NAME: B1=Renom B2=Labate PART A

No calculators, books, or notes. Parts A, B, and C will be graded separately.

1. 4 pts each/ total **12 pts**.

a) Prove the combinatorial identity $\binom{r+1}{j+1} = \binom{r}{j+1} + \binom{r}{j}$ *by using a counting argument* (no credit for an algebraic proof. write clear sentences).

a) Suppose we want to choose j+1 items from r+1. Label one item as special; either it will be chosen, or it will not be chosen. If the special item is not chosen, there are $\binom{r}{j+1}$ ways to choose j+1 items from the remaining r items. If the special item is chosen, there are $\binom{r}{j}$ ways to choose the remaining j items from r items.

remark: This is Theorem 2 on page 311, in section 4.5 on Combinatorial Identities. We did several homework problems of this nature on page 314.

b) Find an antidifference for $a_i = \binom{i}{k}$ (i=k,k+1,...).

b) Let $A_i = \binom{i}{k+1}$ for i=k+1,k+2,... (and $A_k=0$). Then,

$$\Delta A_i = A_{i+1}-A_i = \binom{i+1}{k+1} - \binom{i}{k+1} = \binom{i}{k} = a_i$$

and therefore $\Delta^{-1} a_i = A_i$.

remark: There is little to do here since part a directly tells you the answer to this question, provided you know what an antidifference means. There is no need to have memorized any formulas since the answer has essentially been handed to you.

c) Evaluate $\Sigma_{i=k}^{j} \binom{i}{k}$ *by using antidifferences*.

c) $\Sigma_{i=k}^{j} \binom{i}{k} = \binom{i}{k+1} \Big|_{k}^{j+1} = \binom{j+1}{k+1} - \binom{k}{k+1} = \binom{j+1}{k+1}$

remark: To get credit here, you needed to have part b correct.

#1) #3)

#2) #4)

2. 4 pts each / total **12 pts**

a) What is the coefficient of x^4y^3 in the expansion of $(2x-3y)^7$? (3 pts for correct expression, 1pt for correct simplification to a number).

answer: $\binom{7}{4} 2^4(-3)^3 = \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} 2^4(-3)^3 = -15120$

remark: This is the binomial theorem (page 316). We did several exercises based on it as homework on page 323 (section 4.6). The binomial theorem also reappeared later in the course in our work on antidifferences and differences. For example, we used it to evaluate the difference of x^n.

b) Compute the matrix product $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} (1\ 3\ -2) =$

$$\begin{pmatrix} 2 & 6 & -4 \\ 1 & 3 & -2 \\ -1 & -3 & 2 \end{pmatrix}$$

remark: the product of a 3×1 matrix with a 1×3 matrix is a 3×3 matrix.

c) Find a 3×3 matrix B such that $\begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 0 \\ 5 & 6 & 1 \end{pmatrix} B = \begin{pmatrix} -2 & 4 & 1 \\ -1 & 1 & 1 \\ -6 & 6 & 5 \end{pmatrix}$.

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

remark:

The first column of the answer is (-2,-1,-6), which is exactly -1 times the second column of the first matrix.

The second column of the answer is (4,1,6), which is exactly the sum of the first and third columns of the first matrix.

The third column of the answer is (1,1,5), which is exactly the second column minus the third column of the first matrix.

These three observations tell you the three columns of B.

3. **6 pts.** Find the steady state for the Markov chain whose transition matrix is $A = \begin{pmatrix} 0 & 2/3 & 1 \\ 1 & 0 & 0 \\ 0 & 1/3 & 0 \end{pmatrix}$. Show your work, including all row reduction steps.

solution: We want to find the eigenspace for the eigenvalue $\lambda=1$, which is the same as the null space of A-I. The matrix A-I reduces to $\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$. The null space of A-I is {(x,y,z,w) : x=3z, y=3z, z=any number}. The only member of the null space that is a probability vector is (3/7,3/7,1/7).

remark: problems like this were covered in the homework for April 12.

4. **6 pts.** Find a 2×2 matrix A that has eigenvalues 2 and 3 such that the eigenvector corresponding to $\lambda=2$ is (3,11), and the eigenvector corresponding to $\lambda=3$ is (1,4). (Hint: You have enough information to write down a diagonalization of A.). Show your work.

solution: $\begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ -11 & 3 \end{pmatrix} = \begin{pmatrix} -9 & 3 \\ -44 & 14 \end{pmatrix}$

remark: I showed you how to diagonalize a matrix in class, though we did not do any homework problems based on this.