CS 1050 Homework 1 Solutions

1.1 $\{3\}$.
1.2 $\{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$.
1.3 $\{2, 3, 5, 7, 11, 13\}$.
1.4 empty.

2 a) Lemma 1: The sum of two even numbers is even.
Proof: Let the two even numbers be of the form $2x$ and $2y$ where x, y are natural numbers. Their sum is $2x+2y$, which can be writen as $2(x+y)$, which is of the form $2z$, where $z = x+y$, and z is a natural number. Therefore, it is an even number. Hence sum of two even numbers is an even number. □
b) Lemma 2: The sum of two odd numbers is even.
Proof: Let the two odd numbers be of the form $2x + 1$ and $2y + 1$ where x, y are natural numbers. Their sum is $2x + 2y + 2$, which can be writen as $2(x + y + 1)$, which is of the form $2z$, where $z = x + y + 1$, and z is a natural number. Therefore, it is an even number. Hence sum of two odd numbers is an even number. □
c) Lemma 3: The sum of two integers that are multiples of 3 is also multiples of 3.
Proof: Let the two multiples of 3 be $3x$ and $3y$ where x, y are integers. Their sum is $3x + 3y$ which can also be written as $3(x + y)$, which is of the form $3z$, where $z = x + y$ and z is an integer. Therefore, it is a muliple of 3. Hence the sum of two multiples of 3 is also a multiple of 3. □

3 a) Lemma 4: Let a be an integer such that $a = 3k + 1$ where k is an integer. Then the remainder when a^2 is divided by 3 is 1.
Proof: Assume $a = 3k + 1$. Then $a^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$. Since $3(3k^2 + 2k)$ is divisible by 3, the remainder must be 1. □
b) Lemma 5: Let a be an integer such that $a = 3k + 2$ where k is an integer. Then the remainder when a^2 is divided by 3 is 1.
Proof: Assume $a = 3k+2$. Then $a^2 = 9k^2+12k+4 = 3(3k^2+2k+1)+1$. Since $3(3k^2+2k+1)$ is divisible by 3, the remainder must be 1. □
c) Lemma 6: If a is an integer and a^2 is a multiple of 3, then a is also a multiple of 3.
Proof by contradiction: Assume a is an integer and a^2 is a multiple of 3, but a is NOT a multiple of 3. Then a cannot be written in the form $3k$ for some integer k. Then a must be of the form $3k + 1$ or $3k + 2$ for some integer k.

Case 1: $a = 3k + 1$. Then by Lemma 4, a^2 will have a remainder of 1 when divided by 3 (thus it will not be divisible by 3).

Case 2: $a = 3k + 2$. Then by Lemma 5, a^2 will not be divisible by 3.

Since we assumed that a^2 was divisible by 3, we have reached a contradiction. Thus, our original assumption that a is not a multiple of 3 is false, and we can conclude that a is a multiple of 3. $\square$