

**CS 7535**  
Decomposition Theorems

The decomposition theorems describe how we may bound the speed of convergence of a Markov chain by its speed of convergence of the chain on subsets of the state space, and the speed at which it reaches an equilibrium between these subsets. There are two main versions of the Decomposition theorem we are interested in, the *State Decomposition theorem*[Madras, Randall], which considers subsets with overlap between them, and the *Partition Decomposition theorem*[Martin, Randall], which considers disjoint subsets.

In both cases, we consider subsets  $\Omega_1, \dots, \Omega_n$  whose union is the state space  $\Omega$ . The restriction  $M_i$  of the Markov chain  $M$  to the subset  $\Omega_i$  is the Markov chain that, for any state  $x \in \Omega_i$ , chooses the next state  $y$  according to  $M$ , but then rejects the move if  $y \notin \Omega_i$ .

Both theorems are stated in terms of spectral gaps. Recall the following theorem relates the spectral gap of a Markov chain and its mixing time.

**Theorem.** *Let  $\pi_* = \min_{x \in \Omega} \pi(x)$ . For all  $\varepsilon > 0$  we have*

$$(a) \tau(\varepsilon) \leq \frac{1}{\text{Gap}(P)} \log\left(\frac{1}{\pi_* \varepsilon}\right)$$

$$(b) \tau(\varepsilon) \geq \frac{|\lambda_1|}{2\text{Gap}(P)} \log\left(\frac{1}{2\varepsilon}\right).$$

We are now prepared to state the two versions of the decomposition theorem.

## 1 State Decomposition

(For overlapping sets): In this case, we use the overlap between subsets to define the probability of transitioning from one to the other - the larger the overlap, the larger the probability of transition. We don't want a state to be used too often however, let  $\Theta$  be the maximum number of times any state is present in subset. In other words,  $\Theta = \max_{x \in \Omega} |\{i \text{ s.t. } x \in \Omega_i\}|$ .

The projection chain  $M^*$  will be over the subsets  $\Omega_i$ . The intuition is that if we are watching moves of the Markov chain  $M$ , then when a particle stays in any  $\Omega_i$ , then it is moving according to the restriction  $M_i$ . When a particle is in the overlap between  $\Omega_i$  and  $\Omega_j$ , then we can "switch" from treating an element like it is in  $\Omega_i$  to treating it like it is in  $\Omega_j$ .

To capture this idea, we say that between any two subsets  $\Omega_i, \Omega_j$ ,

$$P_{M^*}(i, j) = \frac{\pi(\Omega_i \cap \Omega_j)}{\Theta \pi(\Omega_i)}$$

For this case, we have

**Theorem** (State Decomposition Theorem).

$$\text{Gap}(M) \geq \frac{1}{\Theta^2} \text{Gap}(M^*) \min_i \text{Gap}(M_i)$$

## 2 Partition Decomposition

Here the subsets form a partition, so we must define the transition probabilities in our projection chain differently. This definition reflects the ergodic flow from  $\Omega_i$  to  $\Omega_j$ :

$$P_{\hat{M}}(i, j) = \frac{\sum_{x \in \Omega_i, y \in \Omega_j} \pi(x) P_M(x, y)}{\pi(\Omega_i)}$$

The restrictions are defined as previously.

For this version,

**Theorem** (Partition Decomposition Theorem).

$$Gap(M) \geq \frac{1}{2} Gap(\hat{M}) \min_i Gap(M_i)$$