1. Let H and K be finite subgroups of a group G.

   (a) Recall that $HK = \{hk : h \in H, k \in K\}$, and show

   $$|HK| = \frac{|H||K|}{|H \cap K|}.$$

   (b) For $x \in G$, the set $HxK = \{hxk : h \in H, k \in K\}$ is called a *double coset* of H and K. Show that G is a disjoint union of double cosets and

   $$|HxK| = \frac{|H||K|}{|H \cap xKx^{-1}|}.$$

   (c) If all double cosets of the form HxH for $x \in G$ have the same number of elements, show that $H \lhd G$.

2. Let G be a p-group where p be a prime integer.

   (a) If $|G| = p^2$, show that G is abelian.

   (b) If $|G| = p^3$, show that either G is abelian or $|Z(G)| = p$, where $Z(G)$ is the center of G.

3. Let p be a prime integer and G be a p-group. If $H \lhd G$ and $|H| = p$, prove that H is contained in the center of G.

4. Let G be an infinite group and H a subgroup of finite index. Show that G has a normal subgroup K of finite index, with $K < H$.

5. Let G be an infinite group containing an element $x \neq e$ having only finitely many conjugates. Prove that G is not simple.

6. Let G be a finite group such that $\mathrm{Aut}(G)$ acts transitively on the set $G \setminus \{e\}$. Show that G is a p-group for some prime p, and that G is abelian.

7. Let G be a group with $|G| = mp^n$ where p is a prime and $m < p$. Show that G has exactly one p-Sylow subgroup, and that this subgroup is normal.

8. Let G be a finite group of odd order which acts transitively on a set S. For $s \in S$, show that the orbits of the action of G_s on $S \setminus \{s\}$ have lengths which are equal in pairs.

9. Let G be a finite group and p a prime number. An element $g \in G$ is called *p-unipotent* if its order is a power of p, and *p-regular* if its order is not divisible by p.

   (a) Let $x \in G$. Show that there exists a unique ordered pair (u, r) of elements of G such that u is p-unipotent, r is p-regular, and $x = ur = ru$.

   (Hint: First consider the case where G is the cyclic group generated by x.)

   (b) Let P be a p-Sylow subgroup of G, C the centralizer of P, and E the set of p-regular elements of G. Show that

   $$|E| \equiv |E \cap C| \mod p.$$

   (c) Deduce that p does not divide the order of E.

   (Hint: Use induction on the cardinality of G to reduce to the case where $C = G$; then use (a)).

10. Let G be a finite group, P a Sylow subgroup of G, and N the normalizer of P. Let X_1 and X_2 be subsets of the center of P which are conjugate, i.e., $sX_1s^{-1} = X_2$ for some element $s \in G$.

    (a) Show that there exists $n \in N$ such that $nxn^{-1} = sxs^{-1}$ for all $x \in X_1$.

    (b) Deduce that two elements of the center of P are conjugate in G if and only if they are conjugates in N.