.2) are centered rarefaction waves (one-mode or two-modes). It is shown that if the initial jump is weak, then the solution for (1.1) exists globally (in time) and converges to the solution of (1.2) uniformly as  $\varepsilon \rightarrow 0$ , except for an initial layer. This will be carried out by a nonlinear stability analysis of the rarefaction waves under discontinuous perturbations after a scaling for (1.1). The piecewise energy estimates and characteristic method are used.

As known, relaxation is important not only as a phenomenon in many physical situations but also as an approach in mathematics to approximate the corresponding systems of conservation laws. Recently, Shi Jin and Zhouping Xin proposed the well-known relaxation schemes for general systems of conservation laws in arbitrary space dimensions (see [4]), which yields satisfactory numerical solutions.

It is Taiping Liu [7] who first analyzed the hyperbolic systems with relaxation, and justified some nonlinear stability criteria for basic waves. Since then, there are a lot of results on  $2 \times 2$  relaxation systems, either the stability analysis or the zero relaxation limit. We refer to [1, 3, 7, 10, 11, 13] for the  $L^\infty$ ,  $L^2$ , and  $L^1$  stability, and [2, 3, 8, 15, 17] for the zero relaxation limit in  $L^\infty$  and  $BV$  frameworks, respectively. We note that the limit equilibrium system is scalar in all the above results.

However, when the corresponding equilibrium system is not a scalar equation, the problem is more difficult and more challenges occur. This is the reason for us to be interested in (1.1) for which the corresponding equilibrium system is the well-known  $p$ -system. We believe our results can be helpful in understanding the relaxation effects in the zero dissipation limit for relaxation systems when the corresponding conservation laws are not scalar.

It should be mentioned that this work is strongly motivated by Xin's work in [20], where the zero viscosity limit to rarefaction waves for the one-dimensional Navier–Stokes equations of the compressible isentropic gases is proved. However, the approach is quite different here, due to the different dissipative effects between the viscosity and the relaxation and the special nonlinearity of (1.1). Since the dissipation of relaxation is much weaker than viscosity, the limit here is much singular than those in [20]. In [20], a key role is the observation of Liu and Hoff in [5] that the initial discontinuities of specific volume will propagate along the particle path and

decay exponentially. In our case, the initial discontinuities will propagate along the three characteristics issued from the origin respectively and the decay estimates will be obtained by a straight characteristic analysis. Recently, Wang and Xin have succeeded in proving the small mean free path limit to centered rarefaction waves for Broadwell model in [19]. Since the form of the Broadwell model is quite similar to (1.1), the framework is similar. But the approach here is different from [19]. In [19], the second order correction of Chapman–Enskog expansion is used in constructing the smooth approximation for rarefaction waves, while we use the first order of the expansion directly. The pointwise estimate for the first order derivatives of the solution is achieved by an iteration argument in [19], while in the present paper we use the characteristic analysis to obtain a better estimate. [19] follows the procedure in [12] for making energy estimate where the Boltzmann entropy function is used. We employ the method of [6] without the need of entropy. The advantage of our approach is that it can be generalized to the general relaxation systems of conservation laws proposed by Jin and Xin in [4].

We would like to mention that the smooth approximation to rarefaction waves for  $p$ -system introduced in [14] firstly enables us to perform the energy estimate for the solutions. It also should be pointed out that [14] is the first work on the stability of rarefaction waves for system of conservation laws with viscosity.

For the nonlinear stability analysis of the rarefaction waves for (1.1) under smooth initial perturbation, we refer to [6].

For simplicity of presentation, we only discuss the case in which the solution of (1.2) is a self similar one containing a single rarefaction wave. We will give our main results in Section 2. The proof is given in Section 3.

## 2. PRELIMINARIES AND MAIN RESULTS

Consider (1.2) with the Riemann data

$$(v(x, 0), u(x, 0)) = (v_0^r(x), u_0^r(x)), \quad (2.1)$$

where

$$(v_0^r(x), u_0^r(x)) = \begin{cases} (v_-, u_-), & x < 0 \\ (v_+, u_+), & x > 0 \end{cases}$$

with  $(v_-, u_-)$  and  $(v_+, u_+)$  being two constant states.

We make the following assumptions: for some constants  $c_1$  and  $d_1$  such that  $-\infty < c_1 < v_-$ ,  $v_+ < d_1 < +\infty$ , it holds

$$(H_1) \quad p'_R(v) < 0,$$

$$(H_2) \quad p''_R(v) > 0,$$

$$(H_3) \quad |p'_R(v)| < E,$$

for  $v \in [c_1, d_1]$ . Here  $(H_3)$  is the so-called sub-characteristic condition (see [7]).

It is easy to know that, under  $(H_1)$ – $(H_2)$ , (1.2) is strictly hyperbolic and genuinely nonlinear, with eigenvalues

$$\lambda_1 = -(-p'_R(v))^{1/2} < 0 < (-p'_R(v))^{1/2} = \lambda_2. \quad (2.2)$$

Furthermore, we assume that  $(v_-, u_-)$  and  $(v_+, u_+)$  are connected by a 1-centered rarefaction wave curve in the phase plane. And the corresponding rarefaction wave solution is  $(v^r, u^r)(x/t)$ .

Consider the Riemann problem

$$\begin{cases} v^e_t - u^e_x = 0 \\ u^e_t + p^e_x = 0 \\ (p^e + Ev^e)_t = \frac{p_R(v^e) - p^e}{\varepsilon}, \quad x \in \mathbf{R}^1, \quad t \geq 0, \end{cases} \quad (2.3)$$

with the initial data

$$S^\varepsilon(x, 0) = \begin{cases} S_+ & \text{if } x > 0 \\ S_- & \text{if } x < 0, \end{cases} \quad (2.4)$$

where  $S^\varepsilon = (v^\varepsilon, u^\varepsilon, p^\varepsilon)$ ,  $S_+ = (v_+, u_+, p_R(v_+))$ , and  $S_- = (v_-, u_-, p_R(v_-))$ .

Let  $S = (v, u, p)$  be the solution of the following rescaled problem of (2.3)–(2.4):

$$\begin{cases} v_t - u_x = 0 \\ u_t + p_x = 0 \\ (p + Ev)_t = p_R(v) - p \\ S(x, 0) = \begin{cases} S_+ & \text{if } x > 0 \\ S_- & \text{if } x < 0. \end{cases} \end{cases} \quad (2.5)$$

It is easy to see that  $S(x, t)$  is a solution of (2.5) if and only if

$$S^\varepsilon(x, t) \equiv S\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \quad (2.6)$$

is a solution of (2.3)–(2.4).

Let  $S^r(x/t) = (v^r, u^r, p^r)$ , where  $p^r = p_R(v^r)$ . We will show the following theorem which is the main result of this paper.

**THEOREM 1.** Under  $(H_1)$ – $(H_3)$ , let the constant states  $(v_-, u_-)$  and  $(v_+, u_+)$  be connected by a centered 1-rarefaction wave,  $(v^r, u^r)(x/t)$ , and there is a suitably small positive constant  $\delta_0$  such that  $|v_+ - v_-| + |u_+ - u_-| \leq \delta_0$ . Then (2.5) has a unique global piecewise smooth solution  $S(x, t)$  such that

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbf{R}} |S(x, t) - S^r(x/t)| = 0. \quad (2.7)$$

In view of (2.6), it follows that (2.3)–(2.4) has a unique global piecewise smooth solution  $S^\varepsilon(x, t)$  satisfying

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbf{R}, t \geq h} |S^\varepsilon(x, t) - S^r(x/t)| = 0, \quad (2.8)$$

for any positive number  $h$ .

*Remark.* (i) Theorem 1 also holds for the solution which is a linear superposition of rarefaction waves from both characteristic families. This follows from the arguments and the results of [6].

(ii) The approach used here strongly depends on the semi-linearity of the system, but we believe it can be generalized to the general relaxation system in [4] in view of the semilinearity of the relaxation systems in [4].

We will end this section by introducing the smooth approximations of the rarefaction wave  $(v^r, u^r)(x/t)$ . This can be carried out by use of the rarefaction waves of inviscid Burgers' equation. As in [6] (See also in [14, 20]), we have

**LEMMA 2.1.** *There exists a smooth function  $(V(x, t), U(x, t))$ , which is the smooth approximation of  $(v^r, u^r)$  in the following sense:*

$$(I) \quad \begin{cases} V_t - U_x = 0 \\ U_t + (p_R(V))_x = 0, \end{cases}$$

$$(II) \quad \lim_{t \rightarrow +\infty} \sup_{x \in \mathbf{R}^1} \{|v^r(x, t) - V(x, t)| + |u^r(x, t) - U(x, t)|\} = 0.$$

Furthermore, we have:

$$(1) \quad \partial V / \partial t > 0, \text{ for any } x \in \mathbf{R}^1, t \geq 0;$$

$$(2) \quad \text{For any } p \in [1, +\infty], \text{ there exists } c_p > 0, \text{ s.t. for any } t \geq 0$$

$$\|(V_x, U_x)\|_{L^p} \leq c_p \delta^{1/p} (1+t)^{-1+(1/p)},$$

and

$$\|(V_x, U_x)\|_{L^\infty} \leq c_\infty \delta;$$

(3) For  $j \geq 2$ , for any  $p \in [1, +\infty)$ , there exists  $c_{p,j} > 0$ , s.t. for any  $t \geq 0$ ,

$$\left\| \frac{\partial^j}{\partial x^j} (V, U) \right\|_{L^p} \leq c_{p,j} \delta (1+t)^{-1};$$

(4) There exists  $c > 0$ , s.t.

$$c^{-1} |V_x| \leq |V_t| \leq c |V_x|, \quad c^{-1} |U_x| \leq |U_t| \leq c |U_x|,$$

where  $\delta$  is the strength of the wave, namely,

$$\delta \equiv |v_+ - v_-| + |u_+ - u_-|.$$

### 3. THE PROOF OF THEOREM 1

We will prove Theorem 1 in this section by use of the piecewise smooth energy estimate. This is done by a careful characteristic analysis on the solution and its first derivatives. Due to the semilinearity and hyperbolicity of (2.5), the discontinuities of  $S(x, t)$  propagate along  $x_1(t) = -\sqrt{E}t$ ,  $x_2(t) = 0$  and  $x_3(t) = \sqrt{E}t$ . The behavior of these jumps plays the key role in the energy estimate.

First of all, let us introduce the notations

$$\Omega_1 = \{x < -\sqrt{E}t, t > 0\}, \quad \Omega_2 = \{-\sqrt{E}t < x < 0, t > 0\},$$

$$\Omega_3 = \{0 < x < \sqrt{E}t, t > 0\}, \quad \Omega_4 = \{x > \sqrt{E}t, t > 0\},$$

$$\mathcal{F}(f(x, t)) = \left( \int_{-\infty}^{x_1(t)-0} + \int_{x_1(t)+0}^{x_2(t)-0} + \int_{x_2(t)+0}^{x_3(t)-0} + \int_{x_3(t)+0}^{+\infty} \right) f(x, t) dx,$$

$$\|f(\cdot, t)\|^2 = \mathcal{F}(f(x, t)^2), \quad \|f(\cdot, t)\|_1^2 = \|f(\cdot, t)\|^2 + \|f_x(\cdot, t)\|^2,$$

$$[f]_i(t) = f(x_i(t) + 0, t) - f(x_i(t) - 0, t), \quad i = 1, 2, 3,$$

$$X(0, T) = \{f \mid f \in C^0(0, T; H^1(\Omega'_i)) \cap C^1(0, T; L^2(\Omega'_i)) \cap C^1(\overline{\Omega}_i),$$

$$i = 1, 2, 3, 4\},$$

where  $\Omega'_i = \Omega_i \cap \{t = \text{constant}\}$ , and  $H^1$  is the usual Sobolev spaces with norm  $\|\cdot\|_1$ .  $C$  will denote the generic positive constant independent of  $t$ .

To prove Theorem 1, we introduce

$$(\phi, \psi, w) = (v, u, p) - (V, U, P), \quad (3.1)$$

where  $P = p_R(V)$ . Then (2.5) and Lemma 2.1 give

$$\begin{cases} \phi_t - \psi_x = 0 \\ \psi_t + w_x = 0 \\ w_t + E\psi_x + w + (p_R(V) - p_R(V + \phi)) + (E + p'_R(V))U_x = 0 \\ (\phi, \psi, w)(x, 0) = (v, u, p)(x, 0) - (V, U, P)(x, 0). \end{cases} \quad (3.2)$$

By virtue of Lemma 2.1, one only needs to show the following Theorem 3.1 in order to prove Theorem 1.

**THEOREM 3.1.** *Under  $(H_1)$ – $(H_3)$ , suppose  $(v_-, u_-)$  and  $(v_+, u_+)$  can be connected by  $(v^r, u^r)$ . Then there exist positive constants  $\delta_0$  and  $\varepsilon_0$ , such that if  $\delta < \delta_0$  and*

$$\|(\phi, \psi, w)(x, 0)\|_1 \leq \varepsilon_0,$$

*then the problem (3.2) has a unique global piecewise smooth solution  $(\phi, \psi, w)$ , which tends to  $(0, 0, 0)$  uniformly in  $x$  as  $t \rightarrow +\infty$ .*

This theorem will be verified by an a priori estimate and the following local result:

**PROPOSITION 3.2.** *If  $(\phi, \psi, w)(x, 0) = g(x)$  for some piecewise  $C^1$  function  $g$  such that for any  $\Gamma > 0$*

$$\|g\|_1 + \sum_{i=1}^4 \|g\|_{C^1(\bar{\Omega}_i)} < \Gamma,$$

*then there exists a positive constant  $t_0 = t_0(\Gamma)$  independent of time such that (3.2) has a unique solution in  $X(0, t_0)$  satisfying*

$$\sup_{0 \leq t \leq t_0} (\|(\phi, \psi, w)\|_1 + \sum_{i=1}^4 \|(\phi, \psi, w)\|_{C^1(\bar{\Omega}_i)}) < C\Gamma,$$

*where  $C$  is a constant independent of  $t$  and  $\Gamma$ .*

*Proof.* The proof of this proposition can be found in [16]; we omit the details.

We denote

$$L_1 \equiv \phi_t - \psi_x = 0. \quad (3.3)$$

Combining (3.2)<sub>2</sub> and (3.2)<sub>3</sub>, one can easily get

$$L_2 \equiv \psi_{tt} - E\psi_{xx} + \phi_t - A(V, \phi)_x - B(V, U_x)_x = 0, \quad (3.4)$$

with

$$A(V, \phi) = p_R(V) - p_R(V + \phi), \quad (3.5)$$

$$(V, U_x) = (E + p'_R(V)) U_x. \quad (3.6)$$

It is clear that (3.3)–(3.4) give a closed system for  $(\phi, \psi)$  with the initial data

$$\begin{cases} \phi(x, 0) = \phi_0(x) = v_0(x) - V(x, 0) \\ \psi(x, 0) = \psi_0(x) = u_0(x) - U(x, 0) \\ \psi_t(x, 0) = \psi_1(x) = P'(x, 0) - p'(x, 0). \end{cases} \quad (3.7)$$

We now proceed the a priori estimate for the solution of (3.3)–(3.4) and (3.7) in the space of  $X(0, T)$  for some  $T > 0$ . Then the bounds on  $w$  can be derived from (3.2). In the following, we assume a priori that there is a solution  $(\phi, \psi, w) \in X(0, T)$  of (3.2).

Let

$$\gamma^2 := \sup_{0 \leq t \leq T} \left( \|(\phi, \psi, w)\|_1^2(t) + \sum_{i=1}^3 (|\phi_i|^2 + |\psi_i|^2 + |w_i|^2)(t) \right).$$

Due to the semilinearity of (2.5) and the sub-characteristic conditions, we can actually show that the jumps decay exponentially in time if we have the a priori estimate on  $\gamma$ . Hereafter, we always assume that  $\gamma < \varepsilon_0$  for some suitably small positive constants  $\varepsilon_0$ . Due to the Sobolev inequality (see (3.17) below for instance), it is clear that there are two constants  $c > c_1$  and  $d < d_1$  such that  $v \in [c, d]$ .

The following lemma is the most important part in our analysis.

**LEMMA 3.3.** *If there exist suitably small positive constants  $\varepsilon_0$  and  $\delta_0$  such that  $\gamma < \varepsilon_0$  and  $\delta < \delta_0$ , then*

$$\begin{aligned} \sum_{i=1}^3 (|\phi_i| + |\psi_i| + |w_i|)(t) &\leq C_1 \delta \exp\{-C_2 t\}, \\ \sum_{i=1}^3 (|\phi_x| + |\psi_x| + |w_x|)(t) &\leq C_1 \delta \exp\{-C_2 t\}, \\ \sum_{i=1}^3 (|\phi_t| + |\psi_t| + |w_t|)(t) &\leq C_1 \delta \exp\{-C_2 t\}, \end{aligned} \quad (3.8)$$

$$\|(\phi, \psi, w)\|_{L^\infty} + \|(\phi_x, \psi_x, w_x)\|_{L^\infty} + \|(\phi_t, \psi_t, w_t)\|_{L^\infty} \leq C_3(\varepsilon_0 + \delta_0),$$

where  $C_1$ ,  $C_2$ , and  $C_3$  are positive constants independent of  $t$ .



*Proof.* First, we will derive the above estimates on the solution of (2.6). It is convenient to use the characteristic variables of (2.5). Let

$$\begin{aligned}
 M &= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & E & 0 \end{pmatrix}, \\
 R &= \begin{pmatrix} 1 & 1 & -1 \\ \sqrt{E} & 0 & \sqrt{E} \\ -E & 0 & E \end{pmatrix}, \\
 R^{-1} &= \begin{pmatrix} 0 & \frac{1}{2\sqrt{E}} & -\frac{1}{2E} \\ 1 & 0 & \frac{1}{E} \\ 0 & \frac{1}{2\sqrt{E}} & \frac{1}{2E} \end{pmatrix}.
 \end{aligned} \tag{3.9}$$

We have

$$R^{-1}MR = \begin{pmatrix} -\sqrt{E} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{E} \end{pmatrix}.$$

Take

$$\begin{pmatrix} v \\ u \\ p \end{pmatrix} = R \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} F_1 + F_2 - F_3 \\ \sqrt{E}(F_1 + F_3) \\ E(F_3 - F_1) \end{pmatrix}. \tag{3.10}$$

Thus

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = R^{-1} \begin{pmatrix} v \\ u \\ p \end{pmatrix} = \begin{pmatrix} \frac{1}{2E}(\sqrt{E}u - p) \\ \frac{1}{E}(Ev + p) \\ \frac{1}{2E}(\sqrt{E}u + p) \end{pmatrix}. \tag{3.11}$$

From (2.5), we have

$$\begin{cases} F_{1t} - \sqrt{E} F_{1x} = -\frac{1}{2E} Q(F) \\ F_{2t} = \frac{1}{E} Q(F) \\ F_{3t} - \sqrt{E} F_{3x} = \frac{1}{2E} Q(F), \end{cases} \quad (3.12)$$

where  $Q(g) = p_R(g_1 + g_2 - g_3) - E(g_3 - g_1)$ , for  $g = (g_1, g_2, g_3)^t$ . The initial data for (3.12) can be obtained from (3.11) and  $S(x, 0)$  directly. The corresponding smooth approximation for rarefaction wave is  $R^{-1}(V, U, P)^t = G$ , and  $Q(G) = 0$ . Define

$$f = F - G = R^{-1}(\phi, \psi, w)^t.$$

It is clear that

$$[F_i]_j = [f_i]_j = 0, \quad \text{for } i \neq j, \quad (3.13)$$

and

$$\begin{aligned} [Q(F)]_1 &= Q(F)(x_1(t) + 0, t) - Q(F)(x_1(t) - 0, t) \\ &= [p_R(F_1 + F_2 - F_3)]_1 + E[F_1]_1 \\ &= \left( \int_0^1 p'_R(v_- + \theta[F_1]_1) d\theta + E \right) [F_1]_1, \end{aligned} \quad (3.14)$$

$$[Q(F)]_2 = \left( \int_0^1 p'_R(v(0-, t) + \theta[F_2]_2) d\theta \right) [F_2]_2, \quad (3.15)$$

$$[Q(F)]_3 = - \left( E + \int_0^1 p'_R(v_+ - \theta[F_3]_3) d\theta \right) [F_3]_3. \quad (3.16)$$

By the assumption  $\gamma < \varepsilon_0$ , and the Sobolev inequality, we have that for  $0 \leq t \leq T$ ,

$$\|f\|_{L^\infty}^2(t) \leq C \left( \|f\| \|f_x\| (t) + \sum_{i=1}^3 [f]_i^2(t) \right) \leq C\varepsilon_0^2. \quad (3.17)$$

Thus  $F$  and  $[F_i]_i$  ( $i = 1, 2, 3$ ) are bounded for sufficiently small  $\varepsilon_0$ . Then  $(H_1)$ – $(H_3)$  imply that there exist positive constants  $a_1, a_2$  and  $E_1$  such that

$$p'_R < -a_1, \quad p''_R > a_2, \quad \text{and} \quad |p'_R| < E_1 < E. \quad (3.18)$$

Taking the jump across  $x_1(t)$  on the both sides of the first equation in (3.12), we have

$$[F_{1t} - \sqrt{E} F_{1x}]_1 = -\frac{1}{2E} [Q(F)]_1. \quad (3.19)$$

Since  $F$  has continuous first derivatives up to the boundaries on each side of the jumps, we can interchange the tangential derivative with the jump to get  $[F_{1t} - \sqrt{E} F_{1x}]_1 = (d/dt)[F_1]_1(t)$  and solve (3.18) for  $[F_1]_1$ :

$$|[F_1]_1(t)| \leq C\delta_0 \exp\{-Ct\}. \quad (3.20)$$

Similarly, we have

$$|[F_2]_2(t)| + |[F_3]_3(t)| \leq C\delta_0 \exp\{-Ct\}. \quad (3.21)$$

For the jumps in the first derivatives, we take jump across  $x_3(t)$  on both side of (3.12)<sub>1</sub> to obtain

$$[F_{1t}]_3 - [\sqrt{E} F_{1x}]_3 = -\frac{1}{2E} [Q(F)]_3. \quad (3.22)$$

We know from (3.13) that the tangential derivative of  $F_1$  has no jump across  $x_3(t)$ , namely

$$[F_{1t}]_3 + [\sqrt{E} F_{1x}]_3 = 0. \quad (3.23)$$

It is not difficult to see from (3.16) and (3.20)–(3.23) that

$$|[F_{1t}]_3| + |[F_{1x}]_3| \leq C\delta_0 \exp\{-Ct\}. \quad (3.24)$$

Similarly, we have

$$|[F_{it}]_j| + |[F_{ix}]_j| \leq C\delta_0 \exp\{-Ct\}, \quad \text{for } i \neq j, \quad i, j = 1, 2, 3. \quad (3.25)$$

We now differentiate (3.12)<sub>1</sub> with respect to  $t$ , and then take the jump across  $x_1(t)$ , to get

$$\frac{d}{dt} [F_{1t}]_1(t) = -\frac{1}{2E} [p'_R(F_1 + F_2 - F_3)(F_{1t} + F_{2t} - F_{3t}) - E(F_{3t} - F_{1t})]_1. \quad (3.26)$$

We note that

$$\begin{aligned} [g_1 g_2]_i &= g_1 g_2(x_i(t) + 0, t) - g_1 g_2(x_i(t) - 0, t) \\ &= g_1(x_i(t) + 0, t)[g_2]_i + g_2(x_i(t) - 0, t)[g_1]_i \\ &= g_2(x_i(t) + 0, t)[g_1]_i + g_1(x_i(t) - 0, t)[g_2]_i, \end{aligned} \quad (3.27)$$

and  $F$  is in the constant equilibrium state in  $\Omega_1$  and  $\Omega_4$ . Thus we can reduce (3.26) into

$$\begin{aligned} \frac{d}{dt} [F_{1t}]_1(t) &= -\frac{1}{2E} (E + p'_R(F_1 + F_2 - F_3))(x_1(t) + 0, t) [F_{1t}]_1 \\ &\quad + O(1) [(F_{3t}, F_{2t})]_1, \end{aligned}$$

which implies that

$$|[F_{1t}]_1|(t) \leq C\delta_0 \exp\{-Ct\}. \quad (3.28)$$

Then we can get from (3.12)<sub>1</sub>, (3.13), and (3.20)–(3.25) that

$$|[F_{1x}]_1|(t) \leq C\delta_0 \exp\{-Ct\}. \quad (3.29)$$

Similarly, we have

$$(|[F_{3t}]_3| + |[F_{3x}]_3|)(t) \leq C\delta_0 \exp\{-Ct\}. \quad (3.30)$$

Taking the jump across  $x_2(t)$  on the both sides of (3.12)<sub>2</sub>, we have

$$|[F_{2t}]_2|(t) \leq C\delta_0 \exp\{-Ct\}. \quad (3.31)$$

Now, we turn to make the pointwise estimate on the first derivatives of  $F$ . We will begin with  $I(x, t) = F_{1t}$  and  $J(x, t) = F_{3t}$ . Taking the time derivative of (3.12)<sub>1,3</sub>, we have

$$\begin{cases} I_t - \sqrt{E} I_x = -a(x, t)I + a(x, t)J - H \\ J_t + \sqrt{E} J_x = a(x, t)I - a(x, t)J + H, \end{cases} \quad (3.32)$$

where

$$a(x, t) = \frac{1}{2E} (E + p'_R(F_1 + F_2 - F_3)),$$

and

$$H(x, t) = \frac{1}{2E^2} p'_R(F_1 + F_2 - F_3) Q(F).$$

It is clear that

$$I = J = 0 = H, \quad \text{in } \{|x| > \sqrt{Et}, t \leq 0\}. \quad (3.33)$$

The jumps estimates on  $I$  and  $J$  imply that

$$\begin{aligned} |(I, J)(x_1(t) + 0, t)| &\leq C\delta_0 \exp\{-Ct\}, \\ |(I, J)(x_3(t) - 0, t)| &\leq C\delta_0 \exp\{-Ct\}. \end{aligned}$$

We also note that

$$|Q(F)| = |p_R(V + \phi) - p_R(V) - w| = O(1)\gamma.$$

Applying Lemma 3.4 below, we can easily show

$$|(I, J)(x, t)| \leq C(\delta_0 + \varepsilon_0). \quad (3.34)$$

and then

$$|(F_{1x}, F_{3x})| \leq C(\delta_0 + \varepsilon_0). \quad (3.35)$$

Based on these estimates, we can bound  $F_{2t}$  by the relations

$$(F_{2t})_t = \frac{1}{E} p'_R(F_1 + F_2 - F_3) F_{2t} + O(1)(\delta_0 + \varepsilon_0)$$

and the jump estimates on  $F_{2t}$  to obtain

$$|F_{2t}| \leq C(\delta_0 + \varepsilon_0). \quad (3.36)$$

Then we can use the above results to derive the bound on  $[F_{2x}]_2(t)$  as

$$\begin{aligned} \frac{d}{dt}[F_{2x}]_2(t) &= \frac{1}{E} [(p'_R(F_1 + F_2 - F_3)(F_{1x} + F_{2x} - F_{3x}) - E(F_{3x} - F_{1x}))]_2 \\ &= \frac{1}{E} (p'_R(F_1 + F_2 - F_3)[F_{2x}]_2 + O(1)\delta_0 \exp\{-Ct\}), \end{aligned}$$

and then

$$|[F_{2x}]_2| \leq C\delta_0 \exp\{-Ct\}. \quad (3.37)$$

Thus, similar to  $F_{2t}$ , we can obtain

$$|F_{2x}| \leq C(\delta_0 + \varepsilon_0). \quad (3.38)$$

Since  $G$  is a smooth function, it is clear that the above estimates on  $F$  are also valid for  $f$  in view of Lemma 2.1. Then, Lemma 3.3 can be shown by a linear transformation from  $f$  to  $(\phi, \psi, w)$ .

The following lemma is important in the proof of the pointwise estimates on the first derivatives of solution.

**LEMMA 3.4.** *Suppose  $0 < b_1 < a(x, t) < b_2 < 1$ . The solutions of the following Goursat problem*

$$\begin{cases} I_t - \sqrt{E} I_x = -a(x, t)I + a(x, t)J - H \\ J_t + \sqrt{E} J_x = a(x, t)I - a(x, t)J + H \\ (I, J)(x_1(t) + 0, t) = (I_1, J_1)(x, t), \quad (I, J)(x_3(t) - 0, t) = (I_2, J_2)(x, t), \end{cases}$$

satisfy the estimate

$$|(I, J)| \leq C(I_0 + J_0 + H_0),$$

where

$$(I_0, J_0, H_0) = (\sup(|I_1| + |I_2|), \sup(|J_1| + |J_2|), \sup(|H|)).$$

*Proof.* A similar result can be found in [10] for the Cauchy problem. We present another proof here.

For any  $(x, t) \in \{|x| \leq \sqrt{E}t, t \geq 0\} \equiv \Omega$ , let  $\xi_1(\tau; x, t)$  be the first family characteristic across  $(x, t)$  and intersecting  $x_3(t)$  at the point  $(\sqrt{E}t_2, t_2)$ , while  $\xi_2(\tau; x, t)$  be the third family characteristic across  $(x, t)$  and intersecting  $x_1(t)$  at the point  $(-\sqrt{E}t_1, t_1)$ . Then we can give the following formulas for  $(I, J)$ :

$$\begin{aligned} I(x, t) &= I(\sqrt{E}t_2, t_2) \exp \left\{ - \int_{t_2}^t a(\xi_1(\tau; x, t), \tau) d\tau \right\} \\ &\quad + \int_{t_2}^t (aJ - H)(\xi_1(\tau; x, t), \tau) \exp \left\{ - \int_{\tau}^t a(\xi_1(\theta; x, t), \theta) d\theta \right\} d\tau, \end{aligned} \tag{3.39}$$

$$\begin{aligned} J(x, t) &= J(-\sqrt{E}t_1, t_1) \exp \left\{ - \int_{t_1}^t a(\xi_2(\tau; x, t), \tau) d\tau \right\} \\ &\quad + \int_{t_1}^t (aI + H)(\xi_2(\tau; x, t), \tau) \exp \left\{ - \int_{\tau}^t a(\xi_2(\theta; x, t), \theta) d\theta \right\} d\tau. \end{aligned} \tag{3.40}$$

Let  $M_1(t) = \|I(\cdot, t)\|_{L^\infty}$ ,  $M_2(t) = \|J(\cdot, t)\|_{L^\infty}$ , and

$$M(t) = \max\{M_1(t), M_2(t)\}. \tag{3.41}$$

Without loss of generality, we assume that  $M(t)$  is reached by  $I(x, t)$  at some point  $(x, t)$ , the case for  $J(x, t)$  can be treated similarly. We see from (3.39) that

$$\begin{aligned} M(t) &\leq I_0 + \int_0^t (b_2 M(\tau) + H) \exp\{-b_1(t-\tau)\} d\tau \\ &\leq \left(I_0 + \frac{H_0}{b_1}\right) + \int_0^t b_2 M(\tau) \exp\{-b_1(t-\tau)\} d\tau. \end{aligned} \quad (3.42)$$

Applying the Gronwall's inequality, we have

$$M(t) \leq \left(I_0 + \frac{H_0}{b_1}\right) \exp\left\{\frac{b_2}{b_1}t\right\}. \quad (3.43)$$

This completes the proof of this lemma.

We now proceed the main energy estimate. The method is a modified version of [6] in studying the nonlinear stability of rarefaction waves under smooth perturbation.

**LEMMA 3.5.** *Suppose (H<sub>1</sub>)–(H<sub>3</sub>) are satisfied,  $\delta < \delta_0$ , and  $\gamma < \varepsilon_0$  for some suitably small  $\delta_0$  and  $\varepsilon_0$ . Then we have*

$$\begin{aligned} &\|(\phi, \phi_x, \psi, \psi_t, \psi_x)(t)\|^2 + \int_0^t \|(\psi_x, \psi_t, V_t^{1/2} \phi, \phi_x)(\tau)\|^2 d\tau \\ &\leq C(\|(\phi, \psi, w)(\cdot, 0)\|_1^2 + \delta_0). \end{aligned} \quad (3.44)$$

*Proof.* We consider the equality

$$AL_1 + (\mu\psi_t + \psi) L_2 = 0 \quad (3.45)$$

with a positive constant  $\mu = (E_1 + E)/2E_1$ . Equation (3.45) can be reduced to

$$\begin{aligned} &\left(\frac{1}{2}\psi^2 + \frac{\mu}{2}\psi_t^2 + \psi\psi_t\right)_t + A\phi_t - \mu A_x\psi_t - \mu E\psi_t\psi_{xx} - E\psi\psi_{xx} \\ &+ (\mu - 1)\psi_t^2 - B_x(\psi + \mu\psi_t) - (A\psi)_x = 0. \end{aligned} \quad (3.46)$$

We perform the calculations

$$-\mu E\psi_t\psi_{xx} = -(\mu E\psi_t\psi_x)_x + \left(\frac{1}{2}\mu E\psi_x^2\right)_x, \quad (3.47)$$

$$-E\psi\psi_{xx} = -(E\psi\psi_x)_x + E\psi_x^2. \quad (3.48)$$

By Taylor's formula, it follows

$$A = -p'_R(V)\phi + g(V, \phi)\phi^2, \quad (3.49)$$

where  $g(V, \phi)$  is a smooth function. Thus

$$\begin{aligned} A\phi_t &= -p'_R(V)\phi\phi_t + g\phi^2\phi_t \\ &= \left(\frac{1}{2}(D - \frac{1}{3}g\phi)\phi^2\right)_t + \frac{1}{3}g_\phi\phi^3\psi_x + \left(\frac{1}{2}p''_R(V) + \frac{1}{3}g_V\phi\right)V_t\phi^2, \end{aligned} \quad (3.50)$$

where  $D = -p'_R(V) + g\phi$ . We see that

$$-\mu A_x\psi_t = -(\mu A\psi_t)_x + (\mu A\psi_x)_t - \mu A_t\psi_x, \quad (3.51)$$

and

$$\mu A\psi_x = \mu D\phi\psi_x, \quad (3.52)$$

$$-\mu A_t\psi_x = \mu(p'_R(V) - g_\phi\phi^2 - 2g\phi)\psi_x^2 + \mu(p''_R(V) - g_V\phi)V_t\phi\psi_x. \quad (3.53)$$

Hence,

$$\begin{aligned} -\mu A_x\psi_t &= (\mu D\phi\psi_x)_t - (\mu A\psi_t)_x + \mu(p'_R(V) - g_\phi\phi^2 - 2g\phi)\psi_x^2 + \mu(p''_R(V) \\ &\quad - g_V\phi)V_t\phi\psi_x. \end{aligned} \quad (3.54)$$

Therefore, (3.47) turns into

$$(G_4 + G_5) + \sum_{l=6}^{10} G_l + G_{11x} = 0, \quad (3.55)$$

where

$$\begin{aligned} G_4 &= \frac{1}{2}\psi^2 + \frac{\mu}{2}\psi_t^2 + \psi\psi_t \\ G_5 &= \frac{1}{2}\left(D - \frac{1}{3}g\phi\right)\phi^2 + \mu D\phi\psi_x + \frac{1}{2}\mu\psi_x^2 \\ G_6 &= (E + \mu(p'_R(V) - g_\phi\phi^2 - 2g\phi))\psi_x^2 \\ G_7 &= (\mu - 1)\psi_t^2 \\ G_8 &= \frac{1}{2}p''_R(V) + \frac{1}{3}g_V\phi)V_t\phi^2 \\ G_9 &= \mu(p''_R(V) - g_V\phi)V_t\phi\psi_x \\ G_{10} &= \frac{1}{3}g_\phi\phi^3\psi_x - B_x(\psi + \mu\psi_t) \\ G_{11} &= -(A\psi + E\psi\psi_x + \mu E\psi_t\psi_x + \mu A\psi_t). \end{aligned} \quad (3.56)$$



Due to  $(H_1)$ – $(H_3)$ , and the smallness of  $\delta_0$  and  $\varepsilon_0$ , it holds that

$$E > E_1 + c_1 \varepsilon_0 > D - \frac{1}{3} g \phi > c_2 > 0, \quad \text{and} \quad E > E_1 + c_1 \varepsilon_0 > D > c_2 > 0,$$

for some positive constants  $c_1$  and  $c_2$ . Thus, there are positive constants  $c_i$  ( $i = 3, \dots, 9$ ) such that

$$\begin{aligned} c_3(\psi^2 + \psi_t^2) &\leq G_4 \leq c_4(\psi^2 + \psi_t^2) \\ c_5(\phi^2 + \psi_x^2) &\leq G_5 \leq c_6(\phi^2 + \psi_x^2) \\ G_6 + G_7 &\geq c_7(\psi_x^2 + \psi_t^2) \\ G_8 &\geq c_8 V_t \phi^2 \\ |G_9| &\leq \frac{1}{2} c_8 V_t \phi^2 + c_9 \delta_0 \psi_x^2. \end{aligned} \tag{3.57}$$

Now we integrate (3.55) over  $[0, t] \times (-\infty, +\infty)$ . Integrating by parts, and using the estimates in Lemma 3.3, we arrive at

$$\begin{aligned} &\|(\phi, \psi, \psi_t, \psi_x)\|^2(t) + \int_0^t \|(\psi_t, \psi_x, V_t \phi^2)\|^2(\tau) d\tau \\ &\leq C \|(\phi, \psi, w)(\cdot, 0)\|_1^2 + C \delta_0 (\delta_0 + \varepsilon_0) + \left| \int_0^t \mathcal{F}(G_{10}) d\tau \right|, \end{aligned} \tag{3.58}$$

where we have used the following formulas in integration by parts:

$$\begin{aligned} \int_0^t \mathcal{F}(h_t) d\tau &= \mathcal{F}(h(t)) - \mathcal{F}(h(0)) + \sqrt{E} \int_0^t ([h]_3 - [h]_1)(\tau) d\tau \\ \int_0^t \mathcal{F}(h_x) d\tau &= \int_0^t \left( h(+\infty, \tau) - h(-\infty, \tau) - \sum_{i=1}^3 [h]_i(\tau) \right) d\tau. \end{aligned}$$

By Lemma 3.3, all the jump terms caused by using integrations by parts can be bounded by  $C \delta_0 (\delta_0 + \varepsilon_0)$ .

We estimate each term in  $G_{10}$  next.

Using Young's inequality and Lemma 2.1, it can be shown that

$$\begin{aligned} \int_0^t \mathcal{F}(|B_x \psi_t|) d\tau &\leq \alpha_1 \int_0^t \mathcal{F}(\psi_t^2) d\tau + C(\alpha_1) \int_0^t \mathcal{F}(B_x^2) d\tau \\ &\leq \alpha_1 \int_0^t \|\psi_t^2\|^2(\tau) d\tau + C(\alpha_1) \delta_0^2, \end{aligned} \tag{3.59}$$

and

$$\begin{aligned}
\left| \int_0^t \mathcal{F}(B_x \psi) d\tau \right| &\leq C \int_0^t (\|\psi\|^{1/2} \|\psi_x\|^{1/2} + \delta_0 \exp\{-C\tau\}) \|B_x\|_{L^1} d\tau \\
&\leq C \int_0^t (\|\psi\|^2 \|\psi_x\|^2 + \|B_x\|_{L^1}^{4/3}) d\tau + C\delta_0^2 \\
&\leq C \left( \delta_0^{4/3} + \delta_0^2 + \varepsilon_0^2 \int_0^t \|\psi_x\|^2 d\tau \right), \tag{3.60}
\end{aligned}$$

where we have used the Sobolev inequality. We also note that

$$\left| \int_0^t \mathcal{F}(\psi_x \phi^3) d\tau \right| \leq \alpha_2 \int_0^t \|\psi_x^2\|^2(\tau) d\tau + C(\alpha_2) \int_0^t \mathcal{F}(\phi^6) d\tau, \tag{3.61}$$

and

$$\begin{aligned}
\int_0^t \mathcal{F}(\phi^6) d\tau &\leq \int_0^t \|\phi\|^4 \|\phi_x\|^2 dx d\tau + C\delta_0^2 \\
&\leq \varepsilon_0^4 \int_0^t \|\phi_x\|^2(\tau) d\tau + C\delta_0^2. \tag{3.62}
\end{aligned}$$

Thus, for suitably small  $\alpha_1$  and  $\alpha_2$ , (3.58)–(3.62) imply that

$$\begin{aligned}
&\|(\psi, \psi_t, \phi, \psi_x)(t)\|^2 + \int_0^t \|\psi_t, \psi_x, V^{1/2}\phi(\tau)\|^2 d\tau \\
&\leq C\varepsilon_0^4 \int_0^t \|\phi_x(\tau)\|^2 d\tau + C\delta_0^{43} + C\delta_0(\delta_0 + \varepsilon_0) + C\|(\phi, \psi, w)(\cdot, 0)\|_1^2. \tag{3.63}
\end{aligned}$$

To bound  $\phi_x$ , we investigate the equation

$$\begin{aligned}
(E\phi_x - \psi_t) \partial_x L_1 - \phi_x L_2 &= (\tfrac{1}{2}E\phi_x^2 - \psi_t \phi_x - \tfrac{1}{2}\psi_x^2)_t + (\psi_t \psi_x)_x \\
&\quad + A_x \phi_x + \phi_x \psi_t + B_x \phi_x = 0, \tag{3.64}
\end{aligned}$$

where

$$A_x \phi_x = (-p'_R(V) + g_\phi \phi^2 + 2g\phi) \phi_x^2 + g_V V_x \phi^2 \phi_x - p''_R(V) V_x \phi \phi_x. \tag{3.65}$$

Then the Cauchy inequality, Lemma 2.1, and (3.63) yield

$$\begin{aligned}
&\|\phi_x\|^2(t) + \int_0^t \|\phi_x\|^2(\tau) d\tau \\
&\leq C\delta_0^{4/3} + C\delta_0(\delta_0 + \varepsilon_0) + C\|(\phi, \psi, w)(\cdot, 0)\|_1^2. \tag{3.66}
\end{aligned}$$

Inequalities (3.63) and (3.66) imply (3.44), and the proof of Lemma 3.5 is completed then.

The next aim is to deduce the estimate on  $w$  by Eqs. (3.2) and the above results. In fact, we can show the following lemma.

LEMMA 3.6. *Under the conditions cited in Lemma 3.5, we have*

$$\begin{aligned} & \| (w, w_x, w_t)(t) \|^2 + \int_0^t \| (w_x, w_t)(\tau) \|^2 d\tau \\ & \leq C(\|(\phi, \psi, w)(\cdot, 0)\|_1^2 + \delta_0). \end{aligned} \quad (3.67)$$

*Proof.* By (3.2), we see that  $w_x = -\psi_t$ . Thus the estimate of  $w_x$  in (3.67) comes from Lemma 3.5 directly. Turn to  $w_t$  next. We know from (3.12) that

$$L_3 \equiv w_{tt} - Ew_{xx} + w_t + A_t + B_t = 0. \quad (3.68)$$

Thus,

$$\begin{aligned} w_t L_3 &= w_t w_{tt} - Ew_t w_{xx} + w_t^2 + A_t w_t + B_t w_t \\ &= \left(\frac{1}{2}w_t^2 + \frac{1}{2}Ew_x^2\right)_t - E(w_x w_t)_x + w_t^2 - A_t w_t - B_t w_t \\ &= 0. \end{aligned} \quad (3.69)$$

Integrating (3.69) over  $[0, t] \times (-\infty, +\infty)$ , integrating by parts, and using the Cauchy inequality with the estimate in Lemma 3.5, we have

$$\|w_t\|^2(t) + \int_0^t \|w_t\|^2(\tau) d\tau \leq C(\|(\phi, \psi, w)(0)\|_1^2 + \delta_0). \quad (3.70)$$

At last, we can get the estimate on  $w$  by taking  $L^2$ -norm in the third equation of (3.2) directly.

We conclude from Lemmas 3.2–3.6 that there exists a unique solution  $(\phi, \psi, w)$  for (3.2) globally in time, satisfying the estimates

$$\sup_{t \geq 0} \|(\phi, \psi, w)(\cdot, t)\|_1^2 + \int_0^{+\infty} \|(\phi_x, \psi_x, w_x, \phi_t, \psi_t, w_t)\|^2 dt \leq C\delta_0, \quad (3.71)$$

$$\sup_{t \geq 0} \left( \sum_{i=1}^4 \|(\phi, \psi, w)\|_{C^1(\bar{\mathcal{Q}}_i)} \right) \leq C\delta_0, \quad (3.72)$$

and the jumps estimates in Lemma 3.3 are valid for all time.

Then we can finish the proof of Theorem 3.1 by the Sobolev inequality as follows. We have from (3.71) that

$$\int_0^{+\infty} \left\{ \|(\phi_x, \psi_x, w_x)\|^2 + \left| \frac{d}{dt} \|(\phi_x, \psi_x, w_x)\|^2 \right| \right\} dt \leq +\infty.$$

This yields

$$\lim_{t \rightarrow +\infty} \|(\phi_x, \psi_x, w_x)(t)\|^2 = 0,$$

and then

$$\lim_{t \rightarrow +\infty} \sup_{x \neq x_i(t)} |(\phi, \psi, w)(x, t)| = 0.$$

This together with the jump estimates in Lemma 3.3 and Lemma 2.1 gives

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbf{R}} |S(x, t) - S'(x/t)| = 0.$$

Thus, we have completed the proof of Theorem 3.1 and therefore Theorem 1 can be proved.

## ACKNOWLEDGMENTS

This work was suggested by Professor Zhouping Xin of Courant Institute when he visited the Morningside Center of Mathematics, Academia Sinica. The authors thank him for this suggestion. We are grateful for the discussion with Mr. Hailiang Li. Special thanks to the referee for the comments in improving the presentation of this paper.

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