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Warning: You will not understand the material on vector space concepts unless you devote ample time to it. This material contains many new ideas and definitions, and is quite abstract. DO NOT UNDERESTIMATE IT.

homework:

go to web site: Linear Algebra : Questions - Hints - Solutions

Then click on link: "vector space concepts"

Do these exercises:

- linear subspaces of a vector space (problems 1-10)
- linear independence (problems 1-5)
- basis for a vector space (problems 1-7)
- subspaces associated with a matrix (all problems)

What is a vector space?

In this course, we only have a few days to talk about vector spaces, so I will not give the definition for lack of time. The only example of a vector space that we will discuss is R^n. Whenever you see the term "vector space" in the notes, book, or problem sets, just think of R^n.

R^n is the set of ordered n-tuples of real numbers.

R^n has two operations that are regarded as "vector space operations": addition and scalar multiplication.

Subspaces

Definition: A subspace M of a vector space V is a subset of V that is closed under addition and scalar multiplication.

This means that M is a subset of V such that

i. $\mathbf{x}+\mathbf{y} \in M$ whenever $\mathbf{x} \in M$ and $\mathbf{y} \in M$

ii. $c\mathbf{x} \in M$ whenever $\mathbf{x} \in M$ and $c \in R$

Subspace spanned by a set of vectors

If S = $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a set of vectors, then the set of all linear combinations of the vectors in S is a subspace. It is called the subspace spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_r$.

example: {(1,1),(2,2),(3,3)} is a spanning set for the line y=x in R^2, but it is not minimal. {(2,2)} is a minimal spanning set.

exammple: {(1,0,0),(0,1,0),(0,0,1)} is a minimal spanning set for R^3, but if you add any more vectors to it it will no longer be minimal.

Linear independence of a set of vectors

If $S=\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a set of vectors, and if there exist numbers $k_1, \ldots, k_r$, not all zero, such that $k_1\mathbf{v}_1+\ldots+k_r\mathbf{v}_r=\mathbf{0}$, then S is called a linearly dependent set. If S is not linearly dependent, then S is called linearly independent.

How to decide if a set of vectors $S=\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent:

Solve the equation $x_1\mathbf{v}_1+\ldots+x_r\mathbf{v}_r = 0$ for the unknowns $x_1,\ldots,x_r$ by row reduction. If there is a unique solution, then S is linearly independent. If there are infinitely many solutions, then S is linearly dependent.

Basis for a subspace

Definition: A set S of vectors in a vector space V is a basis for V if

i. S spans V
ii. S is linearly independent

Theorem: Let S be a subset of a vector space V. There is a number n, called the dimension of V, such that

i. every basis of V has exactly n vectors
ii. every independent set of n vectors is a basis
iii. every spanning set of n vectors is a basis
iv. no set of fewer than n vectors can span V
v. no set of more than n vectors in V can be linearly independent.

subspaces associated with a matrix

Let A be an arbitrary $n \times k$ matrix.

The row space of A is the subspace of R^k spanned by the rows of A.

The column space of A is the subspace of R^n spanned by the columns of A.

The null space of A is the set of all x such that Ax=0. It is a subspace of R^k.

Theorem: For any matrix A, the dimension of the row space is equal to the dimension of the column space.

Definition: The rank of a matrix is the dimension of its row (or column) space.

Definition: The nullity of a matrix is the dimension of its null space.

Theorem: For any matrix A, rank(A) + nullity(A) = number of columns of A.

rank + nullity = number of columns

We reviewed this identity and verified it for several matrices:

$\begin{pmatrix}1\\2\\3\end{pmatrix}$: rank = 1, rowspace = R, column space = {t(1,2,3) : $t \in R$} (which is a line through the origin in R^3), null space = {0}, nullity=0, basis for row space = {1}, basis for column space = {(1,2,3)}, basis for null space = $\varnothing$.

$\begin{pmatrix}1&2\\2&4\\3&6\end{pmatrix}$: rank = 1, row space = {t(1,2):$t \in R$} (which is a line through the origin in R^2, column space = {t(1,2,3):$t \in R$} (a line through the origin in R^3), null space = {t(-2,1):$t \in R$} (a line through the origin in R^2), nullity=1, basis for row space = {(1,2)}, basis for column space = {(1,2,3)}, basis for null space = {(-2,1)}.

what can you learn by reducing a matrix A ?

If D is the reduced form of A, then

1. the nonzero rows of D are a basis for the row space of A
2. the number of nonzero rows of D is the rank of A
3. from D, you can read off the solution set of Ax=0 (the null space of A)
4. by putting the solution set of Ax=0 into "column" format, you'll obtain a basis for the null space of A.
5. If you let b denote the last column of A, so that A = (B|b) (where B denotes all but the last column of A), then you can read the solution of Bx=b from the reduced matrix D.

Example: Consider $A = \begin{pmatrix} 7 & -5 & -1 \\ -2 & 3 & 5 \\ -1 & 4 & 10 \end{pmatrix}$ which reduces to $D = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$. We have

1. the set {(1,0,2),(0,1,3)} is a basis for rowspace(A).
2. rank(A)=2.
3. The solution of the homogeneous system

$$\begin{aligned} 7x - 5y - z &= 0 \\ -2x + 3y + 5z &= 0 \\ -x + 4y + 10z &= 0 \end{aligned}$$

can be read from the reduced matrix D. The null space of A is $\{(x,y,z) : x=-2z,\ y=-3z,\ z \in R\}$.

4. The solution of the homogeneous system can be put into column format as follows:

$$\text{nullspace}(A) = \left\{ z\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} : z \in R \right\}$$

which tells us that {(-2,-3,1)} is a basis for the null space.

5. The solution of the nonhomogeneous system

$$\begin{aligned} 7x - 5y &= -1 \\ -2x + 3y &= 5 \\ -x + 4y &= 10 \end{aligned}$$

can also be read from the reduced matrix D. The solution is

x=2, y=3.

Note that the solution set is **not** a subspace. Although the solution set of a homogeneous system is always a subspace, this is not true for nonhomogeneous systems.