

2005 Georgia Tech High School Mathematics Competition

Junior-Varsity Multiple-Choice Examination – Version B

Problem 1: How many positive integer solutions are there to $1! + 2! + 3! + 4! + \dots + x! \leq x^3$?

- (A) 1 (B) 2 (C) 4 (D) 5 (E) None of the above

Problem 2: Evaluate the expression:

$$\sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=0}^3 i j$$

- (A) 12 (B) 36 (C) 48 (D) 288 (E) None of the above

Problem 3: A function is said to be **even** if $f(x) = f(-x)$ for all values of x . A function is said to be **odd** if $f(x) = -f(-x)$ for all values of x . The function $g(x) = f(x) - f(-x)$ is:

- (A) Even (B) Odd (C) Even only when $x > 0$ (D) Neither even, nor odd

Problem 4: Express $\frac{1}{1-\sqrt{2}}$ in the form $\alpha + \beta\sqrt{2}$, where α and β are rational numbers.

- (A) $1 - \sqrt{2}$ (B) $-1 - \sqrt{2}$ (C) $\frac{-1}{3} - \frac{\sqrt{2}}{3}$ (D) $\frac{1}{3} + \frac{\sqrt{2}}{3}$ (E) $\frac{1}{5} + \frac{\sqrt{2}}{5}$

Problem 5: The Fibonacci numbers are defined by the **recurrence relation:**

$$f_1 = 1 \quad f_2 = 1 \quad f_{n+2} = f_{n+1} + f_n$$

for $n = 1, 2, 3, \dots$. The relation $f_n < 2^n$ is true for:

- (A) Only the integers $n < 2^5$ (B) Only for integers $n < 2005$ (C) Only for integers $n < 2^{2005}$
(D) Only for integers $n < 2^{2005} - 2005$ (E) All positive integers n

Problem 6: If $f(x) = \frac{x^4 - 3x^3 + x^2 - 2}{x - 3}$, what is the value of $f(4)$?

- (A) 62 (B) 70 (C) 78 (D) 81 (E) 90

Problem 7: The number $2^{2005} + 2007$ can be written as the sum of two perfect squares in how many ways?

- (A) 9 (B) 4 (C) 1 (D) 0 (E) None of the above

Problem 8: What rational number can be represented by $0.\overline{0999}$?

- (A) $\frac{11}{101}$ (B) $\frac{11}{111}$ (C) $\frac{11}{121}$ (D) $\frac{11}{131}$ (E) $\frac{99}{1000}$

Problem 9: Given an isosceles right triangle inscribed in a circle where all three vertices of the triangle are located on the circle, determine the ratio of the area of the triangle to the area of the circle, $\frac{\text{Area}\Delta}{\text{Area}O}$.

- (A) $\frac{1}{2\pi}$ (B) $\frac{2}{3\pi}$ (C) $\frac{1}{\pi}$ (D) $\frac{2}{\pi}$ (E) Not enough information given

Problem 10: Let a, b, c, d be real positive numbers, with $\frac{a}{b} < \frac{c}{d}$. Which of the relations are true for:

$$\frac{a}{b} \text{ ————— } \frac{a+c}{b+d} \text{ ————— } \frac{c}{d}$$

- (A) $<, <$ (B) $<, \leq$ (C) $\leq, <$ (D) \leq, \leq (E) None of the above

Problem 11: A triangle has vertices at $(8, 6, 6)$, $(8, 12, 6)$, and $(2, 6, 6)$. What is the area of this triangle?

- (A) 9 (B) 12 (C) 18 (D) 27 (E) None of the above

Problem 12: Find the minimum value of $g(x) = 4x - x^2 + 1$, where x can be any real number.

- (A) -1 (B) 1 (C) 5 (D) 7 (E) None of the above

Problem 13: Let P be a point on the diagonal AC of the rectangle $ABCD$; how does the area of triangle APD compare to the area of triangle APB ?

Area of Δ_{APD} _____ Area of Δ_{APB}

- (A) = (B) > (C) < (D) Not enough information given

Problem 14: How many words of length 8, formed using the alphabet (0,1,2,a,b) have at least one 'a' and one 'b'? For example, one such word is 01221aba.

- (A) $5^8 - 4^8 - 4^8 + 3^8$ (B) $4^8 + 4^8 - 3^8$ (C) 3^8 (D) 4^8 (E) None of the above

Problem 15: What is 342_5 written in base-10?

- (A) 342 (B) 85 (C) 123 (D) 97 (E) None of the above

Problem 16: What digit occupies the 38,885th position when we write out all the integers in succession, beginning with 1 (i.e. 1234567891011121314151617...)?

- (A) 1 (B) 3 (C) 5 (D) 8 (E) 9

Problem 17: The fundamental theorem of algebra states that a polynomial of order n , $p_n(x) = x^n + \alpha_{n-1}x^{n-1} + \alpha_{n-2}x^{n-2} + \dots + \alpha_1x + \alpha_0$, has exactly n roots. If all of the coefficients, $(\alpha_{n-1}, \alpha_{n-2}, \dots, \alpha_1, \alpha_0)$, are real numbers, which of the following statements are true (mark all that apply)?

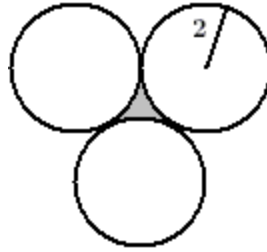
- (A) If n is even, there must exist at least one real root.
 (B) If n is odd, there must exist at least one real root.
 (C) If n is even, there must be at least one complex root.
 (D) The sum of the coefficients, $\sum_{k=1}^{n-1} \alpha_k$, must be positive.
 (E) All complex roots will occur in conjugate pairs.

Problem 18: The geometric sum, $1 + r + r^2 + \dots + r^n + \dots = \sum_{k=0}^{\infty} r^k$ converges to the value $\frac{1}{1-r}$ provided $|r| < 1$.

Calculate $\sum_{k=1}^{\infty} \frac{6 - 2^{k+1}}{3^{k-1}}$

- (A) -13 (B) -12 (C) -3 (D) 6 (E) None of the above

Problem 19: Three mutually tangent circles of equal radius two are shown in the figure below. What is the area of shaded portion between the three circles?



- (A) $\sqrt{3} - \frac{\pi}{2}$ (B) $\frac{4\sqrt{3} - \pi}{3}$ (C) $4\sqrt{3} - 2\pi$
 (D) $2\sqrt{6} - \pi$ (E) Not enough information given

Problem 20: Convergence of the sequence $X_n = (x_0, x_1, x_2, \dots)$ implies that the quantity $|x_{n+1} - x_n|$ tend to zero in the limit as $n \rightarrow \infty$.

The recurrence relation, $z_n = \frac{3}{4} \cdot z_{n-1} + 3$, is a convergent sequence. Determine the value that it converges to.

- (A) 3 (B) $\frac{9}{4}$ (C) 12 (D) 15 (E) None of the above