assumptions: A is an $m \times n$ matrix, $\mathrm{rank}(A)=m$, $A=[a_1,\dots,a_n]$ (so that a_i is the i^{th} column of A).

A solution x to $Ax=b$ is <u>basic</u> if $\{a_i : x_i \neq 0\}$ is linearly independent.

Note: The equation $Ax=b$ has at most $\binom{n}{m}$ basic solutions (thus finitely many). Consider the linear programming problem

(LP) $\max c'x : Ax=b,\ x \geq 0.$

$x \in \mathbf{R}^n$ is <u>feasible</u> for (LP) if $Ax=b$ and $x \geq 0$. $x \in \mathbf{R}^n$ is <u>optimal</u> for (LP) if x is feasible for (LP) and

$$(Ay=b \text{ and } y \geq 0) \Longrightarrow c'x \geq c'y$$

<u>Fundamental Theorem</u>: If (LP) has an optimal solution then it has a basic optimal solution.

<u>Proof</u>: Suppose that (LP) does have an optimal solution. Let p be the smallest integer for which an optimal solution exists having p nonzero entries. Choose an optimal solution x having p nonzero entries. Without loss of generality, we can write $x^t = (x_1,\dots,x_p,0,\dots,0)$ with $x_1>0,\dots,x_p>0$. Let $\mu = c'x$ be the optimal value of (LP).

Our goal is to show that (LP) has a basic optimal solution. If x itself is basic, there is nothing to show, so assume it is not. The proof will be complete when we have show that this assumption leads us to a contradiction.

Since x is not basic, $\{a_i : x_i \neq 0\}$ is linearly dependent, meaning that there are numbers $y_1,\dots,y_p$, not all zero, such that $y_1 a_1+\dots+y_p a_p=0$. Equivalently, we can write $Ay=0$, where $y^t=(y_1,\dots,y_p,0,\dots,0)$.

There is some positive number ε such that

$$x_1+ty_1>0,\dots,x_p+ty_p>0 \text{ whenever } -\varepsilon<t<\varepsilon.$$

Since $Ay=0$ and $Ax=b$, it follows that

$$A(x+ty)=b \text{ for all } t \in \mathbf{R}.$$

Combining these two observations, it follows that

$$x+ty \text{ is feasible for (LP) whenever } -\varepsilon<t<\varepsilon.$$

Since x is optimal for (LP), we have

$$c'(x+ty) \leq c'x = \mu \quad \text{whenever } -\varepsilon<t<\varepsilon.$$

But then $t(c'y) \leq 0$ whenever $-\varepsilon<t<\varepsilon$, which implies $c'y=0$.

Choose t^* as close to zero as possible so that $x+t^*y$ has $p-1$ nonzero entries. Then $x+t^*y \geq 0$ and $A(x+t^*y)=b$ so $x+t^*y$ is feasible for (LP). Also, $c'(x+t^*y) = c'x = \mu$, so $x+t^*y$ is an optimal solution to (LP). This, however, contradicts that p is the smallest number for which an optimal solution exists having p nonzero entries.