An example with eight basic feasible solutions

Consider the LP:

min or max

$x_1 + 2x_2 + 3x_3$

subject to

$2 \le x_1 + x_2 \le 3$

$4 \le x_1 + x_3 \le 5$

$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0$

In standard form, we write this as

$$\begin{aligned} \text{minimize or maximize} \quad & x_1 + 2x_2 + 3x_3 = \text{objective} \\ \text{subject to} \quad & x_1 + x_2 + x_4 = 3 \\ & x_1 + x_2 - x_5 = 2 \\ & x_1 + x_3 + x_6 = 5 \\ & x_1 + x_3 - x_7 = 4 \\ & x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0,\ x_4 \ge 0,\ x_5 \ge 0,\ x_6 \ge 0,\ x_7 \ge 0 \end{aligned}$$

There are $\binom{7}{4} = 35$ ways to choose four columns from the 4×7 coefficient matrix. Each fits into one of 3 categories:

i. The 4 columns do form a basis and the corresponding basic solution is feasible (all variables are nonnegative).

ii. The 4 columns do form a basis (the 4×4 matrix is invertible) but the corresponding basic solution is infeasible (one variable is negative).

iii. The corresponding 4 columns of the coefficient matrix form a singular (not invertible) 4×4 matrix. In other words, these columns do not form a basis.

Below are all 35 possibilities. For basic solutions, we write the column numbers of the basis, the value of the objective, the solution vector, the equivalent system of equations that displays the solution vector. For example, the first basic feasible solution uses columns 1,3,4,6, the corresponding system of equations is

$$\begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

and the solution obtained by setting nonbasic variables equal to 0 is (2,0,2,1,0,1,0), where the value of the objective function is $x_1+2x_2+3x_3=8$.

category i: basic feasible solutions

columns: {1, 3, 4, 6} objective = 8 x = (2,0,2,1,0,1,0) vertex A
{{1, 0, 0, 0}, {1, -1, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0},
{-1, 1, 1, 0}, {0, 0, 0, 1}, {0, -1, 0, 1}, {2, 2, 1, 1}}

columns: {1, 3, 4, 7} objective = 11 x = (2,0,3,1,0,0,1)
{{1, 0, 0, 0}, {1, -1, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0},
{-1, 1, 1, 0}, {0, 1, 0, 1}, {0, 0, 0, 1}, {2, 3, 1, 1}}

columns: {1, 3, 5, 6} objective = 6 *MIN* x = (3,0,1,0,1,1,0)
{{1, 0, 0, 0}, {1, -1, 0, 0}, {0, 1, 0, 0}, {1, -1, 1, 0},
{0, 0, 1, 0}, {0, 0, 0, 1}, {0, -1, 0, 1}, {3, 1, 1, 1}}

columns: {1, 3, 5, 7} objective = 9 x = (3,0,2,0,1,0,1)
{{1, 0, 0, 0}, {1, -1, 0, 0}, {0, 1, 0, 0}, {1, -1, 1, 0},
{0, 0, 1, 0}, {0, 1, 0, 1}, {0, 0, 0, 1}, {3, 2, 1, 1}}

columns: {2, 3, 4, 6} objective = 16 x = (0,2,4,1,0,1,0) vertex B
{{1, 1, 0, 0}, {1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0},
{-1, 0, 1, 0}, {0, 0, 0, 1}, {0, -1, 0, 1}, {2, 4, 1, 1}}

columns: {2, 3, 4, 7} objective = 19 x = (0,2,5,1,0,0,1) vertex C
{{1, 1, 0, 0}, {1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0},

{-1, 0, 1, 0}, {0, 1, 0, 1}, {0, 0, 0, 1}, {2, 5, 1, 1}}

columns: {2, 3, 5, 6} objective = 18 x = (0,3,4,0,1,1,0)
{{1, 1, 0, 0}, {1, 0, 0, 0}, {0, 1, 0, 0}, {1, 0, 1, 0},
 {0, 0, 1, 0}, {0, 0, 0, 1}, {0, -1, 0, 1}, {3, 4, 1, 1}}

columns: {2, 3, 5, 7} objective = 21 *MAX* x = (0,3,5,0,1,0,1) vertex D
{{1, 1, 0, 0}, {1, 0, 0, 0}, {0, 1, 0, 0}, {1, 0, 1, 0},
 {0, 0, 1, 0}, {0, 1, 0, 1}, {0, 0, 0, 1}, {3, 5, 1, 1}}

category ii: basic infeasible solutions

columns: {1, 2, 4, 6} objective = 0 x = (4,-2,0,1,0,1,0)
{{1, 0, 0, 0}, {0, 1, 0, 0}, {1, -1, 0, 0}, {0, 0, 1, 0},
 {0, -1, 1, 0}, {0, 0, 0, 1}, {-1, 1, 0, 1}, {4, -2, 1, 1}}

columns: {1, 2, 4, 7} objective = -1 x = (5,-3,0,1,0,0,1)
{{1, 0, 0, 0}, {0, 1, 0, 0}, {1, -1, 0, 0}, {0, 0, 1, 0},
 {0, -1, 1, 0}, {1, -1, 0, 1}, {0, 0, 0, 1}, {5, -3, 1, 1}}

columns: {1, 2, 5, 6} objective = 2 x = (4,-1,0,0,1,1,0)
{{1, 0, 0, 0}, {0, 1, 0, 0}, {1, -1, 0, 0}, {0, 1, 1, 0},
 {0, 0, 1, 0}, {0, 0, 0, 1}, {-1, 1, 0, 1}, {4, -1, 1, 1}}

columns: {1, 2, 5, 7} objective = 1 x = (5,-2,0,0,1,0,1)
{{1, 0, 0, 0}, {0, 1, 0, 0}, {1, -1, 0, 0}, {0, 1, 1, 0},
 {0, 0, 1, 0}, {1, -1, 0, 1}, {0, 0, 0, 1}, {5, -2, 1, 1}}

columns: {1, 4, 5, 6} objective = 4 x = (4,0,0,-1,2,1,0)
{{1, 0, 0, 0}, {0, 1, -1, 0}, {1, -1, 1, 0}, {0, 1, 0, 0},
 {0, 0, 1, 0}, {0, 0, 0, 1}, {-1, 1, -1, 1}, {4, -1, 2, 1}}

columns: {1, 4, 5, 7} objective = 5 x = (5,0,0,-2,3,0,1)
{{1, 0, 0, 0}, {0, 1, -1, 0}, {1, -1, 1, 0}, {0, 1, 0, 0},
 {0, 0, 1, 0}, {1, -1, 1, 1}, {0, 0, 0, 1}, {5, -2, 3, 1}}

columns: {1, 4, 6, 7} objective = 2 x = (2,0,0,1,0,3,-2)
{{1, 0, 0, 0}, {1, 0, -1, 1}, {0, 0, 1, -1}, {0, 1, 0, 0},
 {-1, 1, 1, -1}, {0, 0, 1, 0}, {0, 0, 0, 1}, {2, 1, 3, -2}}

columns: {1, 5, 6, 7} objective = 3 x = (3,0,0,0,1,2,-1)
{{1, 0, 0, 0}, {1, 0, -1, 1}, {0, 0, 1, -1}, {1, 1, -1, 1},
 {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}, {3, 1, 2, -1}}

columns: {2, 4, 6, 7} objective = 4 x = (0,2,0,1,0,5,-4)
{{1, 0, 1, -1}, {1, 0, 0, 0}, {0, 0, 1, -1}, {0, 1, 0, 0},
 {-1, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}, {2, 1, 5, -4}}

columns: {2, 5, 6, 7} objective = 6 x = (0,3,0,0,1,5,-4)
{{1, 0, 1, -1}, {1, 0, 0, 0}, {0, 0, 1, -1}, {1, 1, 0, 0},
 {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}, {3, 1, 5, -4}}

columns: {3, 4, 5, 6} objective = 12 x = (0,0,4,3,-2,1,0)
{{1, 1, -1, 0}, {0, 1, -1, 0}, {1, 0, 0, 0}, {0, 1, 0, 0},
 {0, 0, 1, 0}, {0, 0, 0, 1}, {-1, 0, 0, 1}, {4, 3, -2, 1}}

columns: {3, 4, 5, 7} objective = 15 x = (0,0,5,3,-2,0,1)
{{1, 1, -1, 0}, {0, 1, -1, 0}, {1, 0, 0, 0}, {0, 1, 0, 0},
 {0, 0, 1, 0}, {1, 0, 0, 1}, {0, 0, 0, 1}, {5, 3, -2, 1}}

columns: {4, 5, 6, 7} objective = 0 x = (0,0,0,3,-2,5,-4)
{{1, -1, 1, -1}, {1, -1, 0, 0}, {0, 0, 1, -1}, {1, 0, 0, 0},
 {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}, {3, -2, 5, -4}}

category iii: not basic

{1, 2, 3, 4}, {1, 2, 3, 5}, {1, 2, 3, 6}, {1, 2, 3, 7}, {1, 2, 4, 5}, {1, 2,
6, 7}, {1, 3, 4, 5}, {1, 3, 6, 7}, {2, 3, 4, 5}, {2, 3, 6, 7}, {2, 4, 5, 6},

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{2, 4, 5, 7}, {3, 4, 6, 7}, {3, 5, 6, 7}
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