# CS 1050 Homework 4 and 5 Solutions

**1.a Proof:** Let $x, y$ be two integers such that $2^x = 3^y$. Let us consider the following cases which together handle all possible pairs $(x, y)$ with $x \neq 0$.

Case a. $x > 0, y > 0$. We have $2 \equiv 0(mod\ 2)$. Since $x > 0$, we have $2^x \equiv 0^x(mod\ 2)$, that is, $2^x \equiv 0(mod\ 2)$. In other words $2^x$ is an even natural number. Now, $3 \equiv 1(mod\ 2)$, and since $y > 0$, we have $3^y \equiv 1^y(mod\ 2)$, that is, $3^y \equiv 1(mod\ 2)$. In other words, $3^y$ is an odd number. Therefore $2^x \neq 3^y$.

Case b. $x > 0, y \leq 0$. In this case, $2^x > 1$ and $3^y \leq 1$, and, again, we cannot have equality.

Case c. $x \leq 0, y > 0$. Similarly, $2^x \leq 1$ and $3^y > 1$.

Case d. $x < 0, y < 0$. If $2^x = 3^y$, it follows that $2^{-x} = 3^{-y}$, where $-x > 0, -y > 0$. But, by case a, this is not possible.

Since in all of these cases, we cannot have $2^x = 3^y$, and since these cases cover all possibilities with $x \neq 0, y \neq 0$, it follows that if $2^x = 3^y$, then $x = y = 0$.

**b. Proof:** Choose two ordered pairs $(a, b), (a', b') \in \mathbb{N} \times \mathbb{N}$ such that $f(a, b) = f(a', b')$. We want to show that $(a, b) = (a', b')$, that is, $a = a'$ and $b = b'$. The relation $f(a, b) = f(a', b')$ implies $2^a 3^b = 2^{a'} 3^{b'}$, and this can be rewritten as $2^{a-a'} = 3^{b-b'}$. From part a), this implies that $a - a' = 0$ and $b - b' = 0$, or $a = a'$ and $b = b'$.

**2.a Proof**. Let any three even numbers be $2x, 2y$ and $2z$, where $x, y, z \in \mathbb{Z}$. Their sum is $2x + 2y + 2z = 2(x + y + z) = 2w$ where $w = x + y + z$, and since $x, y, z \in \mathbb{Z}$, $w \in \mathbb{Z}$. So, $2w$ is an even integer. Hence the sum of any three even integers is an even integer.

**b. Counterexample**. Consider 1, 3, and 5, which are three odd integers. Their sum is 9 which is an odd integer. Hence, the statement is false.

**c. Counterexample**. Consider 2 and 3 which are two primes. Their sum is 5 which is prime. Hence, the statement is false.

**d. Counterexample.** Consider 1, 2, 3 and 4, which are four consecutive integers. Their sum is 10 which is not divisible by 4. Hence, the statement is false.

**e. Proof.** Take any 5 consecutive integers. They are of the form $n, n +$

$1, n+2, n+3$ and $n+4$. Their sum is $5n+10 = 5(n+2)$, which is divisible by 5. Hence, the statement is true.

**3.a** $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z}\ [y^2 = 5 + x^2]$.

**b. Counterexample** Let $x = 0$. Then for the statement to be true there must exist a $y \in \mathbb{Z}$ such that $y^2 = 5$. That is, $y \in \mathbb{Z}$ is the square root of 5. But 5 does not have an integer square root. So, no such $y$ can exist for $x = 0$. So, the statement is false.

**c.** $\exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z}\ [y^2 \neq 5 + x^2]$.

**4.a** $\exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z}\ \exists z \in \mathbb{Z}\ [(x^2 > 4) \Rightarrow (y + z = x)]$.

**b. Proof:** Consider $x = 3$. Clearly $x^2 > 4$. Also, for every integer $y$, let $z = x - y = 3 - y$. So, $y + z = 3 = x$. So, for $x = 3$, for every integer $y$ there is an integer $z = x - y$, such that $y + z = x$. So the statement is true.

**c.** $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z}\ \forall z \in \mathbb{Z}\ [(x^2 > 4)\ and\ (y + z \neq x)]$

**5.a Proof.** Consider $a,\ b,\ ,c \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$. $a \equiv b\ (mod\ m) \Leftrightarrow a - b = km$ for some $k \in \mathbb{Z}$. Multiplying the equation of both sides by $c$, we get, $ac - bc = kmc$, which implies that $ac - bc = hm$ for some $h \in \mathbb{Z}$. This implies that $a \equiv b\ (mod\ m)$.

**b. Counterexample** Let $a = 3, b = 4, c = 2\ and\ m = 2$. $ac = 12,\ bc = 8$. Clearly $ac \equiv bc\ (mod\ m)$. However, $a \not\equiv b\ (mod\ m)$.

**6.a** We have that $3^3 \equiv 9\ (mod\ 10)$. Squaring both sides, we get $3^4 \equiv 81\ (mod\ 10)$. That is, $3^4 \equiv 1\ (mod\ 10)$. Raising both sides to the power of 20, we have $3^{80} \equiv 1\ (mod\ 10)$. Muliplying by $3^3$ on both sides (see 5.a on why we can do that), we get $3^{83} \equiv 27\ (mod\ 10)$. Simplifying, we get $3^{83} \equiv 7\ (mod\ 10)$. That is, the remainder when $3^{83}$ is divided by 10 is 7, which is exaclty the units digit of $3^{83}$.

**b.** We have $3^2 \equiv 2\ (mod\ 7)$. Cubing both sides, we get $3^6 \equiv 8\ (mod\ 7)$. That is, $3^6 \equiv 1\ (mod\ 7)$. Raising both sides to the power of 13, we get $3^{78} \equiv 1\ (mod\ 7)$. Multiplying both sides by $3^5$, we get $3^{83} \equiv 243\ (mod\ 7)$. Simplifying, we get $3^{83} \equiv 5\ (mod\ 7)$, which shows that 5 is the remainder when $3^{83}$ is divided by 7.

**7. Proof.** We have that $9 \equiv 1 \ (mod\ 4)$. That means that $9^n \equiv 1 \ (mod\ 4)$ for any integer $n > 0$. That is, for every $n > 0$, $9^n = 4k + 1$ for some $k \in \mathbb{Z}$, which means for every $n > 0$, $9^n + 3 = 4k + 4 = 4(k + 1)$, for some $k \in \mathbb{Z}$. This shows that $9^n + 3$ is divisible for all positive integers $n$.

**8.a** The truth table is given below.
From this table we can see that the truth values for $p \to q$ and $((\neg p) \vee q)$

| $p$ | $q$ | $p \to q$ | $((\neg p) \vee q)$ |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 |

agree for all values of $p$ and $q$. So, we have $p \to q \Leftrightarrow ((\neg p) \vee q)$.

**b.** The truth table is given below.

| $p$ | $q$ | $r$ | $q \to r$ | $p \to q$ | $p \to r$ | $[p \to (q \to r)]$ | $[(p \to q) \to (p \to r)]$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

In the truth table the values of $[p \to (q \to r)]$ and $[(p \to q) \to (p \to r)]$ agree for all values of $p$ and $q$ so we have that $[p \to (q \to r)] \Leftrightarrow [(p \to q) \to (p \to r)]$

**c.** The truth table is given below.

| $p$ | $q$ | $r$ | $p \leftrightarrow q$ | $q \leftrightarrow r$ | $[(p \leftrightarrow q) \leftrightarrow r]$ | $[p \leftrightarrow (q \leftrightarrow r)]$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 0 | 1 | 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

In the truth table the values of $[(p \leftrightarrow q) \leftrightarrow r]$ and $[p \leftrightarrow (q \leftrightarrow r)]$ agree for all values of $p$ and $q$. So we have $[(p \leftrightarrow q) \leftrightarrow r] \Leftrightarrow [p \leftrightarrow (q \leftrightarrow r)]$.

**9.a** The truth table is given below.

| $p$ | $q$ | $r$ | $p \to q$ | $q \to r$ | $[(p \to q) \to r]$ | $[p \to (q \to r)]$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 0 | 1 |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

In the truth table the values of $[(p \to q) \to r]$ and $[p \to (q \to r)]$ disagree in the first and third rows, so the statement $[(p \to q) \to r] \Leftrightarrow [p \to (q \to r)]$ is false.

**b.** Let $p$ be the statement "$x$ is an even positve integer", let $q$ be the statement "$x$ is an integer" and let $r$ be the statement that "$x$ is greater than 1". Consider the statements $[(p \to q) \to r]$ and $[p \to (q \to r)]$ for

$x = 1$. $p$ is false, $q$ is true and $r$ is false. So, $p \to q$ is true and $q \to r$ is true. We get that $[(p \to q) \to r]$ is false and $[p \to (q \to r)]$ is true. So, for the value of $x = 1$, the two statements are inconsistent.

**10.a** Four more examples of twin primes are (11, 13), (17, 19), (41, 43) and (59, 61).

**b.** Unfortunately, this statement cannot be proved. 3, 5 and 7 are a set of three consecutive odd primes. However, this is the only set of triple primes. It can be shown by contradiction as follows. Suppose that $m$, $m+2$ and $m+3$ are three consecutive odd prime odd numbers and $m \neq 3$. Since $m$ is prime then it cannot be divided by 3. So, either $m = 3i + 1$ for some $i \in \mathbb{Z}^+$ or $3j + 2$ for some $j \in \mathbb{Z}^+$. In the first case, $m + 2 = 3i + 3$ which makes $m+2$ divisible by 3, which is a contradiction to the assumption that $m+2$ is prime, and in the second case $m+1 = 3j+3$ which makes $m+1$ divisible by 3, which is a contradiction to the assumption that $m+1$ is a prime. Hence, apart from 3, 5, 7 there are no other triple primes.

**11.a** $\exists c \in \mathbb{R}\ \exists N \in \mathbb{R}\ \forall n \in \mathbb{Z}^+\ [n \geq N \to f(n) \leq c \cdot g(n)]$.

**b.** $\forall c \in \mathbb{R}\ \forall N \in \mathbb{R}\ \exists n \in \mathbb{Z}^+\ [n \geq N \not\to f(n) \leq c \cdot g(n)]$. In english, it can be stated as: Forall real numbers $c$ and $N$, there exists a positive integer $n$ such that $n \geq N$ does not imply $f(n) \leq c \cdot g(n)$.