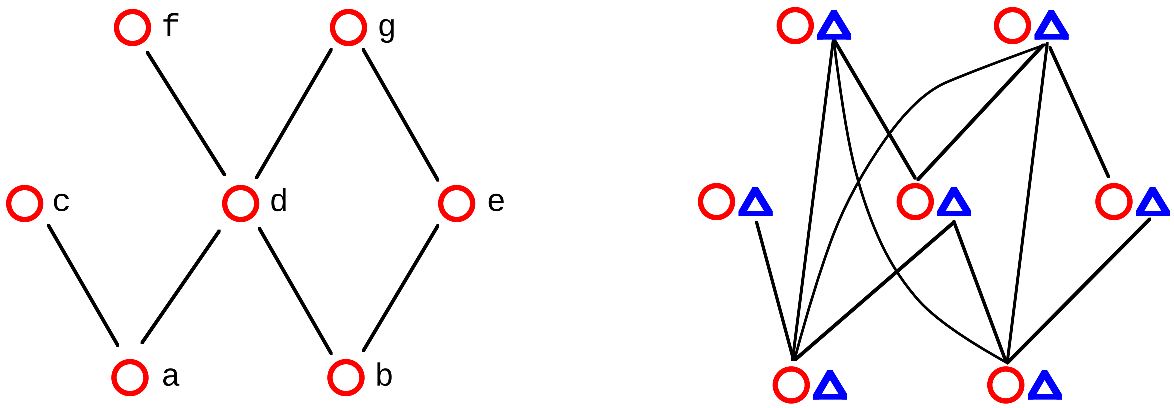
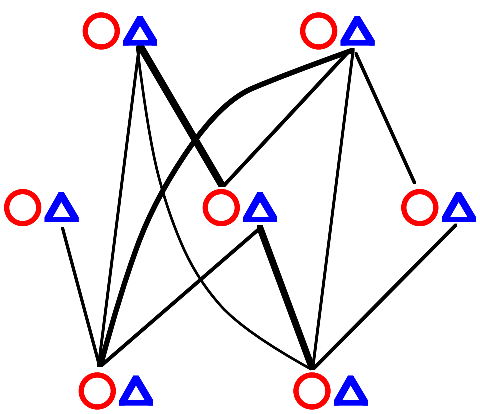
Dilworth's Theorem

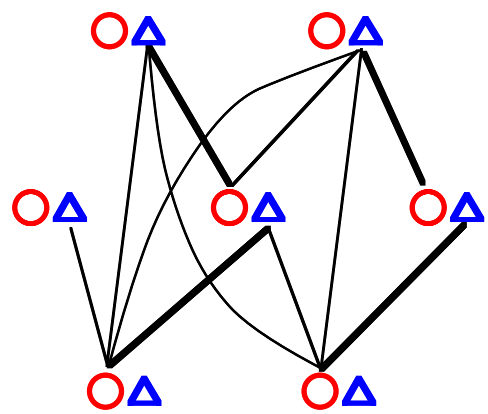


A partially ordered set P of size n=7 and its corresponding bipartite graph

Each matching M for the bipartite graph yields a chain decomposition D of size |D| = n-|M| for P. Here are two examples:



This matching of size 3 corresponds to a chain decomposition into 4 sets:
D = {{a,g},{b,d,f},{c},{e}}



This matching of size 4 corresponds to a chain decomposition into 3 sets:
D = {{c},{a,d,f},{b,e,g}}

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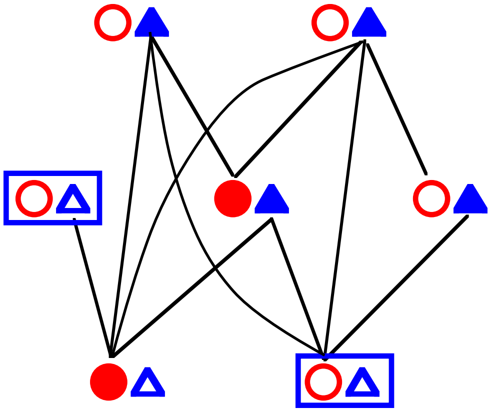
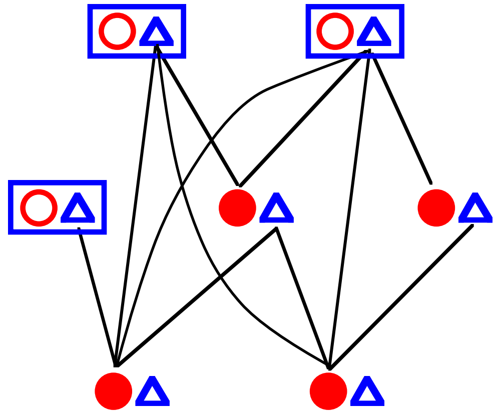
Each cover C for the bipartite graph yields an antichain A for P such that n-|A|≤|C|. (A consists of all elements of P such that neither of the two corresponding bipartite nodes belongs to C). Here are two examples:



|C| = 6, |A| = 2.



|C| = 4, |A| = 3.

Lemma: If A is an antichain and D is a chain decomposition in a bipartite graph, then |A| ≤ |D|.

Proof of Dilworth's Theorem: By the König-Egerváry theorem, the bipartite graph has a cover C and a matching M such that |C| = |M|. Let A and D be the associated antichain and chain decomposition. Then

$$|A| \geq n - |C| = n - |M| = |D| \geq |A|$$

so that |A| = |D|.