1. Let G be a finite group and, for each prime p, choose a p-Sylow subgroup of G. Prove that G is generated by these subgroups (that is every element of G is expressible as a product of some elements of these subgroups.)

2. If p and q are primes, prove that a group of order p^2q cannot be simple.

3. Let G be a finite group with an automorphism φ such that $\varphi(x) = x$ if and only if $x = e$.

   (a) Show that every element of G can be written as $x^{-1}\varphi(x)$.

   (b) Suppose φ has order two, i.e., $\varphi^2(x) = x$ for all $x \in G$. Prove that $\varphi(x) = x^{-1}$ for all $x \in G$, and conclude that G is abelian.

4. Let $p < q$ be prime numbers such that p divides $q - 1$. Show that there exists a non-abelian group of order pq.

5. Let p, q be distinct prime numbers. Prove that a group of order p^2q is solvable.

6. Let G be a finite group, $K \lhd G$ a normal subgroup, and P a p-Sylow subgroup of K. Prove that $G = KN_P$, where N_P is the normalizer of P in G.

7. Let $|G| = p^k m$ where p is a prime number. Let S be the set of p^k-element subsets of G, and so
$$|S| = \binom{p^k m}{p^k}, \quad \text{and therefore} \quad \frac{|S|}{m} = \binom{p^k m - 1}{p^k - 1}.$$

   (a) Show that $(1/m)|S| \equiv 1 \mod p$.

   (b) Let G act on S by left translation. If $A \in S$, prove that the order of the isotropy group G_A divides p^k.

   (c) Let $S_0 = \{A \in S : |G_A| = p^k\}$, and show that
   $$|S| \equiv |S_0| \mod pm.$$
   (Hint: Note that $S \setminus S_0$ is a disjoint union of orbits.)

   (d) Prove that $S_0 = \{Hx : H \text{ is a subgroup of } G \text{ with } |H| = p^k, \text{ and } x \in G\}$.

   (e) Conclude that the number of subgroups of G of order p^k is $1 \mod p$. (This extends the Sylow theorems, since we did not assume that m is relatively prime to p.)

8. Let K be an abelian group of order m and let Q be an abelian group of order n. If $(m, n) = 1$, then every extension G of K by Q is a semi-direct product.

9. (**extra credit) Previous question without the assumption that K and Q are abelian.