January 12, 2000

Example showing all basic solutions

Click on the link "Example showing all basic solutions" to see a sheet (the "sheet") showing all of the basic solutions to the equation Ax=b, for the matrices A and b above. Ignore, for the moment, all references to the objective function; we are only considering the equation Ax=b.

Look at the entry that looks like this:

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columns: {1, 2, 4, 6}   objective = 0   x = (4,-2,0,1,0,1,0)
{{1, 0, 0, 0}, {0, 1, 0, 0}, {1, -1, 0, 0}, {0, 0, 1, 0},
 {0, -1, 1, 0}, {0, 0, 0, 1}, {-1, 1, 0, 1}, {4, -2, 1, 1}}
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This is nothing mysterious. It is just the information we just computed. The 4-tuples are just the columns of the matrix we computed above, and the last 4-tuple is the right-hand-side vector we computed above. The vector x=(4,-2,0,1,0,1,0) is the basic solution we computed above.

The sheet shows the results of exactly the same computation performed for every four columns of A that are linearly independent.

Note the distinction between "feasible basic solutions" and "basic solutions". The ones labeled as "feasible" have the property that the right-hand-side vector is nonnegative.

some convenient notation

If Ax=b is a system of linear equations, where A is $m\times n$ of rank m, and if $B\subset\{1,...,m\}$ is an index set of size n that corresponds to a basic set of columns of A, then A_B refers to the square submatrix of A consisting of those columns whose indices are in the set B. A_N is the submatrix consisting of the other (nonbasic) columns.

The system Ax=b can then be rewritten as $A_Bx_B+A_Nx_N=b$. Multiplying this equation by A_B^{-1} yields the equivalent system $x_B+(A_B^{-1}A_N)x_N=A_B^{-1}b$ that makes evident the basic solution corresponding to B.

For example, if A and b, and if B={1,2,4,6},

$$A = \begin{pmatrix} 1&1&0&1&0&0&0\\ 1&1&0&0&-1&0&0\\ 1&0&1&0&0&1&0\\ 1&0&1&0&0&0&-1 \end{pmatrix},\ \text{and } b = \begin{pmatrix}3\\2\\5\\4\end{pmatrix}\ \text{(as before), then } A_B=\begin{pmatrix} 1&1&1&0\\ 1&1&0&0\\ 1&0&0&1\\ 1&0&0&0 \end{pmatrix},\ A_N=\begin{pmatrix} 0&0&0\\ 0&-1&0\\ 1&0&0\\ 1&0&-1 \end{pmatrix}$$

$$,\ x_N=\begin{pmatrix}x_3\\x_5\\x_7\end{pmatrix},\ x_B=\begin{pmatrix}x_1\\x_2\\x_4\\x_6\end{pmatrix},\ \text{and } A_B^{-1}A_N = \begin{pmatrix} 1&0&-1\\ -1&-1&1\\ 0&1&0\\ 0&0&1 \end{pmatrix}.$$

The equation $x_B+(A_B^{-1}A_N)x_N=A_B^{-1}b$ is the same as the equation

$$\begin{pmatrix} 1&0&1&0&0&0&-1\\ 0&1&-1&0&-1&0&1\\ 0&0&0&1&1&0&0\\ 0&0&0&0&0&1&1 \end{pmatrix} x = \begin{pmatrix}4\\-2\\1\\1\end{pmatrix}$$

that we derived before. When we write the equation like this, we refer to it as "tableau" format. The examples on the sheet are in tableau format. When we put x_B on the left hand side and write

$$x_B = A_B^{-1}b - (A_B^{-1}A_N)x_N$$

then Chvátal refers to this as "dictionary" format. In our example, this would look like this:

$$\begin{aligned} x_1 &= 4 - x_3 + x_7 \\ x_2 &= -2 + x_3 + x_5 - x_7 \\ x_4 &= 1 - x_5 \\ x_6 &= 1 - x_7 \end{aligned}$$

HW #6 Suppose we have a system $\begin{pmatrix} 1 & 3 & 2 & 5 & 4 \\ 4 & 4 & -1 & 7 & 0 \end{pmatrix} x = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$. Rewrite this system in both tableau and dictionary formats to emphasize the basic solution for which x_1, x_3, and x_5 are zero. Does this basic solution satisfy $x \geq 0$?

visualization of basic feasible solutions

Consider the system of linear inequalities:

$$\begin{aligned} 2 \leq x_1 + x_2 \leq 3 \\ 4 \leq x_1 + x_3 \leq 5 \\ x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0 \end{aligned}$$

If we introduce slack variables x_4, x_5, x_6, x_7, the system becomes

$$\begin{aligned} x_1 + x_2 + x_4 &= 3 \\ x_1 + x_2 - x_5 &= 2 \\ x_1 + x_3 + x_6 &= 5 \\ x_1 + x_3 - x_7 &= 4 \end{aligned}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0,\ x_6 \geq 0,\ x_7 \geq 0$$

Notice that the system of equations is exactly the same as our example. (Remark: in class, I called this standard form. Sorry, that's a mistake -- see Chvatal for the correct definition of "standard form")

We can draw a picture to visualize the feasible set in $x_1x_2x_3$-space.

HW #7 Analogous to the picture we drew in class, draw a picture of the same feasible set in $x_2x_3x_7$-space, labelling the vertices A, B, C, and D.

Hint: On the sheet, find the tableau representation for the basic solution for which x_1, x_4, x_5, and x_6 are the basic variables. Write out the four equations. Then, interpreting x_1, x_4, x_5, and x_6 as slack variables, changes these four equations into four inequalities involving only the nonbasic variables x_2, x_3, and x_7. Since there are just three variables, you are in home territory (3 dimensional space), where you can visualize things easily.

Here's another method that is easier but won't teach you quite as much. You can read off the eight vertices of the feasible set from the sheet. The sheet shows that the eight basic feasible solutions are: (2,0,2,1,0,1,0), (2,0,3,1,0,0,1), and so forth. Looking at the second, third, and seventh components of these vectors, we see that the values of (x_2,x_3,x_7) at the vertices are (0,2,0), (0,3,1), and so forth. In this way you get all eight vertices of the set, and you can draw the picture from that.

Remark: You may wonder what exactly you are drawing when you draw this picture. And what is its relation to the picture we drew in class in $x_1x_2x_3$-space ? Both of these are drawings of the exact same polyhedron, the one that is described by $Ax=b$ and $x\geq 0$. This is a polyhedron that lives in R^7. The picture we drew in class is the projection of this object onto the subspace of R^7 spanned by the x_1, x_2, and x_3-axes. The one you are drawing in the homework assignment is the projection onto the subspace spanned by the x_2, x_3, and x_7-axes. Of course, in order to draw a picture on a piece of paper, we must project *again* the three-dimensional object onto a two-dimensional space! We are so used to doing this that it never confuses us -- but the projection of the seven dimensional object onto three-dimensional spaces is a very similar process.

HOMEWORK PROBLEMS 1-7 WILL BE COLLECTED FROM CAMPUS STUDENTS ON WEDNESDAY **JANUARY 19.** You may talk to anyone about homework problems. So get help if you get stuck -- the idea is to learn as much as possible.

For video students, all homeworks and exams are due two weeks after the on-campus students. In this case, that would be **February 2**.