2005 Georgia Tech High School Mathematics Competition
Junior-Varsity Multiple-Choice Examination – Version A

Problem 1: If $f(x) = \frac{x^4-3x^3+x^2-2}{x-3}$, what is the value of $f(4)$?

(A) 62 (B) 70 (C) 78 (D) 81 (E) 90

Problem 2: Evaluate the expression:

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=0}^{3} i\,j$$

(A) 12 (B) 36 (C) 48 (D) 288 (E) None of the above

Problem 3: Express $\frac{1}{1-\sqrt{2}}$ in the form $\alpha + \beta\sqrt{2}$, where α and β are rational numbers.

(A) $1-\sqrt{2}$ (B) $-1-\sqrt{2}$ (C) $\dfrac{-1}{3}-\dfrac{\sqrt{2}}{3}$ (D) $\dfrac{1}{3}+\dfrac{\sqrt{2}}{3}$ (E) $\dfrac{1}{5}+\dfrac{\sqrt{2}}{5}$

Problem 4: A function is said to be **even** if $f(x) = f(-x)$ for all values of x. A function is said to be **odd** if $f(x) = -f(-x)$ for all values of x. The function $g(x) = f(x) - f(-x)$ is:

(A) Even (B) Odd (C) Even only when $x > 0$ (D) Neither even, nor odd

Problem 5: Find the minimum value of $g(x) = 4x - x^2 + 1$, where x can be any real number.

(A) -1 (B) 1 (C) 5 (D) 7 (E) None of the above

Problem 6: How many positive integer solutions are there to $1! + 2! + 3! + 4! + ... + x! \leq x^3$?

(A) 1 (B) 2 (C) 4 (D) 5 (E) None of the above

Problem 7: What is 342_5 written in base-10?

(A) 342 (B) 85 (C) 123 (D) 97 (E) None of the above

Problem 8: What rational number can be represented by $0.\overline{099}$?

$(A)\ \dfrac{11}{101}$ $(B)\ \dfrac{11}{111}$ $(C)\ \dfrac{11}{121}$ $(D)\ \dfrac{11}{131}$ $(E)\ \dfrac{99}{1000}$

Problem 9: A triangle has vertices at $(8,6,6)$, $(8,12,6)$, and $(2,6,6)$. What is the area of this triangle?

(A) 9 (B) 12 (C) 18 (D) 27 (E) None of the above

Problem 10: Let P be a point on the diagonal AC of the rectangle $ABCD$; how does the area of triangle APD compare to the area of triangle APB?

$$\text{Area of } \Delta_{APD} \;\underline{\hspace{2cm}}\; \text{Area of } \Delta_{APB}$$

$(A)\ >$ $(B)\ =$ $(C)\ <$ (D) Not enough information given

Problem 11: Given an isosceles right triangle inscribed in a circle where all three vertices of the triangle are located on the circle, determine the ratio of the area of the triangle to the area of the circle, $\frac{\text{Area}\Delta}{\text{Area}O}$.

$(A)\ \dfrac{1}{2\pi}$ $(B)\ \dfrac{2}{3\pi}$ $(C)\ \dfrac{1}{\pi}$ $(D)\ \dfrac{2}{\pi}$ (E) Not enough information given

Problem 12: The Fibonacci numbers are defined by the **recurrence relation**:

$$f_1 = 1 \qquad f_2 = 1 \qquad f_{n+2} = f_{n+1} + f_n$$

for $n = 1, 2, 3, \ldots$. The relation $f_n < 2^n$ is true for:

(A) Only the integers $n < 2^5$ (B) Only for integers $n < 2005$ (C) Only for integers $n < 2^{2005}$
(D) Only for integers $n < 2^{2005} - 2005$ (E) All positive integers n

Problem 13: How many words of length 8, formed using the alphabet (0,1,2,a,b) have at least one 'a' and one 'b'? For example, one such word is 01221aba.

$(A)\ 5^8 - 4^8 - 4^8 + 3^8$ $(B)\ 4^8 + 4^8 - 3^8$ $(C)\ 3^8$ $(D)\ 4^8$ (E) None of the above

Problem 14: Let a, b, c, d be real positive numbers, with $\frac{a}{b} < \frac{c}{d}$. Which of the relations are true for:

$$\frac{a}{b} \text{ ——— } \frac{a+c}{b+d} \text{ ——— } \frac{c}{d}$$

$(A)\ <, <$ $(B)\ <, \leq$ $(C)\ \leq, <$ $(D)\ \leq, \leq$ (E) None of the above

Problem 15: The number $2^{2005} + 2007$ can be written as the sum of two perfect squares in how many ways?

$(A)\ 0$ $(B)\ 1$ $(C)\ 4$ $(D)\ 9$ (E) None of the above

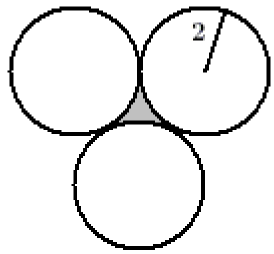
Problem 16: What digit occupies the $38,885^{\text{th}}$ position when we write out all the integers in succession, beginning with 1 (i.e. 1234567891011121314151617...)?

$(A)\ 1$ $(B)\ 3$ $(C)\ 5$ $(D)\ 8$ $(E)\ 9$

Problem 17: The fundamental theorem of algebra states that a polynomial of order n, $p_n(x) = x^n + \alpha_{n-1}x^{n-1} + \alpha_{n-2}x^{n-2} + ... + \alpha_1 x + \alpha_0$, has exactly n roots. If all of the coefficients, $(\alpha_{n-1}, \alpha_{n-2}, ..., \alpha_1, \alpha_0)$, are real numbers, which of the following statements are true (mark all that apply)?

(A) If n is even, there must exist at least one real root.
(B) If n is odd, there must exist at least one real root.
(C) If n is even, there must be at least one complex root.
(D) The sum of the coefficients, $\sum_{k=1}^{n-1} \alpha_k$, must be positive.
(E) All complex roots will occur in conjugate pairs.

Problem 18: Three mutually tangent circles of equal radius two are shown in the figure below. What is the area of shaded portion between the three circles?



$(A)\ \sqrt{3} - \dfrac{\pi}{2}$ $(B)\ \dfrac{4\sqrt{3} - \pi}{3}$ $(C)\ 4\sqrt{3} - 2\pi$
$(D)\ 2\sqrt{6} - \pi$ (E) Not enough information given

Problem 19: Convergence of the sequence $X_n = (x_0, x_1, x_2, ...)$ implies that the quantity $|x_{n+1} - x_n|$ tend to zero in the limit as $n \to \infty$.

The recurrance relation, $z_n = \frac{3}{4} \cdot z_{n-1} + 3$, is a convergent sequence. Determine the value that it converges to.

(A) 3 $(B)\ \dfrac{9}{4}$ (C) 12 (D) 15 (E) None of the above

Problem 20: The geometric sum, $1 + r + r^2 + ... + r^n + ... = \sum_{k=0}^{\infty} r^k$ converges to the value $\frac{1}{1-r}$ provided $|r| < 1$.

Calculate $$\sum_{k=1}^{\infty} \frac{6 - 2^{k+1}}{3^{k-1}}$$

$(A)\ -13$ $(B)\ -12$ $(C)\ -3$ (D) 6 (E) None of the above