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The zero relaxation behavior of piecewise smooth solutions to the reacting flow model in the presence of shocks

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1. Introduction

The goal of this paper is to investigate the asymptotic behavior of a reacting flow of two modes with source terms as the relaxation time (or specially the reaction time) goes to zero. The system in Eulerian form can be written as

$$\begin{cases} (\rho r)_t + (\rho u r)_x = S, \\ (\rho s)_t + (\rho u s)_x = -S, \\ (\rho u)_t + (p + \rho u^2)_x = 0, \end{cases} \tag{R.F.}$$

which was proposed by LeVeque et al. [5] to model the motion of reacting gas with two modes. Where, ρr is the density of the major mode and ρs corresponds to the minor mode, $r + s = 1$. u is the velocity, and $p = \rho c^2(r + \beta s)$ is the pressure which can be derived by Avogadro's Law. Here, c is the sound speed of the major mode. The parameter β provides some tenuous link with real physics, if it is considered as the number of molecules of the minor species produced from one molecule of the major species. S is the source term

$$S = \frac{\rho(r_{\mathrm{E}}(\rho) - r)}{\varepsilon} = -\frac{\rho(s_{\mathrm{E}}(\rho) - s)}{\varepsilon},$$

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where ε is the reaction time, $r_E(\rho)$ and $s_E(\rho)$ are equilibrium distributions. The reader is refered to [5] for more physical and numerical background.

The equations in Lagrangian form of (R.F) can be written as (see [3])

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \quad x \in \mathbf{R}^1, t > 0, \\ p_t + pv^{-1}v_t = \dfrac{1}{\varepsilon}(p_R(v) - p), \end{cases} \tag{1.1}$$

where $v > 0$ is the specific volume, $p > 0$, $p_R(v)v = c^2(r_E + \beta(1 - r_E))$.

As the relaxation time ε goes to zero, we obtain formally the following well-known p-system:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_R(v)_x = 0, \end{cases} \tag{1.2}$$

which is exactly the leading order of the Hilbert expansion of the smooth solutions to (1.1) about ε. Therefore, we expect that (1.2) controls the evolution of the solutions to (1.1) as $\varepsilon \to 0$. For smooth flow, this expectation can be easily realized by Hilbert expansion and a standard energy estimate argument. However, as for the case when discontinuities occur in the solutions to (1.2), the analysis is much more complicated and difficulties appear.

In this paper, inspired by [1, 9], we use the method of matched multiscale asymptotic analysis introduced in [1] to conquer the difficulties, i.e., constructing approximate solutions by matched multiscale asymptotic expansion and analyzing the stability of the approximate solutions. More precisely, we first construct a formal solution to (1.1) by matching the truncated Hilbert expansion (outer expansion, see Section 2.1) and shock layer expansion (inner expansion, see Section 2.2). Then the existence of solution to (1.1) and its convergence to the solution to (1.2) are reduced to the stability analysis of the approximate solution (see [1]). However, because the dissipation of relaxation is much weaker than viscosity, the asymptotic behavior here is much more singular than that in [1]. So we have to use higher-order correction to weaken the nonlinearity in the error equation between (1.1) and (1.2). On the other hand, different from [9], where the equation is semilinear and the smooth steady shock profile of it can be constructed explicitly, system (1.1) is quasilinear and only an abstract result on the existence of shock profiles can be obtained. Besides, the leading order of shock layer expansion, the time-dependent shock profile, is a steady shock profile with parameters varying with time. This leads to difficulties. Furthermore, the quasilinearity of Eq. (1.1) also leads to more difficulties than the semilinear case discussed in [9]. Therefore, much more information on the shock profiles of (1.1) and more careful analysis related to the quasilinearity of (1.1) are required when we construct the approximate solutions and make the corresponding energy estimates in H^3 space. The main result in the present paper is to show that the piecewise smooth solutions to (1.2) with finitely noninteracting shocks satisfying the Lax-entropy condition and subcharacteristic condition are strong limits of the solutions to (1.1) as $\varepsilon \to 0$.

For $v>0$, we make the following assumptions:
(H_1) $p_R'(v)<0$,
(H_2) $p_R''(v)>0$.

It is easy to know that, under (H_1) and (H_2), (1.2) is strictly hyperbolic and genuinely nonlinear, with eigenvalues

$$\lambda_1=-(-p_R'(v))^{1/2}<0<(-p_R'(v))^{1/2}=\lambda_2. \tag{1.3}$$

For simplicity, we only discuss the case when the piecewise smooth solution of (1.2) is a single-shock solution. A function $(v_0(x,t),u_0(x,t))$ is called a single-shock solution of (1.2) up to time $T>0$ if

(i) $(v_0(x,t),u_0(x,t))$ is a distributional solution of (1.2) in $\mathbf{R}^1\times[0,T]$.
(ii) There is a smooth curve, the shock, $x=s(t)$, $0\le t\le T$, such that $(v_0(x,t),u_0(x,t))$ is sufficiently smooth away from $x=s(t)$ and the left and right limits of $(v_0(x,t), u_0(x,t))$ and its derivatives at the shock $x=s(t)$ exist.
(iii) Across the shock $x=s(t)$, the Rankine–Hugoniot condition holds

$$\dot{s}(v_0^l-v_0^r)=u_0^l-u_0^r,$$
$$\dot{s}(u_0^l-u_0^r)=-(p_R(v_0^l)-p_R(v_0^r)). \tag{1.4}$$

(iv) The Lax-entropy condition

$$\lambda_1^l<\dot{s}<\lambda_1^r \text{ or } \lambda_2^r<\dot{s}<\lambda_2^l$$

are satisfied. Here and in the following, we always use the notations $f^l=f(s(t)-0,t)$ and $f^r=f(s(t)+0,t)$.

Without loss of generality, we only consider the case when the shock is in the second family, i.e.

$$\lambda_2^r(t)<\dot{s}(t)<\lambda_2^l(t). \tag{1.5}$$

We also make another assumption: for some constants v_1,v_2 satisfying

$$0<v_1<\min_{0\le t\le T} v_0^l(t), \quad \max_{0\le t\le T} v_0^r(t)<v_2,$$

(H_3) $|p_R'(v)|<\inf_{0\le t\le T}\min\{p_R(v_0^l(t))/v_0^l(t), p_R(v_0^r(t))/v_0^r(t)\}$, $v\in[v_1,v_2]$,
holds, where (H_3) is the so-called subcharacteristic condition (see [7]).

Theorem 1. *Assume that* $(v_0,u_0)(x,t)$ *is a single-shock solution of* (1.2) *up to time* $T>0$, *and let* $p_0=p_R(v_0)$. *Under* (H_1)–(H_3), *there exist positive constants* η_0 *and* ε_0, *such that if*

$$\sum_{1\le\alpha\le 7}\left(\int_0^T\int_{x<s(t)}+\int_0^T\int_{x>s(t)}\right)|\partial_x^\alpha(v_0,u_0,p_0)(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t<+\infty \tag{1.6}$$

and

$$|v_0^r-v_0^l|+|u_0^r-u_0^l|+|p_0^r-p_0^l|\le\eta_0, \quad \forall t\in[0,T], \tag{1.7}$$

then for any $\varepsilon \in (0, \varepsilon_0]$, *there is a smooth solution* $(v^\varepsilon, u^\varepsilon, p^\varepsilon)(x,t)$ *of* (1.1) *satisfying*

$$(v^\varepsilon, u^\varepsilon, p^\varepsilon) \in L^\infty([0,T], H^2). \tag{1.8}$$

Moreover, *for any given* $\alpha \in (0,1)$,

$$\sup_{0\le t\le T} \int_{\mathbf{R}^1} |(v^\varepsilon - v_0, u^\varepsilon - u_0, p^\varepsilon - p_0)(x,t)|^2 \, \mathrm{d}x \le C_1 \varepsilon^\alpha \tag{1.9}$$

and

$$\sup_{\substack{0\le t\le T \\ |x-s(t)|\ge h}} |(v^\varepsilon - v_0, u^\varepsilon - u_0, p^\varepsilon - p_0)(x,t)| \le C_h \varepsilon, \quad \forall h > 0, \tag{1.10}$$

where C_1 *and* C_h *are positive constants independent of* ε.

Remark. (i) A careful analysis can show that conditions (H_2) and (H_3) are only required to hold in a domain of $v_0(x,0)$.

(ii) The advantages of the matched asymptotic analysis method are that the structure of the solution $(v^\varepsilon, u^\varepsilon, p^\varepsilon)$ in the Theorem 1 will be clear, since it is a perturbation of a formal solution which will be constructed explicitly.

(iii) The solutions $(v^\varepsilon, u^\varepsilon, p^\varepsilon)$ have carefully chosen initial data which are essentially those of the Hilbert expansion and the shock-layer expansion.

(iv) In particular, we have that away from the shock, $(v^\varepsilon, u^\varepsilon, p^\varepsilon)$ approximates (v_0, u_0, p_0) at an optimal rate in ε, i.e. (1.10).

(v) The same results hold for finitely noninteracting shock solutions of (1.2).

(vi) Due to our analysis, it is clear that the technique used here can be extended to the general relaxation systems proposed by [4].

There are other results about relaxation system and its zero relaxation limit such as [2–4, 6–10] and references therein.

In the next section, we construct the approximate solutions by use of the matched multiscale asymptotic expansion method. The existence and asymptotic behavior of the solutions to (1.1) are proved in Section 3.

2. Construction of approximate solution

In this section we will construct an approximate solution for (1.1) by using the methods of matched multiscale asymptotic expansions. The outer solutions are constructed by the Hilbert expansion and inner solutions are obtained by shock layer expansion. By matching the outer and inner solutions in an appropriate "matching zone", we can get the outer and inner functions with an appropriate order and make a formal approximate solution to (1.1).

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ p_t + pv^{-1}u_x = \dfrac{1}{\varepsilon}(p_R(v) - p). \end{cases} \tag{2.1}$$

2.1. Outer expansion

Let $\chi^0(x,t)=(v^0,u^0,p^0)(x,t)$ and $\chi_i(x,t)=(v_i,u_i,p_i)(x,t)$, $i=0,1,2,\dots$. In the domain away from the shock $x=s(t)$, solutions to (2.1) may be approximated by

$$\chi^0(x,t)+\varepsilon\chi_1(x,t)+\varepsilon^2\chi_2(x,t)+\dots,\quad x\neq s(t). \tag{2.2}$$

Substituting (2.2) into (2.1) and comparing the coefficients of power ε, we get

$$O(\varepsilon^{-1})\, p_R(v^0(x,t))=p^0, \tag{2.3}$$

$$O(1)\begin{cases}\begin{cases} v^0_t-u^0_x=0,\\ u^0_t+p^0_x=0,\end{cases}\\ p^0_t+p_0v_0^{-1}u^0_x=-p_1+p_R'(v^0)v_1,\end{cases} \tag{2.4}$$

$$O(\varepsilon)\begin{cases}\begin{cases} v_{1t}-u_{1x}=0,\\ u_{1t}+p_{1x}=0,\end{cases}\\ p_{1t}+p_0v_0^{-1}u_{1x}+[p_1v_0^{-1}-p_0v_2^{-1}v_1]u^0_x\\ \quad =-p_2+p_R'(v^0)v_2+\frac{1}{2}p_R''(v^0)v_1^2,\end{cases} \tag{2.5}$$

$$O(\varepsilon^2)\begin{cases}\begin{cases} v_{2t}-u_{2x}=0,\\ u_{2t}+p_{2x}=0,\end{cases}\\ \text{etc.}\end{cases} \tag{2.6}$$

Combined with (2.3), first and second equations of (2.4) becomes a closed system for (v^0,u^0). This system is equivalent to (1.2). Its solution can be chosen as the single-shock solution of (1.2). Hence we set

$$\chi^0=(v_0,u_0,p_0)(x,t)\equiv\chi_0(x,t),\quad x\neq s(t). \tag{2.7}$$

Next, we can obtain from the third equation of (2.4) that

$$\begin{aligned} p_1 &= -(p^0_t+p_0v_0^{-1}u^0_x)+p_R'(v_0)v_1\\ &= -(p_0v_0^{-1}+p_R'(v_0))u_{0x}+p_R'(v_0)v_1,\end{aligned}\quad x\neq s(t). \tag{2.8}$$

We conclude from (2.8) that the first and second equations of (2.5) becomes the following closed system of $(v_1,u_1)(x,t)$:

$$\begin{cases} v_{1t}-u_{1x}=0,\\ u_{1t}+(p_R'(v_0)v_1)_x=((p_0v_0^{-1}+p_R'(v_0))u_{0x})_x,\end{cases}\quad x\neq s(t) \tag{2.9}$$

and (2.8) and (2.9) are equations for $\chi_1(x,t)$.

Similarly, $\chi_2(x,t)$ satisfies

$$\begin{aligned} p_2=&-(p_{1t}+p_0v_0^{-1}u_{1x}+[p_1v_0^{-1}-p_0v_2^{-1}v_1]u^0_x)\\ &+p_R'(v_0)v_2+\frac{1}{2}p_R''(v_0)v_1^2\end{aligned} \tag{2.10}$$

and

$$\begin{cases} v_{2t} - u_{2x} = 0, \\ u_{2t} + (p_R'(v_0)v_2)_x = p_{1t} + p_0 v_0^{-1} u_{1x} \qquad x \neq s(t), \\ \quad + (p_1 v_0^{-1} - p_0 v_2^{-1} v_1) u_x^0 - \frac{1}{2} p_R''(v_0) v_1^2)_x, \end{cases} \tag{2.11}$$

The above process can be used to find higher-order outer functions $\chi_i(x,t)$, $i \geq 3$ which are expected to be smooth away from the shock uniformly up to $x = s(t)$.

2.2. Inner expansion and matching conditions

Near the shock, the solution of (2.1) will be represented by a shock layer expansion of the form

$$X_0(\xi,t) + \varepsilon X_1(\xi,t) + \varepsilon^2 X_2(\xi,t) + \cdots, \tag{2.12}$$

where $X_i = (V_i, U_i, P_i)$, $i = 0,1,2,3,\ldots,\xi$ is given by

$$\xi = \frac{x - s(t)}{\varepsilon} + \delta(t,\varepsilon) \tag{2.13}$$

and $\delta(t,\varepsilon)$ is a perturbation of the shock position to be determined later and has the form of

$$\delta(t,\varepsilon) = \delta_0(t) + \varepsilon\delta_1(t) + \varepsilon^2\delta_2(t) + \cdots . \tag{2.14}$$

Substituting (2.12)–(2.14) into (2.1) and matching the powers of ε, we have

$$O(\varepsilon^{-1}) \begin{cases} -\dot{s}V_{0\xi} - U_{0\xi} = 0, \\ -\dot{s}U_{0\xi} + P_{0\xi} = 0, \\ -\dot{s}P_{0\xi} + P_0 V_0^{-1} U_{0\xi} = p_R(V_0) - P_0, \end{cases} \tag{2.15}$$

$$O(1) \begin{cases} -\dot{s}V_{1\xi} - U_{1\xi} + V_{0t} + \dot{\delta}_0(t)V_{0\xi} = 0, \\ -\dot{s}U_{1\xi} + P_{1\xi} + U_{0t} + \dot{\delta}_0(t)U_{0\xi} = 0, \\ -\dot{s}P_{1\xi} + P_0 V_0^{-1} U_{1\xi} + (P_1 V_0^{-1} - P_0 V_0^{-2} V_1) U_{0\xi} \\ \quad = -(P_{0t} + \dot{\delta}_0(t) P_{0\xi}) + p_R'(V_0)V_1 - P_1, \end{cases} \tag{2.16}$$

$$O(\varepsilon) \begin{cases} -\dot{s}V_{2\xi} - U_{2\xi} + V_{1t} + \dot{\delta}_0(t)V_{1\xi} + \dot{\delta}_1(t)V_{0\xi} = 0, \\ -\dot{s}U_{2\xi} + P_{2\xi} + U_{1t} + \dot{\delta}_0(t)U_{1\xi} + \dot{\delta}_1(t)U_{0\xi} = 0, \\ -\dot{s}P_{2\xi} + P_0 V_0^{-1} U_{2\xi} + (P_1 V_0^{-1} - P_0 V_0^{-2} V_1) U_{1\xi} \\ \quad + (P_2 V_0 + P_0 V_0^{-3} V_1^2 - 2P_1 V_0^{-2} V_1 - P_0 V_0^{-2} V_2) U_{0\xi} \\ \quad + P_{1t} + \dot{\delta}_0(t) P_{1\xi} + \dot{\delta}_1(t) P_{0\xi} = p_R'(V_0)V_2 - P_2 + \frac{1}{2} p_R''(V_0) V_1^2, \end{cases} \tag{2.17}$$

etc.

The above inner expansion is to be true in a zone of size $O(\varepsilon)$ around $x = s(t)$.

In order to construct a smooth solution to (1.1), the outer expansion and the inner expansion are required to be valid and to agree with each other in the "matching zone", where $|\xi|\to\infty$ and $|x-s(t)|$ is small. Expressing the outer solutions in terms of ξ and using Taylor's expansion, we easily get the following "matching conditions" of the outer expansion and the inner expansion as $\xi\to\mp\infty$:

$$X_0(\xi,t)=\chi_0(s(t)\mp 0,t)+o(1), \tag{2.18}$$

$$X_1(\xi,t)=\chi_1(s(t)\mp 0,t)+(\xi-\delta_0)\chi_{0x}(s(t)\mp 0,t)+o(1), \tag{2.19}$$

$$X_2(\xi,t)=\chi_2(s(t)\mp 0,t)+(\xi-\delta_0)\chi_{1x}(s(t)\mp 0,t)$$
$$-\delta_1\chi_{0x}(s(t)\mp 0,t)+\tfrac{1}{2}(\xi-\delta_0)^2\chi_{0xx}(s(t)\mp 0,t)+o(1), \tag{2.20}$$

etc.

Eqs. (2.18)–(2.20) require that inner functions have algebraic decay rates as $\xi\to\mp\infty$.

2.3. Constructions of the outer and inner functions

We need to construct the outer and inner functions order by order at the same time, make sure the corresponding matching conditions, and determine the value of $\delta(t,\varepsilon)$.

The leading order of outer functions, $\chi_0(x,t)$, is exactly the single-shock solution to (1.2) in Theorem 1. For any fixed t (viewed as a parameter), $X_0(\xi,t)$ determined by (2.15) is just the travelling wave solution to (1.1) with the boundary conditions (2.18). Up to a phase shift, $X_0(\xi,t)$ can be uniquely determined (see [3]). Since the shift can be absorbed by $\delta(t,\varepsilon)$, we can take it as zero. Although, we could not get the explicit formula for $X_0(\xi,t)$ as in [9], we have the following properties of the travelling wave (see [3,8]).

Lemma 2.1 (Shock profile). *Under the Lax-entropy condition and the subcharacteristic condition,* (2.15) *and* (2.18) *has a smooth solution* $\Phi(v_0^l,u_0^l,p_0^l,\xi,\dot{s})=(V_0,U_0,P_0)$, *which is unique up to a shift in* ξ *and satisfies* $V_{0\xi}>0$, *and*

$$|\Phi-(v_0^l,u_0^l,p_0^l)|\le O(1)|v_0^r-v_0^l|\exp(-C_3|\xi|),$$

$$|V_{0\xi\xi}|\le O(1)|v_0^r-v_0^l|V_{0\xi},$$

$$|\partial_\xi^k\Phi|<C_1|v_0^r-v_0^l|\exp(-C_2|\xi|),\quad 1\le k\le 3,\ \text{as } \xi\to-\infty,$$

where C_i $(i=1,2,3)$ *are positive constants. Similar results hold as* $\xi\to+\infty$ *if we replace* v_0^l,u_0^l,p_0^l *by* v_0^r,u_0^r,p_0^r *and make some revisions.*

So far, we have constructed the leading order functions $\chi_0(x,t)$ and $X_0(\xi,t)$. Now, we continue to construct the second-order functions $\chi_1(x,t)$ and $X_1(\xi,t)$. It will be shown in the following that χ_1, X_1 and $\delta_0(t)$ must be determined at the same time.

Integrating first and second equations of (2.16) over $[0,\xi]$, we have

$$\begin{cases} \dot{s}V_1 + U_1 = \dot{\delta}_0 V_0 + \int_0^\xi V_{0\xi}\,\mathrm{d}\xi + c_1(t), \\ \dot{s}U_1 - P_1 = \dot{\delta}_0 U_0 + \int_0^\xi U_{0\xi}\,\mathrm{d}\xi + c_2(t), \end{cases} \tag{2.21}$$

where $c_1(t)$ and $c_2(t)$ are integration constants to be determined. Eqs. (2.16) and (2.21) lead to

$$U_{1\xi} = fU_1 + Q, \tag{2.22}$$

where

$$f = (\dot{s}^2 + p_R'(V_0) + (\dot{s}V_0^{-1} + P_0V_0^{-2})U_{0\xi})(\dot{s}(\dot{s}^2 - P_0V_0^{-1}))^{-1},$$

$$Q = \frac{1}{\dot{s}(\dot{s}^2 - P_0V_0^{-1})}\left(-\dot{s}\int_0^\xi U_{0t}\,\mathrm{d}\xi - p_R'(V_0)\right)$$

$$\int_0^\xi V_{0t}\,\mathrm{d}\xi - (\dot{s}U_0 + p_R'(V_0)V_0)\dot{\delta} + \dot{s}(\dot{s}U_{0t} + P_{0t}) + \dot{s}c_2(t) - p_R'(V_0)c_1(t)$$

$$+U_{0\xi}\left(\dot{s}V_0^{-1}\left(c_2(t) - \int_0^\xi U_{0t}\,\mathrm{d}\xi\right) - P_0V_0^{-2}\left(c_1(t) + \int_0^\xi V_{0t}\,\mathrm{d}\xi\right)\right.$$

$$\left.\left. -\ \dot{\delta}(\dot{s}V_0^{-1}U_0 + P_0V_0^{-1})\right)\right).$$

From (2.22) we obtain

$$U_1(\xi,t) = \left(\exp\left\{\int_0^\xi f(\eta,t)\,\mathrm{d}\eta\right\}\right)\left(\int_0^\xi \exp\left\{\int_0^\eta -f(\lambda,t)\,\mathrm{d}\lambda\right\}Q(\eta,t)\,\mathrm{d}\eta\right). \tag{2.23}$$

Then X_1 can be determined, provided that $c_1(t)$, $c_2(t)$ and $\delta_0(t)$ have been determined, because we have

$$\begin{cases} V_1 = \dfrac{1}{\dot{s}}\left(\dot{\delta}_0 V_0 + \int_0^\xi V_{0\xi}\,\mathrm{d}\xi + c_1(t) - U_1\right), \\ P_1 = \dot{s}U_1 - \dot{\delta}_0 U_0 - \int_0^\xi U_{0\xi}\,\mathrm{d}\xi - c_2(t) + \dfrac{1}{\dot{s}}h_1. \end{cases} \tag{2.24}$$

$c_1(t)$, $c_2(t)$ and $\delta_0(t)$ will be determined in such a way that $X_1(\xi,t)$ constructed above satisfies the matching conditions (2.19). Similarly to [9], we have:

Lemma 2.2. *The third equation of* (2.19) *will be satisfied if first and second equations of* (2.19) *hold.*

Proof. We only need to check that as $\xi \to \mp\infty$

$$P_1(\xi,t) = p_1(s(t)\mp 0,t) + (\xi - \delta_0)p_{0x}(s(t)\mp 0,t) + o(1) \tag{2.25}$$

holds. Let us only consider the case of $\xi \to +\infty$. The case of $\xi \to -\infty$ is similar. By virtue of the second equation of (2.16) $\times \dot{s}+$ third equation of (2.16), we get

$$P_1 = p_R'(V_0)V_1 + (\dot{s}^2 - P_0V_0^{-1})U_{1\xi} - (\dot{s}\dot{\delta}_0 + P_1V_0^{-1} - P_0V_0^{-2}V_1)U_{0\xi} \\ - \dot{s}U_{0t} - \dot{\delta}_0 P_{0\xi} - P_{0t}. \tag{2.26}$$

Using the first and second equations of (2.19), (2.18), second equation of (2.4), (2.3), (2.8) and Lemma 2.1, noticing the relations $\dot{f}^r = f_t^r + \dot{s} f_x^r$, we find, as $\xi \to +\infty$, that the right-hand side of (2.26) is equivalent to

$$\begin{aligned} &p_R'(v_0^r)(v_1^r + (\xi - \delta_0)v_{0x}^r) + (\dot{s}^2 - P_0V_0^{-1})u_{0x}^r(t) - \dot{s}\dot{u}_0^r(t) - \dot{p}_0^r + o(1) \\ &=(-P_0V_0^{-1}u_{0x}^r - p_{0t}^r + p_R'(v_0^r)v_1^r) + (\xi - \delta_0)p_R'(v_0^r)v_{0x}^r + o(1) \\ &= p_1(s(t)+0,t) + (\xi - \delta_0)p_R'(v_0)v_{0x}(s(t)+0,t) + o(1) \\ &= p_1(s(t)\mp 0,t) + (\xi - \delta_0)p_{0x}(s(t)\mp 0,t) + o(1), \end{aligned}$$

which is exactly the right-hand side of (2.25). The proof is completed. □

In the following, we will also use the notations $f^r(t) \equiv \lim_{\xi\to+\infty} f(\xi,t)$ and $f^l(t) \equiv \lim_{\xi\to-\infty} f(\xi,t)$.

Due to entropy condition, subcharacteristic condition and Lemma 2.1, we have

$$f_1(\xi,t) = \begin{cases} f_1^l(t) + O(1)\exp\{-\alpha_0|\xi|\} & \text{as } \xi \to -\infty, \\ f_1^r(t) + O(1)\exp\{-\alpha_0|\xi|\} & \text{as } \xi \to +\infty \end{cases} \tag{2.27}$$

and

$$f_1^l(t) = \frac{\dot{s}^2 + p_R'(v_0^l)}{\dot{s}(\dot{s}^2 - P_R'(V_0)V_0^{-1})} > 0, \qquad f_1^r(t) = \frac{\dot{s}^2 + p_R'(v_0^r)}{\dot{s}(\dot{s}^2 - P_R'(V_0)V_0^{-1})} < 0, \tag{2.28}$$

where $\alpha_0 > 0$ is a suitable constant.

By Lemma 2.1, we see that

$$X_{0t}(\xi,t) = \begin{cases} \dot{\chi}_0^l(t) + O(1)\exp\{-\alpha_0|\xi|\} & \text{as } \xi \to -\infty, \\ \dot{\chi}_0^r(t) + O(1)\exp\{-\alpha_0|\xi|\} & \text{as } \xi \to +\infty. \end{cases} \tag{2.29}$$

Let

$$A = \dot{s}^2 + p_R'(V_0), \qquad B = \frac{p_R'(V_0)}{\dot{s}}. \tag{2.30}$$

By (1.2), we can make the following relations:

$$\begin{aligned} &v_{0x}^l = (\dot{s}\dot{v}_0^l - \dot{u}_0^l)(A^l)^{-1}, \qquad v_{0x}^r = (\dot{s}\dot{v}_0^r - \dot{u}_0^l)(A^r)^{-1}, \\ &u_{0x}^l = \dot{s}(A^l)^{-1}(B^l\dot{v}_0^l + \dot{u}_0^l), \qquad u_{0x}^r = \dot{s}(A^r)^{-1}(B^r\dot{v}_0^r + \dot{u}_0^r). \end{aligned}$$

Then by (2.23), the first equation of (2.24) and the above relations, we can get the following asymptotic behavior of $V_1(\xi,t)$ and $U_1(\xi,t)$:

$$\begin{aligned} V_1^l &= \frac{1}{A^l}(\dot\delta_0(\dot s v_0^l - u_0^l) + \dot s c_1(t) + c_2(t)) + \xi\partial_x v_0^l + \partial_x u_0^l \\ &\quad + O_{11}(t) + O(1)\exp\{-\alpha_0|\xi|\} \quad \text{as } \xi \to -\infty, \\ V_1^r &= \frac{1}{A^r}(\dot\delta_0(\dot s v_0^r - u_0^r) + \dot s c_1(t) + c_2(t)) + \xi\partial_x v_0^r + \partial_x u_0^r \\ &\quad + O_{12}(t) + O(1)\exp\{-\alpha_0|\xi|\} \quad \text{as } \xi \to +\infty \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} U_1^l &= \frac{1}{A^l}(\dot\delta_0(\dot s u_0^l + p_R'(V_0)v_0^l) + p_R'(V_0^l)c_1(t) - \dot s c_2(t)) + \xi\partial_x u_0^l - \dot s\partial_x u_0^l \\ &\quad + O_{21}(t) + O(1)\exp\{-\alpha_0|\xi|\} \quad \text{as } \xi \to -\infty, \\ U_1^r &= \frac{1}{A^r}(\dot\delta_0(\dot s u_0^r + p_R'(V_0)v_0^r) + p_R'(V_0^r)c_1(t) - \dot s c_2(t)) + \xi\partial_x u_0^r - \dot s\partial_x u_0^r \\ &\quad + O_{22}(t) + O(1)\exp\{-\alpha_0|\xi|\} \quad \text{as } \xi \to +\infty, \end{aligned} \tag{2.32}$$

where O_{11}, O_{12}, O_{21} and O_{22} are known functions.

Therefore, the matching conditions (first and second of (2.19)) will be satisfied if we choose $c_1(t)$ and $c_2(t)$ such that

$$\begin{cases} v_1^l - \delta_0 v_{0x}^l = \dfrac{1}{A^l}(\dot\delta_0(\dot s v_0^l - u_0^l) + \dot s c_1(t) + c_2(t)) + \partial_x u_0^l + O_{11}(t), \\ v_1^r - \delta_0 v_{0x}^r = \dfrac{1}{A^r}(\dot\delta_0(\dot s v_0^r - u_0^r) + \dot s c_1(t) + c_2(t)) + \partial_x u_0^r + O_{12}(t), \end{cases} \tag{2.33}$$

$$\begin{cases} u_1^l - \delta_0 u_{0x}^l = \dfrac{1}{A^l}(\dot\delta_0(\dot s u_0^l + p_R'(V_0)v_0^l) + p_R'(V_0^l)c_1(t) \\ \qquad\qquad -\dot s c_2(t)) - \dot s\partial_x u_0^l + O_{21}(t), \\ u_1^r - \delta_0 u_{0x}^r = \dfrac{1}{A^r}(\dot\delta_0(\dot s u_0^r + p_R'(V_0)v_0^r) + p_R'(V_0^r)c_1(t) \\ \qquad\qquad -\dot s c_2(t)) - \dot s\partial_x u_0^r + O_{22}(t). \end{cases} \tag{2.34}$$

By virtue of (2.33) and (2.34), we obtain, respectively,

$$\begin{cases} c_1(t) = (\dot s v_1^l + u_1^l) - \delta_0 \dot v_0^l - \dot\delta_0 v_0^l + O_{31}(t), \\ c_2(t) = -\dot s(u_1^l - B^l v_1^l) + \delta_0 \dot u_0^l + \dot\delta_0 u_0^l - A^l u_{0x}^l + O_{32}(t), \end{cases} \tag{2.35}$$

$$\begin{cases} c_1(t) = (\dot s v_1^r + u_1^r) - \delta_0 \dot v_0^r - \dot\delta_0 v_0^r + O_{41}(t), \\ c_2(t) = \dot s(u_1^r - B^r v_1^r) + \delta_0 \dot u_0^r + \dot\delta_0 u_0^r - A^r u_{0x}^r + O_{42}(t), \end{cases} \tag{2.36}$$

where $O_{31}(t)$, $O_{32}(t)$, $O_{41}(t)$, $O_{42}(t)$ are known functions.

From (2.35) and (2.36), we get the following relations:

$$\dot{s}(v_1^l - v_1^r) + (u_1^l - u_1^r) - (\delta_0(v_0^l - v_0^r))^{\dot{}} + O_{51}(t) = 0, \tag{2.37}$$

$$\dot{s}(u_1^l - u_1^r) - \dot{s}(B^l v_1^l - B^r v_1^r) - (\delta_0(u_0^l - u_0^r))^{\dot{}} + (A^l u_{0x}^l - A^r u_{0x}^r) + O_{52}(t) = 0, \tag{2.38}$$

where $O_{51}(t)$ and $O_{52}(t)$ are known functions.

Define

$$e_{11} \equiv -[\dot{s}v_1 + u_1] = \dot{s}(v_1^l - v_1^r) + (u_1^l - u_1^r), \tag{2.39}$$

$$\begin{aligned} e_{12} &\equiv -[\dot{s}u_1 - p_R'(V_0)v_1 + (p_0 v_0^{-1} + p_R'(v_0))u_{0x}] \\ &= \dot{s}(u_1^l - u_1^r) - \dot{s}(B^l v_1^l - B^r v_1^r) + (p_0^l v_0^{l-1} + p_R'(v_0^l))u_{0x}^l \\ &\quad - (p_0^r v_0^{r-1} + p_R'(v_0^r))u_{0x}^r, \end{aligned} \tag{2.40}$$

then (2.37) and (2.38) become

$$e_{11} = \delta_0(\dot{v}_0^l - \dot{v}_0^r) + \dot{\delta}_0(v_0^l - v_0^r) + O_{61}(t), \tag{2.41}$$

$$\begin{aligned} e_{12} &= \delta_0(\dot{u}_0^l - \dot{u}_0^r) + \dot{\delta}_0(u_0^l - u_0^r) + O_{62}(t) \\ &= -\dot{s}e_{11} - \ddot{s}(v_0^l - v_0^r)\delta_0 + O_{62}(t), \end{aligned} \tag{2.42}$$

where we have used the relation $(\dot{u}_0^l - \dot{u}_0^r) = \dot{s}(\dot{v}_0^l - \dot{v}_0^r) - \ddot{s}(v_0^l - v_0^r)$, and $-O_{61} = O_{51}$, $-O_{62} = O_{52} - ((p_0^l v_0^{l^{-1}} + p_R'(v_0^l))u_{0x}^l - (p_0^r v_0^{r^{-1}} + p_R'(v_0^r))u_{0x}^r)$. Now we find that the matching conditions (first and second of (2.19)) will be satisfied if the boundary values crossing the shock for $\chi_1(x,t)$ satisfy (2.41) and (2.42).

Next, we will show that (2.41) and (2.42) are exactly the relation between the boundary data of $\chi_1(x,t)$ needed to solve the initial boundary value problem for linear hyperbolic equations (2.9) in Ω_+ and Ω_-, respectively, where

$$\Omega_- = \{(x,t)\colon x < s(t), 0 \le t \le T\}, \quad \Omega_+ = \{(x,t)\colon x > s(t), 0 \le t \le T\}.$$

System (2.9), i.e.,

$$\begin{cases} v_{1t} - u_{1x} = 0, \\ u_{1t} + (p_R'(v_0)v_1)_x = ((p_R(v_0)/v_0 + p_R'(v_0))u_{0x})_x \end{cases}$$

has eigenvalues $\lambda_1(v_0) = -\sqrt{-p_R'(v_0)}$ and $\lambda_2(v_0) = \sqrt{-p_R'(v_0)}$ with corresponding right eigenvectors $r_1 = (-1, \lambda_1)^{\mathrm{T}}$ and $r_2 = (-1, \lambda_2)^{\mathrm{T}}$, respectively.

Let

$$\begin{pmatrix} v_1 \\ u_1 \end{pmatrix} = M \begin{pmatrix} n_1 \\ z_1 \end{pmatrix} \tag{2.43}$$

with

$$M = \begin{pmatrix} -1 & -1 \\ \lambda_1 & \lambda_2 \end{pmatrix}.$$

Then we can diagonalize system (2.9) to obtain

$$\begin{pmatrix} n_1 \\ z_1 \end{pmatrix}_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} n_1 \\ z_1 \end{pmatrix}_x$$

$$= -M^{-1}(M_t + (JM)_x) \begin{pmatrix} n_1 \\ z_1 \end{pmatrix} + M^{-1} \begin{pmatrix} 0 \\ ((p_0 v_0^{-1} + p_R'(v_0))u_{0x})_x \end{pmatrix}, \tag{2.44}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ p_R'(v_0) & 0 \end{pmatrix}.$$

By the characteristic method, we see, due to the entropy condition, that z_1^l, z_1^r and n_1^r will be determined by integrating along appropriate characteristics, and only n_1^l is required to be specified at the left-hand side of $x=s(t)$. The value of n_1^l can be obtained by (2.41) and (2.42). Rewrite (2.41) and (2.42) in terms of (n_1, z_1) as

$$\begin{pmatrix} e_{11} \\ e_{12} \end{pmatrix} = \left[(-\dot{s}M + JM) \begin{pmatrix} n_1 \\ z_1 \end{pmatrix}\right] + \left[\begin{pmatrix} 0 \\ p_0 v_0^{-1} + p_R'(v_0))u_{0x})_x \end{pmatrix}\right]. \tag{2.45}$$

This is a system of two equations about an unknown n_1^l, for which the solvability condition is

$$\lambda^l e_{11} + e_{12} = z_1^l(\lambda_2^l - \lambda_1^l)(\dot{s} - \lambda_2^l) + n_1^r(\dot{s} - \lambda_1^r)(\lambda_1^l - \lambda_1^r)$$
$$+ z_1^r(\dot{s} - \lambda_2^r)(\lambda_1^l - \lambda_2^r) - [p_0 v_0^{-1} + p_R'(v_0))u_{0x}]$$
$$\equiv F_1(t). \tag{2.46}$$

If (2.46) is satisfied, one can solve (2.45) to obtain

$$n_1^l = (\dot{s} - \lambda_1^l)^{-2}((n_1^r(\dot{s} - \lambda_1^r)^2 + z_1^r(\dot{s} - \lambda_2^r)^2 - z_1^l(\dot{s} - \lambda_2^l)^2)$$
$$+ \ddot{s}(v_0^l - v_0^r)\delta_0 + O(t), \tag{2.47}$$

where we have used (2.42). By (2.42), we can also reduce (2.46) to

$$(\lambda_1^l - \dot{s})\frac{\mathrm{d}}{\mathrm{d}t}\{\delta_0(v_0^l - v_0^r)\} - \ddot{s}(v_0^l - v_0^r)\delta_0 = F_1(t) + O(t), \tag{2.48}$$

which is a linear ODE on $\delta_0(v_0^l - v_0^r)$ and which completely determines δ_0 up to a constant. Then we get n_1^l by (2.46). And the standard theory of mixed problems for linear hyperbolic systems gives the smooth solutions to (2.9) in Ω_- and Ω_+, respectively. The outer functions $\chi_1(x,t)$ are determined then.

In terms of the construction of χ_1 and δ_0, we know the inner functions $X_1(\xi,t)$ and first and second equations of (2.19) are satisfied as well. We combine all the results obtained so far to conclude the following theorem

Theorem 2.3. *$\chi_1(x,t)$, $X_1(\xi,t)$ and δ_0 can be determined such that*

(i) *$\chi_1(x,t)$ and its derivatives are uniformly continuous up to $x=s(t)$ and*

$$\sum_{|\alpha|\le 3}\int\int_{x\neq s(t)} |\partial_x^\alpha \chi_1(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t < +\infty, \tag{2.49}$$

(ii) *$X_1(\xi,t)$ is smooth and for some $\alpha_0>0$*

$$\begin{aligned} X_1(\xi,t) &= \chi_1(s(t)\mp 0,t) + (\xi-\delta_0)\chi_{0x}(s(t)\mp 0,t) \\ &\quad + O(1)\exp\{-\alpha_0|\xi|\} \quad \text{as } \xi\to\mp\infty \end{aligned} \tag{2.50}$$

holds.

It is easy to see that the above procedure can be carried out to any order. In particular, we can construct $\chi_2(x,t)$, $X_2(\xi,t)$, $\chi_3(x,t)$, $X_3(\xi,t)$, δ_1 and δ_2 such that similar properties in Theorem 2.3 hold.

2.4. Approximate solutions

Now, we construct a smooth approximate solution to (1.1) by patching the inner and outer solutions discussed in the above.

Set

$$I(x,t) = (X_0+\varepsilon X_1+\varepsilon^2 X_2+\varepsilon^3 X_3)\left(\frac{x-s(t)}{\varepsilon}+\delta_0+\varepsilon\delta_1+\varepsilon^2\delta_2,t\right) \tag{2.51}$$

and

$$O(x,t) = (\chi_0+\varepsilon\chi_1+\varepsilon^2\chi_2+\varepsilon^3\chi_3)(x,t), \quad x\neq s(t). \tag{2.52}$$

Let $m(y)\in C_0^\infty(\mathbf{R}^1)$ such that $0\le m(y)\le 1$ and

$$m(y) = \begin{cases} 1, & |y|<1, \\ 0, & |y|\ge 2. \end{cases} \tag{2.53}$$

Choose $\alpha\in(\frac{2}{3},1)$ as a constant. We set

$$S^\varepsilon(x,t) = m\left(\frac{x-s(t)}{\varepsilon^\alpha}\right)I(x,t) + \left(1-m\left(\frac{x-s(t)}{\varepsilon^\alpha}\right)\right)O(x,t) + d(x,t), \tag{2.54}$$

where $d(x,t)=(d_1,d_2,d_3)^t(x,t)$ is a higher-order correction to be determined later. We use the following notations:

$$S^\varepsilon = (v^\varepsilon,u^\varepsilon,p^\varepsilon)^{\mathrm{T}}, \qquad I=(I_1,I_2,I_3)^{\mathrm{T}}, \qquad O=(O_1,O_2,O_3)^{\mathrm{T}}.$$

Using the structure of the various orders of inner and outer solutions, we have

$$\begin{cases} v_t^\varepsilon - u_x^\varepsilon = F_2 + d_{1t} - d_{2x}, \\ u_t^\varepsilon + p_x^\varepsilon = F_3 + d_{2t} + d_{3x}, \\ p_t^\varepsilon + p^\varepsilon (v^\varepsilon)^{-1} u_x^\varepsilon - q(S^\varepsilon) = F_4 + d_{3t} + p^\varepsilon (v^\varepsilon)^{-1} d_{2x} - \Delta \equiv R(x,t), \end{cases} \tag{2.55}$$

where $q(f) = (1/\varepsilon)(p_R(f_1) - f_3)$ for $f = (f_1, f_2, f_3)^{\mathrm{T}}$, and

$$\begin{aligned} F_2 &= \varepsilon^3 m(\dot\delta_2 V_{1\xi} + (\dot\delta_1 + \varepsilon\dot\delta_3) V_{2\xi} + (\dot\delta_0 + \varepsilon\dot\delta_1 + \varepsilon^2\dot\delta_2) V_{3\xi} + V_{3t}) \\ &\quad + m_t(I_1 - O_1) - m_x(I_2 - O_2), \\ F_3 &= \varepsilon^3 m(\dot\delta_2 U_{1\xi} + (\dot\delta_1 + \varepsilon\dot\delta_3) U_{2\xi} + (\dot\delta_0 + \varepsilon\dot\delta_1 + \varepsilon^2\dot\delta_2) U_{3\xi} + U_{3t}) \\ &\quad + m_t(I_2 - O_2) + m_x(I_3 - O_3), \\ F_4 &= \varepsilon^3 m(\dot\delta_2 P_{1\xi} + (\dot\delta_1 + \varepsilon\dot\delta_3) P_{2\xi} + (\dot\delta_0 + \varepsilon\dot\delta_1 + \varepsilon^2\dot\delta_2) P_{3\xi} + P_{3t}) \\ &\quad + m_t(I_3 - O_3) + p^\varepsilon (v^\varepsilon)^{-1} m_x (I_2 - O_2), \\ \Delta &= q(S^\varepsilon) - m q(I) - (1-m) q(O) - m(p^\varepsilon (v^\varepsilon)^{-1} - I_3 I_1^{-1}) I_{2x} \\ &\quad - (1-m)(p^\varepsilon (v^\varepsilon)^{-1} - O_3 O_1^{-1}) O_{2x}. \end{aligned} \tag{2.56}$$

The construction of approximate solutions leads to

$$\frac{\partial^k}{\partial x^k}(I - O)(x,t) = O(1)\varepsilon^{(4-k)\alpha} \quad \text{in } \varepsilon^\alpha < |x - s(t)| < 2\varepsilon^\alpha, \tag{2.57}$$

which can be obtained from the matching conditions (2.18)–(2.20).

Choosing (d_1, d_2, d_3) such that

$$\begin{cases} d_{1t} - d_{2x} = -F_2, \\ d_{2t} + d_{3x} = -F_3, \\ d_3 = C_0 d_1, \\ d_1(x,0) = d_2(x,0) = 0 \end{cases} \tag{2.58}$$

with $C_0 = 2\max_{t\in[0,T]} \lambda_2^l(t)$.

Due to the following facts

(i) F_2, F_3 and m have their supports in $\{(x,t)\colon |x - s(t)| \le 2\varepsilon^\alpha, 0 \le t \le T\}$,
(ii) $|\partial_x^k (F_2, F_3)(x,t)| \le O(1)\varepsilon^{(3-k)\alpha}$, $0 \le k \le 2$,

we can obtain, by the characteristic method the following lemma.

Lemma 2.4. *The solution* (d_1, d_2) *to* (2.58) *has a compact support and for integers* $0 \le k \le 3$ *satisfies*

$$\left| \frac{\partial^k}{\partial x^k} d \right| \le O(1)\varepsilon^{(4-k)\alpha} \quad \forall (x,t) \in \mathbf{R}^1 \times [0,T]. \tag{2.59}$$

Furthermore, we can estimate Δ *as follows*:

$$\left|\frac{\partial^k}{\partial x^k}\Delta\right| \le O(1)\varepsilon^{(4-k)\alpha-1}, \quad 0 \le k \le 2. \tag{2.60}$$

Now, we conclude with the following theorem.

Theorem 2.5. *Let* $S^\varepsilon(x,t)$ *be the smooth function defined in* (2.54) *with* $d(x,t)$ *determined in* (2.58). *Then* S^ε *satisfies*

$$\begin{cases} v_t^\varepsilon - u_x^\varepsilon = 0, \\ u_t^\varepsilon + p_x^\varepsilon = 0, \\ p_t^\varepsilon + p_\varepsilon v_\varepsilon^{-1} u_x^\varepsilon - q(S^\varepsilon) = R(x,t) \end{cases} \tag{2.61}$$

with $R(x,t)$ *satisfying*

$$\begin{aligned} &|\partial_x^k R| \le O(1)\varepsilon^{(4-k)\alpha-1}, \quad 0 \le k \le 2, \\ &\int_0^T \int_{-\infty}^{+\infty} |\partial_x^k R|^2 \,\mathrm{d}x\,\mathrm{d}t \le O(1)\varepsilon^{2(4-k)\alpha-2}, \quad 0 \le k \le 2. \end{aligned} \tag{2.62}$$

So far, we have finished the construction of the formal approximate solutions to (1.1).

3. Stability analysis

We now prove that there exists a smooth solution to (1.1) in a neighborhood of $S^\varepsilon(x,t)$ and, for sufficiently small ε, the asymptotic behavior of the solution to (1.1) is governed by $S^\varepsilon(x,t)$.

Let $\overline{S^\varepsilon}$ be an exact solution to (1.1) with initial data $\overline{S^\varepsilon}(x,0)=S^\varepsilon(x,0)$. We decompose the solution as

$$\overline{S^\varepsilon}(x,t) = S^\varepsilon(x,t) + (\bar\phi, \bar\psi, \bar w)^{\mathrm{T}}(x,t), \quad (x,t) \in \mathbf{R}^1 \times [0,T]. \tag{3.1}$$

It is easy to show that

$$\begin{cases} \bar\phi_t - \bar\psi_x = 0, \\ \bar\psi_t + \bar w_x = 0, \\ \bar w_t + (p_\varepsilon + \bar w)(v_\varepsilon + \bar\phi)^{-1}\bar\psi_x = \dfrac{1}{\varepsilon}(p_R(v^\varepsilon + \bar\phi) - p_R(v^\varepsilon) - \bar w) - R(x,t), \\ \bar\phi(x,0) = \bar\psi(x,0) = \bar w(x,0) = 0. \end{cases} \tag{3.2}$$

Setting

$$\bar\phi = \tilde\phi_x, \quad \bar\psi = \tilde\psi_x, \quad \bar w = \tilde w, \tag{3.3}$$

we have

$$\begin{cases} \tilde{\phi}_t - \tilde{\psi}_x = 0, \\ \tilde{\psi}_t + \tilde{w} = 0, \\ \tilde{w}_t + (p_\varepsilon + \tilde{w})(v_\varepsilon + \tilde{\phi}_x)^{-1}\tilde{\psi}_{xx} = \dfrac{1}{\varepsilon}(p_R(v^\varepsilon + \tilde{\phi}_x) - p_R(v^\varepsilon) - \tilde{w}) - R(x,t), \\ \tilde{\phi}(x,0) = \tilde{\psi}(x,0) = \tilde{w}(x,0) = 0, \end{cases}$$

which yields

$$\begin{cases} \tilde{\phi}_t - \tilde{\psi}_x = 0, \\ \tilde{\psi}_{tt} - (p_\varepsilon + \tilde{\psi}_t)(v_\varepsilon + \tilde{\phi}_x)^{-1}\tilde{\psi}_{xx} \\ \quad = -\dfrac{1}{\varepsilon}(p_R(v^\varepsilon + \tilde{\phi}_x) - p_R(v^\varepsilon) + \tilde{\psi}_t) + R(x,t), \\ \tilde{\phi}(x,0) = \tilde{\psi}(x,0) = \tilde{\psi}_t(x,0) = 0. \end{cases} \tag{3.4}$$

Using the following scalings

$$\tilde{\phi} = \varepsilon\phi, \quad \tilde{\psi} = \varepsilon\psi, \tag{3.5}$$

and

$$y = \frac{x - s(t)}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon}, \tag{3.6}$$

we simplify (3.4) into

$$\begin{cases} L_1(\phi,\psi) = 0, \\ L_2(\phi,\psi) = F(x,t), \\ \phi(y,0) = \psi(y,0) = \psi_\tau(y,0), \end{cases} \tag{3.7}$$

where

$$L_1 = \phi_\tau - \dot{s}\phi_y - \psi_y,$$

$$L_2 = (\psi_\tau - \dot{s}\psi_y)_\tau - \dot{s}(\psi_\tau - \dot{s}\psi_y)_y - E\psi_{yy} + (\psi_\tau - \dot{s}\psi_y) - D\phi_y,$$

$$E = (p^\varepsilon - (\psi_\tau - \dot{s}\psi_y))(v^\varepsilon + \phi_y)^{-1},$$

$$D = -p_R'(v^\varepsilon), \tag{3.8}$$

$$F = \varepsilon R - (p_R(v^\varepsilon + \phi_y) - p_R(v^\varepsilon) - p_R'(v^\varepsilon)\phi_y)$$
$$\quad + ((p^\varepsilon - (\psi_\tau - \dot{s}\psi_y))(v^\varepsilon + \phi_y)^{-1} - p^\varepsilon(v^\varepsilon)^{-1})u_y^\varepsilon.$$

To study the existence and the asymptotic behavior of $\overline{S^\varepsilon}(x,t)$ for sufficiently small ε, we only need to show that, for sufficiently small ε, (3.7) has a smooth “small” solution up to $\tau = T/\varepsilon$. This will be realized by an argument similar to the stability analysis of the shock profiles to (1.1) (see [3]). The different part lies in that S^ε depends on ε and t here.

From now on, we use H^l $(l \geq 1)$ to denote the usual Sobolev space with the norm $\|\cdot\|_l$ and $\|\cdot\|$ to denote the usual L^2-norm. We also use the following notation for simplicity:

$$\|(f_1, f_2, \dots, f_k)\|_m^2 \equiv \sum_{i=1}^{k} \|f_i\|_m^2.$$

Let us define the solution space of (3.7) by

$$X(0,\tau_0) = \{(\phi,\psi) \in C^0(0,\tau_0; H^3), \quad \psi_\tau \in C^0(0,\tau_0; H^2)\} \tag{3.9}$$

with $0 < \tau_0 \leq T/\varepsilon$. Suppose, for some $0 < \tau_0 \leq T/\varepsilon$, there exists a solution (ϕ,ψ) to (3.7) such that $(\phi,\psi) \in X(0,\tau_0)$. Denote the norm for (ϕ,ψ) by

$$N(\tau) = \sup_{0 \leq s \leq \tau} (\|(\phi,\psi)(s)\|_3 + \|\psi_\tau(s)\|_2). \tag{3.10}$$

The main result in this section is the following a priori estimate.

Theorem 3.1. *Suppose* (H_1)–(H_3) *are satisfied. There exist positive constants* ε_0, η_0, δ_0 *and* K_0 *which are independent of* ε *and* τ_0 *such that if*

(i) $0 < \varepsilon \leq \varepsilon_0$,
(ii) $|v_0^r - v_0^l| + |u_0^r - u_0^l| + |p_0^r - p_0^l| \leq \eta_0$,
(iii) $N(\tau_0) \leq \delta_0$,
then for $(\phi,\psi) \in X(0,\tau_0)$ *the following inequality holds.*

$$\sup_{0 \leq \tau \leq \tau_0} (\|(\phi,\psi)(\tau)\|_3^2 + \|\psi_\tau(\tau)\|_2^2) + \int_0^{\tau_0} (\|(\phi_y,\psi_y)(\tau)\|_2^2 + \|\psi_\tau(\tau)\|_2^2)\,\mathrm{d}\tau$$
$$\leq K_0 \varepsilon^{8\alpha-3}. \tag{3.11}$$

To prove Theorem 3.1, we need the following lemmas.

First, by the construction of S^ε and with a method similar to the one used in [9], we get some useful properties of S^ε.

Lemma 3.2. *Let* $S^\varepsilon(x,t)$ *be defined as in* (2.54), *then*

(i) $S^\varepsilon(x,t) = \begin{cases} \chi_0 + O(1)\varepsilon^{\min(4\alpha,1)}, & |x - s(t)| \geq \varepsilon^\alpha, \\ X_0 + O(1)\varepsilon^\alpha, & |x - s(t)| \leq 2\varepsilon^\alpha, \end{cases}$
(ii) $S_y^\varepsilon(y,t) = mX_{0y} + O(1)\varepsilon$, $S_\tau^\varepsilon = O(1)\varepsilon$.
(iii) $v^\varepsilon > 0, \quad p^\varepsilon > 0$.

Now, we begin with the energy estimates. First, we establish the following basic energy estimate.

Lemma 3.3. *Suppose the conditions in Theorem* 3.1 *are satisfied, then, for all* $\tau \in (0,\tau_0]$, *we have*

$$\|(\phi,\psi)(\tau)\|_1^2+\|(\psi_\tau-\dot{s}\psi_y)(\tau)\|^2+\int_0^\tau\|(\phi_y,\psi_y,\psi_\tau-\dot{s}\psi_y)(s)\|^2\,\mathrm{d}s$$
$$+\int_0^\tau\int_{-\infty}^{+\infty}m|V_{0y}|\psi^2\,\mathrm{d}y\,\mathrm{d}s$$
$$\le O(1)\varepsilon^{8\alpha-3}.$$

Proof. We consider the equality

$$(k_0\phi+\psi_y)L_1+D^{-1}((\psi_\tau-\dot{s}\psi_y)+k_0\psi)L_2=D^{-1}(k_0\psi+(\psi_\tau-\dot{s}\psi_y))F, \tag{3.12}$$

which can be reduced to

$$[G_1+G_2]_\tau+G_3+G_4=D^{-1}(k_0\psi+(\psi_\tau-\dot{s}\psi_y))F, \tag{3.13}$$

where

$$G_1=\tfrac{1}{2}k_0\phi^2+\phi\psi_y+\frac{1}{2}D^{-1}E\psi_y^2,$$

$$G_2=\tfrac{1}{2}k_0(D^{-1}+\dot{s}D_y^{-1})\psi^2+\frac{1}{2}D^{-1}(\psi_\tau-\dot{s}\psi_y)^2+k_0D^{-1}\psi(\psi_\tau-\dot{s}\psi_y),$$

$$G_3=((1-k_0)D^{-1}-\tfrac{1}{2}D_\tau^{-1}+\tfrac{1}{2}\dot{s}D_y^{-1})(\psi_\tau-\dot{s}\psi_y)^2$$
$$+(k_0ED^{-1}-\tfrac{1}{2}ED_\tau^{-1}+\tfrac{1}{2}\dot{s}ED_y^{-1}-\tfrac{1}{2}D^{-1}(E_\tau-\dot{s}E_y)-1)\psi_y^2$$
$$+(ED^{-1})_y(\psi_\tau-\dot{s}\psi_y)\psi_y,$$

$$G_4=\tfrac{1}{2}k_0(\dot{s}D_y^{-1}-D_\tau^{-1})\psi^2-k_0D_\tau^{-1}\psi(\psi_\tau-\dot{s}\psi_y)$$
$$+k_0(E-\dot{s}^2D_y^{-1}+D^{-1}E_y)\psi\,\psi_y+\tfrac{1}{2}(\dot{s}D_y^{-1})_\tau\psi^2+\{\dots\}_y$$

and $\{\dots\}_y$ denotes the terms which disappear after integrations with respect to y.

By Sobolev embedding theorem and Lemma 3.2, we know that $v^\varepsilon+\phi_y>0$ and $p^\varepsilon-(\psi_\tau-\dot{s}\psi_y)>0$ are bounded. Thus we can choose k_0 such that

$$1-k_0>k_1>0,\qquad \inf\{ED^{-1}\}k_0-1>k_2>0. \tag{3.14}$$

We see from Lemma 3.2 that

$$\begin{aligned}&D_y^{-1}=D^{-2}p_R''(v^\varepsilon)(mV_{0y}+O(1)\varepsilon),\\&D_\tau^{-1}=O(1)\varepsilon,\end{aligned} \tag{3.15}$$

which implies

$$D_y^{-1}=O(1)(\eta_0+\varepsilon). \tag{3.16}$$

We easily get, by virtue of (3.7), that

$$E_\tau - \dot{s}E_y = (v^\varepsilon + \phi_y)^{-1}(-2E\psi_{yy} - D\phi_y + \psi_\tau - \dot{s}\psi_y - F), \tag{3.17}$$

which implies, by (3.8), (2.62), and Sobolev embedding theorem, that

$$E_\tau - \dot{s}E_y = O(1)(N(\tau) + \eta_0 + \varepsilon). \tag{3.18}$$

Therefore, for δ_0, η_0 and ε_0 suitably small, there exist positive constants a_1, a_2, b_1, b_2 and b_3 such that

$$\begin{aligned} &a_1(\phi^2 + \psi_y^2) \le G_1 \le b_1(\phi^2 + \psi_y^2),\\ &a_2(\psi^2 + (\psi_\tau - \dot{s}\psi_y)^2) \le G_2 \le b_2(\psi^2 + (\psi_\tau - \dot{s}\psi_y)^2),\\ &G_3 \ge b_3(\psi_y^2 + (\psi_\tau - \dot{s}\psi_y)^2). \end{aligned} \tag{3.19}$$

We now estimate G_4. By (3.15) and (3.16), we have, for a positive constant b_4, that

$$\tfrac{1}{2}\dot{s}D_y^{-1}\psi^2 \ge b_4 m V_{0y}\psi^2 + O(1)\varepsilon\psi^2, \tag{3.20}$$

$$\begin{aligned} &|-k_0 D_\tau^{-1}\psi(\psi_\tau - \dot{s}\psi_y) + k_0(E - \dot{s}^2 D_y^{-1} + D^{-1}E_y)\psi\,\psi_y + \tfrac{1}{2}(\dot{s}D_y^{-1})_\tau\psi^2|\\ &\quad \le O(1)\varepsilon(\psi^2 + \psi_y^2 + (\psi_\tau - \dot{s}\psi_y)^2). \end{aligned} \tag{3.21}$$

With the help of (3.17)–(3.21), we integrate (3.13) over $[0,\tau]\times(-\infty,+\infty)$ to obtain

$$\begin{aligned} &\|\phi(\tau)\|^2 + \|\psi(\tau)\|_1^2 + \|(\psi_\tau - \dot{s}\psi_y)(\tau)\|^2\\ &\quad + \int_0^\tau \|(\psi_y, \psi_\tau - \dot{s}\psi_y)(s)\|^2\,\mathrm{d}s + \int_0^\tau\int_{-\infty}^{+\infty} m|V_{0y}|\psi^2\,\mathrm{d}y\,\mathrm{d}s\\ &\quad \le O(1)\varepsilon\int_0^\tau (\|\psi(s)\|_1^2 + \|(\psi_\tau - \dot{s}\psi_y)(s)\|^2)\,\mathrm{d}s\\ &\qquad + O(1)\int_0^\tau\int_{-\infty}^{+\infty} |D^{-1}(\psi + (\psi_\tau - \dot{s}\psi_y))F|\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{3.22}$$

The last term of (3.22) can be estimated, with the help of (3.8), as follows:

$$\begin{aligned} &\int_0^\tau\int_{-\infty}^{+\infty} |D^{-1}(\psi + \mu(\psi_\tau - \dot{s}\psi_y))F|\,\mathrm{d}y\,\mathrm{d}s\\ &\quad \le O(1)\int_0^\tau\int_{-\infty}^{+\infty} |D^{-1}(\psi + \mu(\psi_\tau - \dot{s}\psi_y))|(\varepsilon|R(x,t)|\\ &\qquad + \phi_y^2 + \eta_0|\psi_\tau - \dot{s}\psi_y| + \eta_0|\phi_y|)\,\mathrm{d}y\,\mathrm{d}s\\ &\quad \le O(1)(N(\tau) + \eta_0)\int_0^\tau \|\phi_y(s)\|^2\,\mathrm{d}s + \varepsilon\int_0^\tau\int_{-\infty}^{+\infty} R^2\,\mathrm{d}y\,\mathrm{d}s\\ &\qquad + O(1)(\varepsilon + \eta_0)\int_0^\tau (\|\psi(s)\|^2 + \|(\psi_\tau - \dot{s}\psi_y)(s)\|^2)\,\mathrm{d}s \end{aligned}$$

and

$$\varepsilon \int_0^\tau \int_{-\infty}^{+\infty} R^2 \,\mathrm{d}y\,\mathrm{d}s = \varepsilon^{-1} \int_0^{\varepsilon\tau} \int_{-\infty}^{+\infty} R^2(x,\eta)\,\mathrm{d}x\,\mathrm{d}\eta$$
$$\leq O(1)\varepsilon^{8\alpha-3},$$

which together with (3.22) yield

$$\|\phi(\tau)\|^2 + \|\psi(\tau)\|_1^2 + \|(\psi_\tau - \dot{s}\psi_y)(\tau)\|^2$$
$$+ \int_0^\tau \|(\psi_y, \psi_\tau - \dot{s}\psi_y)(s)\|^2 \,\mathrm{d}s + \int_0^\tau \int_{-\infty}^{+\infty} m|V_{0y}|\psi^2 \,\mathrm{d}y\,\mathrm{d}s$$
$$\leq O(1)\varepsilon \int_0^\tau (\|\psi(s)\|_1^2 + \|(\psi_\tau - \dot{s}\psi_y)(s)\|^2)\,\mathrm{d}s$$
$$+ O(1)N(\tau) \int_0^\tau \|\phi_y(s)\|^2 \,\mathrm{d}s + O(1)\varepsilon^{8\alpha-3}. \tag{3.23}$$

To estimate $\|\phi_y(\tau)\|^2$, we use the following relation:

$$(E\phi_y - (\psi_\tau - \dot{s}\psi_y))\partial_y L_1 - \phi_y L_2$$
$$= [\tfrac{1}{2}E\phi_y^2 - (\psi_\tau - \dot{s}\psi_y)\phi_y - \tfrac{1}{2}\psi_y^2]_\tau - \phi_y(\psi_\tau - \dot{s}\psi_y)$$
$$+ (D - \tfrac{1}{2}(B_\tau - \dot{s}E_y))\phi_y^2 + \{\ldots\}_y. \tag{3.24}$$

Integrating (3.24) over $[0,\tau] \times (-\infty, +\infty)$, using (3.16) and Young's inequality, we can obtain, similar to (3.23), that

$$\|\phi_y(\tau)\|^2 + \int_0^\tau \|\phi_y(s)\|^2 \,\mathrm{d}s$$
$$\leq O(1)(\|(\psi_y, \psi_\tau - \dot{s}\psi_y)(\tau)\|^2 + \int_0^\tau \|(\psi_y, \psi_\tau - \dot{s}\psi_y)(s)\|^2 \,\mathrm{d}s$$
$$+ O(1)(N(\tau) + \varepsilon) \int_0^\tau \|\phi_y(s)\| \,\mathrm{d}s + O(1)\varepsilon^{8\alpha-3}. \tag{3.25}$$

Combining (3.23) and (3.25), we have, for suitably small δ_0, ε_0 and η_0 and for a positive constant K, that

$$\|(\phi,\psi)(\tau)\|_1^2 + \|(\psi_\tau - \dot{s}\psi_y)(\tau)\|^2$$
$$+ \int_0^\tau \|(\phi_y, \psi_y, \psi_\tau - \dot{s}\psi_y)(s)\|^2 \,\mathrm{d}s + \int_0^\tau \int_{-\infty}^{+\infty} m|V_{0y}|\psi^2 \,\mathrm{d}y\,\mathrm{d}s$$
$$\leq O(1)\varepsilon^{8\alpha-3} + K\varepsilon \int_0^\tau \|\psi(s)\| \,\mathrm{d}s, \tag{3.26}$$

which implies

$$\|\psi(\tau)\|^2 \le O(1)\varepsilon^{8\alpha-3} + K\varepsilon \int_0^\tau \|\psi(s)\|^2\,\mathrm{d}s.$$

Thus, the Gronwall's inequality gives that

$$\int_0^\tau \|\psi(s)\|^2\,\mathrm{d}s \le O(1)\varepsilon^{8\alpha-3} \int_0^\tau \exp\{K\varepsilon(\tau-\varepsilon)\}\,\mathrm{d}s. \tag{3.27}$$

Substituting (3.27) into (3.26), we have

$$\begin{aligned}
&\|(\phi,\psi)(\tau)\|_1^2 + \|(\psi_\tau - \dot{s}\psi_y)(\tau)\|^2 \\
&\quad + \int_0^\tau \|(\phi_y,\psi_y,\psi_\tau - \dot{s}\psi_y)(s)\|^2\,\mathrm{d}s + \int_0^\tau \int_{-\infty}^{+\infty} m|V_{0y}|\psi^2\,\mathrm{d}y\,\mathrm{d}s \\
&\quad \le O(1)\varepsilon^{8\alpha-3}\left(1 + K\varepsilon \int_0^\tau \exp\{K\varepsilon(\tau-\varepsilon)\}\,\mathrm{d}s\right) \\
&\quad \le O(1)\varepsilon^{8\alpha-3}.
\end{aligned}$$

The proof of Lemma 3.3 is complete. □

Now, we deal with higher-order estimates. We have:

Lemma 3.4. *Suppose the conditions in Theorem* 3.1 *are satisfied, then, for all* $\tau \in [0,\tau_0]$, *we have*

$$\begin{aligned}
&\|(\phi,\psi)(\tau)\|_2^2 + \|(\psi_\tau - \dot{s}\psi_y)_y(\tau)\|^2 + \int_0^\tau \|(\phi_y,\psi_y,(\psi_\tau - \dot{s}\psi_y))(s)\|_1^2\,\mathrm{d}s \\
&\quad + \int_0^\tau \int_{-\infty}^{+\infty} m|V_{0y}|\psi_y^2\,\mathrm{d}y\,\mathrm{d}s \\
&\quad \le O(1)\varepsilon^{8\alpha-3}.
\end{aligned}$$

Proof. Using the following relations:

$$(k_0\phi_y + \psi_{yy})\partial_y L_1 + D^{-1}((\psi_\tau - \dot{s}\psi_y)_y + k_0\psi_y)\partial_y L_2 = D^{-1}(\psi_y + \mu(\psi_\tau - \dot{s}\psi_y)_y)F_y \tag{3.28}$$

and

$$(E\phi_{yy} - (\psi_\tau - \dot{s}\psi_y)_y)\partial_{yy}L_1 - \phi_{yy}\partial_y L_2 = -\phi_{yy}F_y \tag{3.29}$$

with the help of (3.14)–(3.18), we get, by repeating the procedure in the proof of Lemma 3.3, that

$$\|(\phi_y,(\psi_\tau - \dot{s}\psi_y),\psi_y)(\tau)\|_1^2 + \int_0^\tau \|(\phi_y,(\psi_\tau - \dot{s}\psi_y),\psi_y)(s)\|_1^2\,\mathrm{d}\tau$$

$$+\int_0^\tau\int_{-\infty}^{+\infty} m|V_{0y}|\psi_y^2\,\mathrm{d}y\,\mathrm{d}s$$

$$\le O(1)\varepsilon^{8\alpha-3}$$

$$+\int_0^\tau\int_{-\infty}^{+\infty}(|D^{-1}(k_0\psi_y+(\psi_\tau-\dot{s}\psi_y)_y)F_y|+|\phi_{yy}F_y|)\,\mathrm{d}y\,\mathrm{d}s. \quad (3.30)$$

Using the following estimate:

$$|F_y|\le O(1)\varepsilon|R_y|+O(1)(\eta_0+|\phi_{yy}|)\phi_y^2+O(1)|\phi_y||\phi_{yy}|$$

$$+\eta_0(|(\psi_\tau-\dot{s}\psi_y)_y|+|\phi_{yy}|+|(\psi_\tau-\dot{s}\psi_y)|+|\phi_y|) \quad (3.31)$$

and

$$O(1)\varepsilon^2\int_0^\tau\int_{-\infty}^{+\infty}R_y^2\,\mathrm{d}y\,\mathrm{d}s=\varepsilon\int_0^{\varepsilon\tau}\int_{-\infty}^{+\infty}R_x^2(x,\eta)\,\mathrm{d}x\,\mathrm{d}\eta$$

$$\le O(1)\varepsilon^{6\alpha-1}$$

$$\le O(1)\varepsilon^{8\alpha-3}, \quad (3.32)$$

we can estimate the last term of (3.30), for a positive and suitably small α_1, as

$$\int_0^\tau\int_{-\infty}^{+\infty}|D^{-1}(k_0\psi_y+(\psi_\tau-\dot{s}\psi_y)_y)F_y|+|\phi_{yy}F_y|)\,\mathrm{d}y\,\mathrm{d}s$$

$$\le\alpha_1\int_0^\tau\|(\phi_{yy},\psi_y,(\psi_\tau-\dot{s}\psi_y)_y)(s)\|^2\,\mathrm{d}s+O(1)\varepsilon^{8\alpha-3}$$

$$+O(1)(\eta_0+N(\tau))\int_0^\tau\|(\phi_y,\psi_y,\psi_\tau-\dot{s}\psi_y)(s)\|_1^2\,\mathrm{d}s, \quad (3.33)$$

which, together with (3.30) and Gronwall's inequality, leads to Lemma 3.4. □

Lemma 3.5. *Suppose the conditions in Theorem* 3.1 *are satisfied, then, for all* $\tau\in[0,\tau_0]$, *we have*

$$\|(\phi,\psi)(\tau)\|_3^2+\|(\psi_\tau-\dot{s}\psi_y)(\tau)\|_2^2+\int_0^\tau\|(\phi_y,\psi_y,(\psi_\tau-\dot{s}\psi_y))(s)\|_2^2\,\mathrm{d}s$$

$$+\int_0^\tau\int_{-\infty}^{+\infty}m|V_{0y}|\psi_{yy}^2\,\mathrm{d}y\,\mathrm{d}s$$

$$\le O(1)\varepsilon^{8\alpha-3}.$$

Proof. Using the following relations:

$$(k_0\phi_{yy}+\psi_{yyy})\partial_{yy}L_1+D^{-1}((\psi_\tau-\dot{s}\psi_y)_{yy}+k_0\psi_{yy})\partial_yL_2$$

$$=D^{-1}(\psi_y+\mu(\psi_\tau-\dot{s}\psi_y)_{yy})F_{yy}, \tag{3.34}$$

$$(E\phi_{yyy}-(\psi_\tau-\dot{s}\psi_y)_{yy})\partial_{yyy}L_1-\phi_{yyy}\partial_{yy}L_2=-\phi_{yyy}F_{yy} \tag{3.35}$$

and the following estimates:

$$\begin{aligned}|F_{yy}|\le{}& O(1)|\varepsilon R_{yy}|+O(1)(1+(\eta_0+|\phi_{yy}|)|\phi_{yy}|+|\phi_{yyy}|)\phi_y^2\\&+O(1)(\eta_0+|\phi_{yy}|)|\phi_y||\phi_y|\\&+O(1)|\phi_y||\phi_{yy}|^2+O(1)|\phi||\phi_{yy}|\\&+\eta_0(|(\psi_\tau-\dot{s}\psi_y)_{yy}|+|\phi_{yyy}|\\&+(|(\psi_\tau-\dot{s}\psi_y)|+|\phi_y|)(1+|\phi_{yy}|+|\phi_{yyy}|)\\&+(|(\psi_\tau-\dot{s}\psi_y)_y|+|\phi_{yy}|)(1+|\phi_{yy}|)),\end{aligned} \tag{3.36}$$

$$\varepsilon^2\int_0^\tau\int_{-\infty}^{+\infty}R_{yy}^2\,\mathrm{d}y\,\mathrm{d}s\le O(1)\varepsilon^{8\alpha-3}, \tag{3.37}$$

we can prove Lemma 3.5 with the help of (3.14)–(3.18) and a similar argument as in the proof of Lemmas 3.3 and 3.4.

The combination of Lemmas 3.3–3.5 leads to Theorem 3.1.

Now, we turn to the initial value problem (3.7). Since there always exists a unique solution to the initial value problem (3.7) in space X locally (in time), we get, by the a priori estimates, Theorem 3.1 and a standard continuity argument for the hyperbolic systems:

Theorem 3.6. *Suppose* (H_1)–(H_3) *are satisfied. Let* ε_0, η_0, δ_0 *and* K_0 *be the suitable constants as in Theorem* 3.1, *such that* $|v_0^r-v_0^l|+|u_0^r-u_0^l|+|p_0^r-p_0^l|\le\eta_0$. *Then for each* $\varepsilon\in(0,\varepsilon_0]$, *there exists a unique solution* (ϕ,ψ) *to* (3.7) *in* $X(0,T/\varepsilon)$ *satisfying*

$$\sup_{0\le\tau\le T/\varepsilon}(\|(\phi,\psi)(\tau)\|_3^2+\|\psi_\tau(\tau)\|_2^2)+\int_0^{T/\varepsilon}\|(\phi_y,\psi_y,\psi_\tau)(\tau)\|_2^2\,\mathrm{d}\tau\le K_0\varepsilon^{8\alpha-3} \tag{3.38}$$

for $(\phi,\psi)\in X(0,T/\varepsilon)$.

By Theorem 3.6 and the structure of S^ε, we find that, for each $\varepsilon\in(0,\varepsilon_0]$, there exists a smooth solution $\overline{S^\varepsilon}$ to (1.1) on $[0,T]\times\mathbf{R}^1$ such that (1.8) is satisfied.

Turn to the desired asymptotic behavior of $\overline{S^\varepsilon}(x,t)$. We have, from (3.1), (3.3), (3.5), (3.6) and (3.38), that

$$\sup_{0\le t\le T}\|(\overline{S^\varepsilon}-S^\varepsilon)(\cdot,t)\|^2=\sup_{0\le t\le T}\|(\tilde{\phi}_x,\tilde{\psi}_x,\tilde{w})(\cdot,t)\|^2$$

$$= \varepsilon^2 \sup_{0\le t\le T} \|(\phi_x, \psi_x, \psi_t)\|^2$$

$$= \sup_{0\le \tau\le T/\varepsilon} \|(\phi_y, \psi_y, \psi_\tau)\|^2$$

$$\le K_0 \varepsilon^{8\alpha-3}. \tag{3.39}$$

On the other hand, by the construction of the approximate solutions (see Lemma 3.2), we have

$$\sup_{0\le t\le T} \|S^\varepsilon - \chi_0\|^2 \le O(1)\varepsilon^\alpha.$$

Hence, we have

$$\sup_{0\le t\le T} \|\overline{S^\varepsilon} - \chi_0\|^2 \le O(1)\varepsilon^\alpha,$$

which is (1.9).

To prove (1.10), we use Sobolev's inequality and (3.38) to obtain

$$\sup_{x\in\mathbf{R}^1} |(\overline{S^\varepsilon} - S^\varepsilon)| = \sup_{y\in\mathbf{R}^1} |(\phi_y, \psi_y, \psi_\tau)|$$

$$\le O(1)\|(\phi_y, \psi_y, \psi_\tau)\|^{1/2}\|(\phi_{yy}, \psi_{yy}, \psi_{\tau y})\|^{1/2}$$

$$\le O(1)\varepsilon^{(8\alpha-3)/2}. \tag{3.40}$$

Eq. (3.40) and (i) of Lemma 3.2 yield (1.10), if we choose $\alpha > \frac{5}{8}$. Theorem 1 is proved. □

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