# 2005 Georgia Tech High School Mathematics Competition
## Varsity Proof Examination

**Problem 1:** Prove that the following expression holds for all positive integers, $n$.

$$\sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n}$$

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**Problem 2:** How many unique ways could a person make change for a one-dollar bill using only combinations of the most commonly used coin denominations (1-cent, 5-cent, 10-cent, and 25-cent)?

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**Problem 3:** Is the series $\frac{1}{9} + \frac{1}{19} + \frac{1}{29} + ... + \frac{1}{89} + \frac{1}{90} + \frac{1}{91} + ... + \frac{1}{99} + \frac{1}{109} + \frac{1}{119} + ...$, where each denominator contains the digit 9, convergent? If yes, show what value it converges to. If not, show that it is divergent.

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**Problem 4:** Given any integer, $a$, not divisible by $p$, prove that $a^{p-1} \equiv 1 \, mod \, p$, where $p \geq 3$ is any prime number.

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**Problem 5:**

A closed interval includes its endpoints and is denoted $[a, b]$, such that $a$ and $b$ belong in the interval (as do all $x$, $a \leq x \leq b$).

An open interval does not include its endpoints and is denoted $(a, b)$, such that neither $a$ nor $b$ are included in the interval, while every $x$, $a < x < b$ does.

The Cantor middle-thirds set, $\mathbb{C}$, is a subset of the real line from $[0, 1]$, and is obtained from $[0, 1]$ by iteratively removing the open middle-third intervals: $(\frac{1}{3}, \frac{2}{3})$, then removing $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$, and so forth (infinitely many times).

(a) How many points are included in the Cantor middle-thirds set?

(b) Describe whether or not it is possible to map the points in the Cantor set to the natural numbers, $\mathbb{N} = 1, 2, 3, .....$

(c) Calculate the length of the Cantor set. Recall that the length of a closed interval, $[a, b]$, is equal to $b - a$.