1. Let G be a finite group with an automorphism φ such that $\varphi(x) = x$ if and only if $x = e$.
   (a) Show that every element of G can be written as $x^{-1}\varphi(x)$.
   (b) Suppose φ has order two, i.e., $\varphi^2(x) = x$ for all $x \in G$. Prove that $\varphi(x) = x^{-1}$ for all $x \in G$, and conclude that G is abelian.

2. Let G be a finite group with an automorphism φ such that $\varphi(x) = x$ if and only if $x = e$. If p is a prime dividing $|G|$, prove that G has a unique p-Sylow subgroup P satisfying $\varphi(P) = P$.

3. Find a group G with two elements $a, b \in G$ such that a and b have finite order, but ab does not have finite order.

   (Hint: Try $G = GL_2(\mathbb{Z})$, the group of invertible 2×2 matrices over $\mathbb{Z}$.)

4. Let $G = SL_2(\mathbb{R})$, i.e., G is the group of 2×2 matrices over the real numbers which have determinant 1. Prove that $G^{(1)}$ contains every matrix of the form
$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \text{ where } a \in \mathbb{R}.$$

5. Let $p < q$ be prime numbers such that p divides $q-1$. Show that there exists a non-abelian group of order pq.

6. Let p, q be distinct prime numbers. Prove that a group of order p^2q is solvable.

7. Let G be a finite group, $K \lhd G$ a normal subgroup, and P a p-Sylow subgroup of K. Prove that $G = KN_P$, where N_P is the normalizer of P in G.

8. Let $n \geq 3$. Prove that for $x \in S_n$, the element x^3 can never be a 3-cycle.

9. Let $|G| = p^k m$ where p is a prime number. Let S be the set of p^k-element subsets of G, and so
$$|S| = \binom{p^k m}{p^k}, \quad \text{and therefore} \quad \frac{|S|}{m} = \binom{p^k m - 1}{p^k - 1}.$$
   (a) Show that $(1/m)|S| \equiv 1 \mod p$.
   (b) Let G act on S by left translation. If $A \in S$, prove that the order of the isotropy group G_A divides p^k.
   (c) Let $S_0 = \{A \in S : |G_A| = p^k\}$, and show that
$$|S| \equiv |S_0| \mod pm.$$
   (Hint: Note that $S \setminus S_0$ is a disjoint union of orbits.)
   (d) Prove that $S_0 = \{Hx : H \text{ is a subgroup of } G \text{ with } |H| = p^k, \text{ and } x \in G\}$.
   (e) Conclude that the number of subgroups of G of order p^k is $1 \mod p$. (This extends the Sylow theorems, since we did not assume that m is relatively prime to p.)