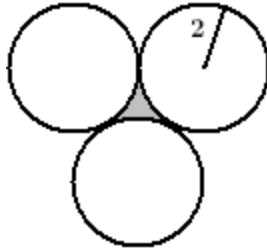


2005 Georgia Tech High School Mathematics Competition

Varsity Multiple-Choice Examination – Version A

Problem 1: Three mutually tangent circles of equal radius two are shown in the figure below. What is the area of shaded portion between the three circles?



- (A) $\sqrt{3} - \frac{\pi}{2}$ (B) $4\sqrt{3} - 2\pi$ (C) $\frac{4\sqrt{3} - \pi}{3}$
 (D) $2\sqrt{6} - \pi$ (E) Not enough information given

Problem 2: What is the value of $\cos(\frac{5\pi}{12})$?

- (A) $\frac{1 + \sqrt{3}}{2\sqrt{2}}$ (B) $\frac{-1 + \sqrt{3}}{2\sqrt{2}}$ (C) $\frac{1 - \sqrt{3}}{2\sqrt{2}}$ (D) $\frac{-1 - \sqrt{3}}{2\sqrt{2}}$ (E) None of the above

Problem 3: Calculate the expression: $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + n \cdot n!$.

- (A) $(n^2 + n + 1) \cdot n!$ (B) $(n + 1)! - 1$ (C) $(n + 2)! - n!$ (D) $(n!)^2 - 1$ (E) None of the above

Problem 4: What is the minimum value of $f(n) = 2n^2 + 3n + 1$, where n can be any real number?

- (A) 1 (B) $\frac{-3}{4}$ (C) $\frac{-1}{8}$ (D) $\frac{-1}{16}$ (E) None of the above

Problem 5: Determine the value of x :

$$-7\ln\left(\frac{9}{10}\right) + 2\ln\left(\frac{24}{25}\right) + 3\ln\left(\frac{81}{80}\right) = \ln(x)$$

- (A) 8 (B) 6 (C) 4 (D) 2 (E) None of the above

Problem 6: Solve the following linear system of equations for the value of x :

$$\begin{aligned} x + 2y + 3z &= 1 \\ 2x + 3y + 4z &= 2 \\ -x + y + 2z &= 1 \end{aligned}$$

- (A) -2 (B) -1 (C) 0 (D) 1 (E) 2

Problem 7: Given the standard coins issued by the U.S. treasury (1-cent, 5-cent, 10-cent, 25-cent, and 50-cent), what is the largest amount of money a person can have in coins and be unable to make change for a dollar?

- (A) 99 cents (B) 1.09 dollars (C) 1.14 dollars (D) 1.19 dollars (E) 1.24 dollars

Problem 8: The equation $x^2 + 2x + y^2 + 6y + 6 = 0$ defines a circle. What is its radius?

- (A) 2 (B) $\sqrt{6}$ (C) $\sqrt{14}$ (D) 6 (E) None of the above

Problem 9: What is $\sum_{n=0}^{360} \sin\left(\frac{n2\pi}{360}\right)$?

- (A) 0 (B) $\frac{1}{2}$ (C) 1 (D) $\frac{\pi}{2}$ (E) None of the above

Problem 10: We surveyed 83 freshman at Georgia Tech one semester and found that:

| | | |
|------------------------------|-------------------------------------|--------------------------|
| 27 take English | 27 take American History | 55 take Calculus |
| 12 take English and Calculus | 7 take English and American History | |
| 12 take Calculus and History | | 2 take all three courses |

How many students take NONE of these three courses?

- (A) 1 (B) 3 (C) 5 (D) 9 (E) None of the above

Problem 11: The fundamental theorem of algebra states that a polynomial of order n , $p_n(x) = x^n + \alpha_{n-1}x^{n-1} + \alpha_{n-2}x^{n-2} + \dots + \alpha_1x + \alpha_0$, has exactly n roots. If all of the coefficients, $(\alpha_{n-1}, \alpha_{n-2}, \dots, \alpha_1, \alpha_0)$, are real numbers, which of the following statements are true (mark all that apply)?

- (A) If n is even, there must exist at least one real root.
 (B) If n is odd, there must exist at least one real root.
 (C) If n is even, there must be at least one complex root.
 (D) The sum of the coefficients, $\sum_{k=1}^{n-1} \alpha_k$, must be positive.
 (E) All complex roots will occur in conjugate pairs.

Problem 12: What is the sum of all the digits in the sequence 1, 2, 3, 4, ..., 2003, 2004, 2005?

- (A) 28,027 (B) 29,327 (C) 35,615 (D) 37,137 (E) 39,213

Problem 13: What are the last two digits of 103^{2005} ?

- (A) 01 (B) 23 (C) 43 (D) 63 (E) None of the above

Problem 14: How many even numbers between 1000 and 9999 have all distinct digits?

- (A) 2296 (B) 2298 (C) 2300 (D) 2302 (E) None of the above

Problem 15: How many positive integers $n < 1000$ satisfy $\lfloor \frac{n/3}{150} \rfloor = \frac{\lfloor n/3 \rfloor}{150}$, where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x ?

- (A) 4 (B) 6 (C) 8 (D) 9 (E) None of the above

Problem 16: Convergence of the sequence $X_n = (x_0, x_1, x_2, \dots)$ implies that the quantity $|x_{n+1} - x_n|$ tend to zero in the limit as $n \rightarrow \infty$.

The recurrence relation, $z_n = \frac{3}{4} \cdot z_{n-1} + 3$, is a convergent sequence. Determine the value that it converges to.

- (A) 3 (B) $\frac{9}{4}$ (C) 12 (D) 15 (E) None of the above

Problem 17: Evaluate the term $\left(\frac{1+i}{2}\right)^{10}$, where $i = \sqrt{-1}$.

- (A) $\frac{1-i}{4}$ (B) $\frac{1}{8}$ (C) $\frac{-1+i}{16}$ (D) $\frac{i}{32}$ (E) None of the above

Problem 18: An element x is said to be in the **null space** of linear operator, A, if $A \cdot x = 0$. If A and B are taken to be two $n \times n$ matrices (linear operators) and x is a length- n vector belonging to the nullspace of A, which of the following are true? Mark all that apply.

- (A) $(A \circ B)x = 0$ (B) $(B \circ A)x = 0$
 (C) The null space of $(A \circ B)$ is a subset of the null space of A
 (D) The null space of $(B \circ A)$ is a subset of the null space of A
 (E) The null space of $(A \circ B)$ is the same as the null space of $(B \circ A)$

Problem 19: Suppose you are presented with a 9x9 grid, where each of the 81 sections contains exactly one yellow jacket. At the sound of a bell, each yellow jacket is supposed to move horizontally or vertically exactly one space in the grid, such that each grid space contains a different yellow jacket (one yellow jacket per section). How many different ways can this be achieved?

- (A) 1 (B) 9! (C) $9 \cdot 8$ (D) $\frac{9!}{2!}$ (E) None of the above

Problem 20: Find the number of distinguishable permutations of the letters in the word:

HELLUVANENGINEER

- (A) 10^{16} (B) $\binom{16}{10}$ (C) $\frac{16!}{2!3!4!}$ (D) 16! (E) None of the above