Throughout, A is a commutative ring with $0 \neq 1$.

1. We say that $a \in A$ is a *zerodivisor* if there exists $b \neq 0$ in A such that $ab = 0$. (This differs from Lang's definition only to the extent that 0 will be called a zerodivisor.)

   Let $\mathcal{F}$ be the set of all ideals of A in which every element is a zerodivisor.

   (a) Prove that $\mathcal{F}$ has maximal elements.

   (b) Prove that every maximal element of $\mathcal{F}$ is a prime ideal.

   (c) Conclude that the set of zerodivisors is a union of prime ideals.

2. Let a be a nilpotent element of A. Prove that $1 + a$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

3. Let $A[x]$ be the ring of polynomials in an indeterminate x with coefficients in a ring A. Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$ where $a_i \in A$.

   (a) Prove that f is a unit in $A[x]$ if and only if a_0 is a unit in A and $a_1, \ldots, a_n$ are nilpotent.
   (Hint: If $b_0 + b_1x + \cdots + b_mx^m$ is the inverse of f, prove by induction on r that $a_n^{r+1}b_{m-r} = 0$.)

   (b) Prove that f is nilpotent if and only if $a_0, a_1, \ldots, a_n$ are nilpotent.

4. Let A be a ring and $\mathfrak{N}$ its nilradical. Prove that the following are equivalent:

   (a) A has exactly one prime ideal;

   (b) every element of A is either a unit or is nilpotent;

   (c) $A/\mathfrak{N}$ is a field.

5. An element $e \in A$ is an *idempotent* if $e^2 = e$. Prove that the only idempotent elements in a local ring are 0 and 1.

6. Let $\mathfrak{N}$ be the nilradical of a ring A. If $a + \mathfrak{N}$ is an idempotent element of $A/\mathfrak{N}$, prove that there exits a unique idempotent $e \in A$ with $e - a \in \mathfrak{N}$.

7. Let A be a ring and let $X = \operatorname{Spec} A$. Recall that for each subset E of A, we defined $V(E) = \{\mathfrak{p} \in X \mid E \subseteq \mathfrak{p}\}$, and that these are precisely the closed sets for the Zariski topology on X. For each $f \in A$, let X_f be the complement of $V(f)$ in X. The sets X_f are open in the Zariski topology. Prove that

   (a) the sets X_f form a basis of open sets, i.e., every open set in X is a union of sets of the form X_f;

   (b) $X_f \cap X_g = X_{fg}$;

   (c) $X_f = \emptyset$ if and only if f is nilpotent;

   (d) $X_f = X$ if and only if f is a unit;

   (e) $X_f = X_g$ if and only if $\text{radical}(f) = \text{radical}(g)$;

8. Prove that $X = \operatorname{Spec} A$ is a compact topological space, i.e., every open cover of X has a finite subcover.

   (Hint: It is enough to consider a cover of X by basic open sets X_{f_i}.)

9. Let A_1, A_2 be rings, and $A_1 \times A_2$ their direct product. What are the prime ideals of the ring $A_1 \times A_2$?

10. Let A be a ring. Prove that the following are equivalent:

    (a) $X = \operatorname{Spec} A$ is disconnected, i.e., X is the disjoint union of two open sets;

    (b) $A \approx A_1 \times A_2$ for rings A_1, A_2, neither of which is the zero ring;

    (c) A contains an idempotent other than 0 and 1.