

The Final Tableau Solves the Dual

Consider the dual linear programming problems:

<p>(P)</p> $\max 9x_1 + 7x_2 + 1x_3$ <p>subject to $x_1, x_2, x_3 \geq 0$ and</p> $5x_1 + 1x_2 + 2x_3 \leq 13$ $2x_1 + 3x_2 + 1x_3 \leq 11$ $1x_1 + 1x_2 + 4x_3 \leq 15$	<p>(D)</p> $\min 13y_1 + 11y_2 + 15y_3$ <p>subject to $y_1, y_2, y_3 \geq 0$ and</p> $5y_1 + 2y_2 + 1y_3 \geq 9$ $1y_1 + 3y_2 + 1y_3 \geq 7$ $2y_1 + 1y_2 + 4y_3 \geq 1$
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The simplex algorithm gives the following sequence of tableaus:

<p>(4 5 6)</p> $\begin{array}{ccccccc} 5 & 1 & 2 & 1 & 0 & 0 & 13 \\ 2 & 3 & 1 & 0 & 1 & 0 & 11 \\ 1 & 1 & 4 & 0 & 0 & 1 & 15 \\ 9 & 7 & 1 & 0 & 0 & 0 & 0 \end{array}$	→	<p>(1,5,6)</p> $\begin{array}{ccccccc} 1 & .2 & .4 & .2 & 0 & 0 & 2.6 \\ 0 & 2.6 & .2 & -.4 & 1 & 0 & 5.8 \\ 0 & .8 & 3.6 & -.2 & 0 & 1 & 12.4 \\ 0 & 5.2 & -2.6 & -1.8 & 0 & 0 & -23.4 \end{array}$	→
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(1,2,6)

$$\begin{array}{ccccccc} 1 & 0 & 5/13 & 3/13 & -1/13 & 0 & 28/13 \\ 0 & 1 & 1/13 & -2/13 & 5/13 & 0 & 29/13 \\ 0 & 0 & 46/13 & -1/13 & -4/13 & 1 & 138/13 \\ 0 & 0 & -3 & -1 & -2 & 0 & -35 \end{array}$$

From the final tableau, we see that $x = (28/13, 29/13, 0, 0, 0, 138/13)$ is the optimal solution and the optimal value is 35.

The final row of the last (and optimal) tableau can be written as

(1)
$$-0x_1 -0x_2 - 3x_3 - \underline{1}x_4 - \underline{2}x_5 - \underline{0}x_6 -z = -35$$

(z stands for the objective function). All of the coefficients in this expression are nonpositive, since negative ones cannot exist in the final tableau. We will show next that $(y_1, y_2, y_3) = (1, 2, 0)$ is an optimal solution for (D). To demonstrate this, it suffices to show first that $(1, 2, 0)$ is dual feasible

(2)
$$\begin{aligned} 5(1) + 2(2) + 1(0) &\geq 9 \\ 1(1) + 3(2) + 1(0) &\geq 7 \\ 2(1) + 1(2) + 4(0) &\geq 1 \\ 1, 2, 0 &\geq 0 \end{aligned}$$

and then that the value of the dual objective function $13y_1 + 11y_2 + 15y_3$ at $(y_1, y_2, y_3) = (1, 2, 0)$ is equal to the optimal value for (P):

(3)
$$13(1) + 11(2) + 15(0) = 35$$

These statements are obviously true in *this particular case* but why will this always work? To see why, rewrite (1) as

$$\begin{aligned}9x_1 + 7x_2 + 1x_3 &= 35 - 0x_1 - 0x_2 - 3x_3 - 1x_4 - 2x_5 - 0x_6 \\ &= 35 - \underline{0}x_1 - \underline{0}x_2 - \underline{3}x_3 - \underline{1}(13-5x_1-1x_2-2x_3) \\ &\quad - \underline{2}(11-2x_1-3x_2-1x_3) \\ &\quad - \underline{0}(15-1x_1-1x_2-4x_3)\end{aligned}$$

Equating the constant term on the left hand side of the equation with the constant term on the right hand side gives $1(13)+2(11)+0(15) = 35$, which is equation (3). Equating the coefficients of x_1 , x_2 , and x_3 on the right and left sides, we get the following equations:

$$\begin{aligned}1(5)+2(2)+0(1) &= 9 + \underline{0} \geq 9 \\ 1(1)+2(3)+0(1) &= 7 + \underline{0} \geq 7 \\ 1(2)+2(1)+0(4) &= 1 + \underline{3} \geq 1\end{aligned}$$

which is the system of inequalities (2). The underlined numbers are nonnegative because they appear as coefficients in (1).

The Final Dictionary Solves the Dual

Consider the dual linear programming problems:

(P)
 $\max 9x_1 + 7x_2 + 1x_3$
 subject to $x_1, x_2, x_3 \geq 0$ and
 $5x_1 + 1x_2 + 2x_3 \leq 13$
 $2x_1 + 3x_2 + 1x_3 \leq 11$
 $1x_1 + 1x_2 + 4x_3 \leq 15$

(D)
 $\min 13y_1 + 11y_2 + 15y_3$
 subject to $y_1, y_2, y_3 \geq 0$ and
 $5y_1 + 2y_2 + 1y_3 \geq 9$
 $1y_1 + 3y_2 + 1y_3 \geq 7$
 $2y_1 + 1y_2 + 4y_3 \geq 1$

The simplex algorithm gives the following sequence of dictionaries:

$$\begin{array}{l|l|l} x_4 = 13 - 5x_1 - 1x_2 - 2x_3 & x_1 = 2.6 - 0.2x_4 - 0.2x_2 - 0.4x_3 & x_1 = (28 - 3x_4 + 1x_5 - 5x_3)/13 \\ x_5 = 11 - 2x_1 - 3x_2 - 1x_3 & x_5 = 5.8 + 0.4x_4 - 2.6x_2 - 0.2x_3 & x_2 = (29 + 2x_4 - 5x_5 - 1x_3)/13 \\ x_6 = 15 - 1x_1 - 1x_2 - 4x_3 & x_6 = 12.4 + 0.2x_4 - 0.8x_2 - 3.6x_3 & x_6 = (138 + 1x_4 + 4x_5 - 46x_3)/13 \\ z = 0 + 9x_1 + 7x_2 + 1x_3 & z = 23.4 - 1.8x_4 + 5.2x_2 - 2.6x_3 & z = 35 - 1x_4 - 2x_5 - 3x_3 \end{array}$$

From the final dictionary, we see that $(x_1, x_2, x_3) = (28/13, 29/13, 0)$ is the optimal solution and the optimal value is $9(28/13) + 7(29/13) + 1(0) = 35$.

The final row of the last (and optimal) dictionary can be written as

(1)
$$z = 35 - 0x_1 - 0x_2 - 3x_3 - \underline{1}x_4 - \underline{2}x_5 - \underline{0}x_6.$$

All of the coefficients in this expression are nonpositive, since negative ones cannot exist in the final dictionary. We will show next that $(y_1, y_2, y_3) = (1, 2, 0)$ is an optimal solution to (D). To demonstrate this, it suffices to show that $(1, 2, 0)$ is feasible:

(2)
$$\begin{aligned} 5(1) + 2(2) + 1(0) &\geq 9 \\ 1(1) + 3(2) + 1(0) &\geq 7 \\ 2(1) + 1(2) + 4(0) &\geq 1 \\ 1, 2, 0 &\geq 0 \end{aligned}$$

and that the value of the dual objective function $13y_1 + 11y_2 + 15y_3$ at $(y_1, y_2, y_3) = (1, 2, 0)$ is equal to the optimal value for (P):

(3)
$$13(1) + 11(2) + 15(0) = 35$$

These statements are obviously true in this particular case. But why will it always work? To see why, rewrite (1) as

$$\begin{aligned} 9x_1 + 7x_2 + 1x_3 &= 35 - 0x_1 - 0x_2 - 3x_3 - 1x_4 - 2x_5 - 0x_6 \\ &= 35 - \underline{0}x_1 - \underline{0}x_2 - \underline{3}x_3 - \underline{1}(13-5x_1-1x_2-2x_3) - \underline{2}(11-2x_1-3x_2-1x_3) - \\ &\underline{0}(15-1x_1-1x_2-4x_3) \end{aligned}$$

Equating the constant term on the left hand side of the equation with the constant term on the right hand side gives $1(13)+2(11)+0(15) = 35$, which is equation (3). Equating the coefficients of x_1 , x_2 , and x_3 on the right and left sides, we get the following equations:

$$\begin{aligned} 1(5)+2(2)+0(1) &= 9 + \underline{0} \geq 9 \\ 1(1)+2(3)+0(1) &= 7 + \underline{0} \geq 7 \\ 1(2)+2(1)+0(4) &= 4 + \underline{3} \geq 4 \end{aligned}$$

which is the system of inequalities (2). The underlined numbers are nonnegative because they appear as coefficients in (1).