Georgia Tech HSMC 2008

Junior Varsity Multiple Choice

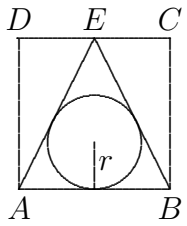
February 23^{rd}, 2008

1. As a New Year's resolution, Alex decided to save some money. He took an empty jar, and on January 1^{st} he put $1^{\text{¢}}$ in the jar. On January 2^{nd} he put $2^{\text{¢}}$ in the jar, and so on (on the n^{th} day of the year he put $n^{\text{¢}}$ in the jar). He uses pennies, nickels, dimes and quarters (and no other denomination), and each day he manages to use the smallest possible number of coins. Today (February 23^{rd}), he places the appropriate amount in the jar, and then wonders which of the following statements are true: In the jar...

   (i) There are more dimes than quarters.
   (ii) There is more money in nickels than in pennies.
   (iii) There are more nickels than dimes.
   (iv) There is more money in quarters than in all the other coins put together.

   (a) (i) and (ii) are true.
   (b) (iii) and (iv) are true.
   (c) (ii) and (iii) are true.
   (d) (i) and (iv) are true.
   (e) (i), (ii), (iii), and (iv) are all true.

2. Let $ABCD$ be a square of side length 2. Let E be the middle point of the segment CD. What is the radius r of the circle inscribed in $\triangle ABE$?



(a) $\dfrac{\sqrt{5}-1}{2}$

(b) $\dfrac{\sqrt{5}+1}{2}$

(c) $\dfrac{3}{4}$

(d) $\dfrac{4}{3}$

(e) $\dfrac{\sqrt{5}}{3}$

3. A palindrome is a string that reads the same both forwards and backwards, such as the word *radar*. In the 26-letter English alphabet, how many palindromes are there that are exactly 11 letters long?

(a) $\dfrac{26!}{5!}$

(b) $\dfrac{26!}{11!}$

(c) 26^5

(d) 26^6

(e) 26^{11}

4. I have three daughters: Anna, Brenda, and Diana. The sum of Anna's age and Brenda's age is 24 more than Diana's age. The sum of Anna's age and Diana's age is 20 more than Brenda's age. The sum of Diana's age and Brenda's age is 10 more than Anna's age. How old is Brenda?

   (*a*) 15

   (*b*) 17

   (*c*) 19

   (*d*) 20

   (*e*) 22

5. Regular-priced amusement park tickets cost \$29 each. Children's tickets cost \$19 each. If a group of people spent a total of \$600, then how many total tickets were bought?

   (*a*) 25

   (*b*) 27

   (*c*) 28

   (*d*) 29

   (*e*) 30

6. How many pairs of positive integers (n, m), with $n \leq m$ satisfy the following equation?

$$\frac{1}{5} = \frac{1}{n} + \frac{1}{m}$$

   (*a*) The equation has no solutions.

   (*b*) 1

   (*c*) 2

   (*d*) 3

   (*e*) 4

7. Let $\triangle XYZ$ be right at Z. Let A_x denote the area of the circle with diameter YZ. Let A_y denote the area of the circle with diameter XZ and let A_z denote the area of the circle with diameter XY. Which of the following relations is true?

   (a) $A_z = A_x - A_y$

   (b) $A_z = A_x + A_y$

   (c) $A_z = A_x^2 + A_y^2$

   (d) $A_z^2 = A_x^2 + A_y^2$

   (e) $A_z^2 = A_x^2 - A_y^2$

8. What numeral is in the 100th decimal place in the decimal representation of 6/7?

   (a) 1

   (b) 2

   (c) 4

   (d) 5

   (e) 8

9. In a league with four teams, each team plays the others exactly once. Each game ends in either a win or a loss. No ties are allowed. What's the greatest number of teams that can end with a losing record (i.e. having lost more games than they won)?

   (a) One team

   (b) Two teams

   (c) Three teams

   (d) It is possible for all four teams to have a losing record.

   (e) Not enough information to determine the answer.

10. What is the largest hypotenuse a right triangle can have if all three sides are integers less than 115 and two of its sides are consecutive integers?

    (a) 25

    (b) 41

    (c) 61

    (d) 85

    (e) 113

11. C_1 and C_2 are two circles of radius 1 such that their centers are 1 unit length appart. What is the area of the intersection of these two circles?

    (a) $\sqrt{3} - \frac{2\pi}{3}$.

    (b) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$.

    (c) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$.

    (d) $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$.

    (e) $\frac{2\pi}{3} - \frac{1}{2}$.

12. A regular hexagon and an equilateral triangle have equal perimeters. What is the ratio of the area of the hexagon to the area of the triangle?

    (a) $\frac{2}{3}$

    (b) $\frac{3}{2}$

    (c) $\frac{4}{3}$

    (d) 3

    (e) 4

13. A triangle and a square are drawn on the plane. What is the greatest number of regions into which they can divide the plane?

    (*a*) 5
    (*b*) 6
    (*c*) 7
    (*d*) 8
    (*e*) 9

14. What is the remainder when $2^{30} \cdot 3^{20}$ is divided by 7?

    (*a*) 1
    (*b*) 2
    (*c*) 4
    (*d*) 5
    (*e*) 6

15. How many numbers from 1 to 1000 are divisible by 60 but not by 24?

    (*a*) 8
    (*b*) 12
    (*c*) 16
    (*d*) 24
    (*e*) 41

16. How many distinct positive divisors does the integer 400400 have?

    (*a*) 5
    (*b*) 8
    (*c*) 120
    (*d*) 240
    (*e*) 360

17. The equation $x^2 + Bx + C = 0$ has 5 as the sum of its roots, and 15 as the sum of the squares of its roots. What is the value of C?

    (*a*) 5

    (*b*) 7.5

    (*c*) 10

    (*d*) 12.5

    (*e*) 15

18. Given the function $f(x) = 5x^3 - 7x^2 + 11x + 13$, find A, B, C, and D such that $f(x) = A(x+1)^3 + B(x+1)^2 + C(x+1) + D$.

    (*a*) $A = 5, B = -22, C = 40, D = -36$

    (*b*) $A = 5, B = -22, C = 40, D = -10$

    (*c*) $A = 5, B = 8, C = 40, D = 12$

    (*d*) $A = 5, B = 1, C = 19, D = 10$

    (*e*) $A = 5, B = 1, C = 19, D = -4$

19. Which of the following pairs of numbers are *not* relatively prime? (Two numbers a and b are called relatively prime if $\mathrm{GCD}(a, b) = 1$.)

    (*a*) (159, 698)

    (*b*) (199, 934)

    (*c*) (201, 1003)

    (*d*) (343, 1008)

    (*e*) (461, 951)

20. Given a regular pentagon with sides of length 2, find the height h.

(a) $\dfrac{\sqrt{3+2\sqrt{3}}}{5}$

(b) $\dfrac{\sqrt{3+2\sqrt{3}}}{2}$

(c) $\dfrac{\sqrt{5+2\sqrt{5}}}{3}$

(d) $\sqrt{3+2\sqrt{3}}$

(e) $\sqrt{5+2\sqrt{5}}$